Exercises - day I

1. Let $\mathcal{I}$ be the set of independent sets of a matroid and let $\mathcal{C}$ consist of those edge sets that are minimal with the property that they are not in $\mathcal{I}$. Prove that $\mathcal{C}$ satisfies the circuit axioms. How can we recover $\mathcal{I}$ from $\mathcal{C}$?

2. Prove that minors of matroids are matroids.

3. The *algebraic circuits* of a graph G are the edge sets of finite cycles and double rays.

   Prove that if G is locally finite, then the set of algebraic circuits is the set of circuits of a matroid. What about non-locally finite graphs G?

4. Let M and N be a *twinned pair of matroids*, that is, the set of circuits of M is the set of finite circuits of N and the set of cocircuits of N is the set of finite cocircuits of M.

   Prove that M and N have a common base.

Reminder:
Independence axioms

A subset $\mathcal{I}$ of $\mathcal{P}(E)$ is called the set of *independence sets of a matroid* if and only if it satisfies the following.

(I1) $\emptyset \in \mathcal{I}$.

(I2) $\lceil \mathcal{I} \rceil = \mathcal{I}$, that is, $\mathcal{I}$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} \mid I \subseteq I' \subseteq X\}$ has a maximal element.

Circuit axioms

A subset $\mathcal{C}$ of $\mathcal{P}(E)$ is called the set of *circuits of a matroid* if and only if it satisfies the following.

(C1) $\emptyset \notin \mathcal{C}$

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) Whenever $X \subseteq C \in \mathcal{C}$ and $\{C_x \mid x \in X\} \subseteq \mathcal{C}$ satisfies $x \in C_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in C \setminus \left(\bigcup_{x \in X} C_x\right)$ there exists a $C' \in \mathcal{C}$ such that $z \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X$.

(CM) Let I be a set not containing any element of $\mathcal{C}$. Let X be a superset of I. Then there a set J with $I \subseteq J \subseteq X$ such that J does not include any element of $\mathcal{C}$ but for any $e \in X \setminus I$ there is some $o \in \mathcal{C}$ with $e \in o \subseteq J + e$.