Exercises - day V

1. Prove the infinite version of Hall's Theorem for locally finite graphs.
2. Prove the infinite version of Hall's Theorem for countable graphs.
3. Let G be a connected graph, and let A and B be disjoint sets of vertices of G such that $G[A]$ and $G[B]$ are connected. Let P and Q be the edge sets of $G[A]$ and $G[B]$ respectively. Describe, in graph-theoretic terms, what waves for the pair $(M_{FC}(G)/P \backslash Q, M_{FC}(G)^*/P \backslash Q$ look like. Prove directly that any union of waves is a wave.
4. A *dicut* in a directed graph is a cut in which all edges are directed the same way across the cut. A *dijoin* in a directed graph is a set of edges meeting all dicuts. The Lucchesi-Younger Theorem states that the smallest dijoin is the same size as the largest family of disjoint dicuts. Find a meaningful generalisation of this theorem to infinite directed graphs (you don't have to prove your generalisation is true).

Reminder (infinite version of Hall's Theorem): Let G be a bipartite graph on (A, B). For $A' \subseteq A$ we denote by $N(A')$ the set of neighbours of A. We say that one set of vertices of G is *matchable* into another if there is an injection of the first set into the second using edges of G. Let G be a bipartite graph as above such that for any $A' \subseteq A$, if $N(A')$ is matchable into A' then the matching hits all vertices of A'. The infinite version of Hall's Theorem states that under these circumstances A is matchable into B.