The Rationality Problem in Invariant Theory

Christian Böhning
Mathematisches Institut
Georg-August-Universität Göttingen

Contents

Introduction

Invariant theory as a mathematical discipline on its own originated in England around the middle of the nineteenth century with Cayley's papers on hyperdeterminants and his famous Memoirs on Quantics, followed by Salmon, Sylvester and Boole, and Aronhold, Clebsch and Gordan in Germany. There was also a third school in Italy associated with the names of Brioschi, Cremona, Beltrami and Capelli. The techniques employed in this early phase, long before Hilbert transformed the subject with his conceptual ideas, were often computational and symbolic in nature. One of the main questions was, given a linear algebraic group G and finite-dimensional G-representation V over $\mathbb{C}$, to describe the algebra of invariant polynomial functions $\mathbb{C}[V]^G$ explicitly; in fact, most attention was given classically to the case where $G = \mathrm{SL}_2(\mathbb{C})$ or $G = \mathrm{SL}_3(\mathbb{C})$ and V is a space of binary or ternary forms of some fixed degree.

Suppose now G to be connected and semisimple. Today we know by work of Popov that the algebra of invariants $\mathbb{C}[V]^G$ can be arbitrarily complicated: a natural measure for its complexity is the length of its syzygy chain or in other words its homological dimension $\mathrm{hd}(\mathbb{C}[V]^G)$. Then (see e.g. [Po92], Chapter 3) it is known that if G is nontrivial, then for any $n \in \mathbb{N}$, there exists a G-module V with $\mathrm{hd}(\mathbb{C}[V]^G) > n$ and there exist, up to isomorphism and addition of trivial direct summands, only finitely many G-modules with $\mathrm{hd}(\mathbb{C}[V]^G) \leq n$. Moreover, the complexity of invariant rings increases quite rapidly: classically, a finite generating set and finite set of defining relations for $\mathbb{C}[\mathrm{Sym}^d(\mathbb{C}^2)^\vee]^{\mathrm{SL}_2(\mathbb{C})}$ was only obtained for $d \leq 6$ to which the 20th century (Dixmier & Lazard, Shioda) contributed just $d = 7, 8$. For $d > 8$ the homological dimension of the algebra of invariants is known to be greater than 10 (cf. [Po-Vi], §8).

Thus, algebraically, one is lead to ask: when is the structure of invariants of a G-module V as simple as possible? If we interpret this as asking when

$\mathbb{C}[V]^G$ is free, i.e. has algebraically independent homogeneous generators, then, by Popov's theorem, the classification of such V is a finite problem and more information on it can be found in [Po92], [Po-Vi]. One can also try to classify G-modules V with $\mathbb{C}[V]^G$ of fixed homological dimension. However, the situation aquires a very interesting different flavour if we shift from a biregular to a birational point of view, and ask

> When is $\mathbb{C}(V)^G$, the field of invariant rational functions, a purely transcendental extension of $\mathbb{C}$ or, as we will say, *rational*?

This is the main question various aspects of which we will treat in this work. $\mathbb{C}(V)^G$ is always of finite transcendence degree over $\mathbb{C}$ (there is no Hilbert's 14th Problem birationally), and we can ask this for any linear algebraic group G whatsoever. If G is not assumed to be connected, there are examples by Saltman [Sa] that $\mathbb{C}(V)^G$ need not even become rational after adjunction of a number of additional indeterminates ($\mathbb{C}(V)^G$ is not *stably rational*). G can be taken as a finite solvable group acting on V through its regular representation. This contradicts a conjecture put forward originally by Emmy Noether.
The quite astonishing fact, though, given the complexity of invariant rings themselves, is that no example with irrational $\mathbb{C}(V)^G$ is known if G is assumed to be connected! Putting $X = V$, we can reinterpret our original question as asking: when is the quotient variety X/G rational? X/G is taken in the sense of Rosenlicht and well-defined up to birational equivalence. One may replace X by e.g. a rational homogeneous variety and ask the same question: again no example of an irrational quotient X/G is known if G is connected. The introduction of the geometric point of view is not only a reformulation, but an indispensible step for any progress on our original algebraic problem. One may add as another example the solution to the Lüroth Problem in dimension 2: is an algebraic function field L of transcendence degree 2 over $\mathbb{C}$ which is contained in a purely transcendental extension of $\mathbb{C}$ itself a purely transcendental extension of $\mathbb{C}$? The affirmative answer follows as a corollary of Castelnuovo's Theorem characterizing smooth projective rational surfaces as those that do not have (non-zero) holomorphic one-forms and whose bicanonical linear system is empty. There is apparently no purely algebraic proof of this fact, though there was a time when some people tried to rewrite the Italian birational theory of algebraic surfaces in terms of function fields. We mention that there are counter-examples to the Lüroth Problem in dimension 3 and higher (cf. Artin and Mumford [A-M]). So there are examples

of *unirational* algebraic function fields L (unirational means contained in a purely transcendental extension of $\mathbb{C}$) of transcendence degree ≥ 3 which are not themselves purely transcendental extensions of $\mathbb{C}$. There are also examples of stably rational non-rational L for transcendence degree 3 and higher [B-CT-S-SwD], which is the solution to the *Zariski problem.* Thus we have the strict inclusions

$$\{ \text{ rational } L\} \subsetneqq \{ \text{ stably rational } L\} \subsetneqq \{ \text{ unirational } L\} \ .$$

Another reason to study quotients of the form V/G (or $\mathbb{P}(V)/G$) is that many moduli spaces in algebraic geometry are of this so-called *linear type.* For example, $\mathfrak{M}_g$, the moduli space of curves of genus g, is known to be of linear type for $1 \leq g \leq 6$. For example, $\mathfrak{M}_1 \simeq \mathbb{P}(\mathrm{Sym}^4(\mathbb{C}^2)^\vee)/\mathrm{SL}_2(\mathbb{C})$ is the ubiquitous moduli space of elliptic curves, and $\mathfrak{M}_2 \simeq \mathbb{P}(\mathrm{Sym}^6(\mathbb{C}^2)^\vee)/\mathrm{SL}_2(\mathbb{C})$ because a genus 2 curve is a double cover of $\mathbb{P}^1$ branched in 6 points via its canonical map. $\mathfrak{M}_3 \simeq \mathbb{P}(\mathrm{Sym}^4(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ since a general (non-hyperelliptic) curve of genus 3 is realized as a smooth quartic in $\mathbb{P}^2$ via the canonical embedding. We do not discuss $\mathfrak{M}_4$, $\mathfrak{M}_5$, $\mathfrak{M}_6$, but just remark that certainly $\mathfrak{M}_g$ ceizes to be of linear type at some point because for $g \geq 23$, $\mathfrak{M}_g$ is not even unirational. Other examples of moduli spaces of linear type are the moduli spaces of polarized K3 surfaces of degree d for $d = 2,\ 4,\ 6,\ 10$ (these classify pairs $(S,\, h)$ where S is a smooth K3 surface and h an ample class with $h^2 = d$ on S), or many moduli spaces of vector bundles. Of course, one should add to this list moduli spaces such as $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee)/\mathrm{SL}_{n+1}(\mathbb{C})$, the moduli space of degree d hypersurfaces in $\mathbb{P}^n$ (for projective equivalence) which are of linear type by definition, and very interesting in their own right.
The transgression in the behaviour of $\mathfrak{M}_g$ from being rational/unirational for small g and of general type for g large illustrates an important point: rational (or unirational) moduli spaces emerge as the most interesting examples (whereas the general curve of genus g for large g is rather hard to put hands on as a mathematical object). In general, rational varieties (or those close to being rational) are those that appear most frequently in applications in mathematics and make up the greatest part of one's motivating examples in algebraic geometry, though they are only a very small portion in the class of all varieties. It is precisely the fact that they are for the most part tangible objects and amenable to concrete study which explains their importance, and the wildness and absence of special features, symmetries etc. which lessens the impact of the rest of varieties on the whole of mathematics.

What methods are there to tackle the rationality problem for $\mathbb{P}(V)/G$? This is discussed in great detail in Chapters 1 and 2, so we content ourselves here with emphasizing some general structural features and recurring problems.

- If V and W are representations of the linear algebraic group G where the generic stabilizer is trivial, then $\mathbb{C}(V)^G$ and $\mathbb{C}(W)^G$ are stably equivalent, i.e. they become isomorphic after adjoining some number of indeterminates to each of them. This is the content of the so-called "no-name lemma" of Bogomolov and Katsylo [Bo-Ka]. So the stable equivalence type is determined by the group G alone in this case, and in many cases one can prove easily that a space V/G is stably rational.

- If one wants to prove rationality for a quotient X/G (X could be a linear G-representation or a more general G-variety), then, after possibly some preparatory reduction steps consisting of taking sections for the G action on X and thus reducing G to a smaller group and replacing X by a subvariety, virtually all the methods for proving rationality consist in introducing some fibration structure in the space X: one finds a G-equivariant rational map $\varphi : X \dashrightarrow Y$ to some base variety Y such that Y/G is stably rational, and the generic fibre of φ is rational, and then one tries to use descent to prove that $X/G \dashrightarrow Y/G$ is birational to a Zariski bundle over Y/G with rational fibre.

- In the examples which occur in practice where, in the situation of the previous item, X is generically a G-vector bundle over Y, the map φ can almost always be viewed as induced from a resolution of singularities map $H \times_P F \to \mathfrak{S}$ where $\mathfrak{S}$ is a stratum of the unstable cone in a representation W of a reductive group $H \supset G$ and $F \subset W$ is some subspace which is stable under a parabolic subgroup $P \subset H$. This method is described in detail in Section 1.3 of Chapter 1.

- In the set-up of the previous two items, one almost always has to prove that the map φ satisfies certain nondegeneracy or genericity conditions, and this is usually a hard part of the proof. As an illustration, one can take a surface S in $\mathbb{P}^4$ which is the intersection of two quadric hypersurfaces Q_1 and Q_2. To prove rationality of S one projects from a line l common to both Q_1 and Q_2, but one has to check that the projection is dominant unto $\mathbb{P}^2$; otherwise S could be a bundle over an elliptic curve which is irrational. Checking nondegeneracy typically involves the use of computer algebra, but special ideas are needed when e.g. one deals with an infinite number of spaces V_n/G, $n = 1, 2, \dots$. A trick used in Chapter 4 is to show that the data for which genericity has to be checked becomes a periodic function of n over a finite field $\mathbb{F}_p$, and then to use upper-semicontinuity over $\mathrm{Spec}(\mathbb{Z})$ to prove nondegeneracy over $\mathbb{Q}$ (or $\mathbb{C}$).

- Finally, it would be very nice to give an example of a space V/G (where V as before is a linear representation of the connected linear algebraic group G) which is *not rational*, if such an example exists at all. A possible candidate could be given by taking V_d the space of pairs of $d \times d$ matrices and $G = \mathrm{PGL}_d(\mathbb{C})$ acting on V_d by simultaneous conjugation. The corresponding invariant function field $\mathbb{C}(V_d)^G$ is not known to be rational or even stably rational in general. For further information see Section 1.2.1 in Chapter 1. To determine the properties of $\mathbb{C}(V_d)^G$ for general d is one of the major open and guiding problems in the subject. One should also remark that if a space V/G is stably rational, then if it were not rational, there would be practically no methods available today to prove this: the Clemens-Griffiths method of intermediate Jacobians (see [Is-Pr], Chapter 8) is limited to threefolds and the quotients V/G quickly have higher dimension, the Noether-Fano-Iskovskikh-Manin method (see *loc. cit.*) based on the study of maximal centers for birational maps has not been put to use in this context and it is hard to see how one should do it, and Brauer-Grothendieck invariants are not sensitive to the distinction between stably rational irrational and rational varieties.

The main known results of rationality for spaces V/G can be summarized as follows: in [Kat83], [Kat84], [Bogo2] and [Bo-Ka] it is proven that all quotients $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2)^\vee)/\mathrm{SL}_2(\mathbb{C})$ are rational, so the problem is solved completely for binary forms. The moduli spaces $C(d) = \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ of plane curves of degree d are rational for $d \equiv 1 \pmod 4$, all d, and for $d \equiv 1 \pmod 9$, $d \geq 19$, by [Shep], and for $d \equiv 0 \pmod 3$, $d \geq 210$, by [Kat89]. This was basically everything that was known for ternary forms prior to the present work, but there were also several rationality results for $C(d)$ for small particular values of d. Though these are somewhat sporadic, they are very valuable and should be rated rather high since rationality of $C(d)$ can be very hard to prove for *small* d, cf. [Kat92/2], [Kat96] and Chapter 3 for the case of $C(4)$. For moduli spaces of hypersurfaces of degree d in $\mathbb{P}^n$ for $n \geq 3$ much less is known: they are rational for $n = 3$, $d = 1, 2, 3$, and $n > 3$, $d = 1, 2$, which is trivial except for $n = d = 3$ cf. [Be]. Likewise, spaces of mixed tensors do not seem to have been studied so far in a systematic way to my knowledge, maybe due to the smaller geometric relevance. We should add, however, that in [Shep], the rationality of $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^2)^\vee \otimes (\mathbb{C}^2)^\vee)/\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, the space of pencils of binary forms of degree d, is proven if d is even and $d \geq 10$. But for connected linear groups G other than $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{C})$, there are again no such good results as far as I know. The reader may turn to the surveys [Dolg1] and [CT-S] for more detailed information to complement our very coarse outline.

We turn to the description of our main results and the contents of the separate chapters. We will be rather brief because detailed information about a certain chapter can be found in the separate introduction preceding the respective chapter in the main body of the text.
Chapter 1 introduces basic notions, gives a detailed geometric discussion of the rationality problem for the quotient of the space of pairs of $n \times n$ matrices acted on by $\mathrm{PGL}_n(\mathbb{C})$ through simultaneous conjugation, and presents known results for specific groups, tori, solvable groups, special groups in the sense of Séminaire Chevalley. We then add a detailed exposition of a unifying technique for proving rationality of spaces V/G that comprises a lot of the known tricks; it uses the Hesselink stratification of the Hilbert nullcone and desingularizations of the strata in terms of homogeneous bundles culminating in Theorem 1.3.2.7. Though the method was sketched in [Shep89], it has not received such a systematic treatment so far. In Proposition 1.3.2.10 we prove a criterion for stable rationality of quotients of Grassmannians by an SL-action which is new, and in combination with Theorem 1.3.2.7 yields rationality of the moduli space of plane curves of degree 34 (Theorem 1.3.2.11) which was previously unknown. In section 1.4 we give a brief summary of further topics, cohomological obstructions to rationality (unramified cohomology) and aspects of the rationality problem over fields other than $\mathbb{C}$.
Chapter 2 contains a discussion of techniques available to prove rationality of spaces V/G; in part they fit in the framework of Theorem 1.3.2.7, but are presented on an elementary level with examples here which is necessary for concrete applications. In Proposition 2.2.1.5 we give a reduction of the field of rational functions of the moduli space of plane curves C of degree d to-

gether with a theta-characteristic θ with $h^0(C,\,\theta)=0$ to a simpler invariant function field which is new. Finally we give an account of a method for proving rationality due to P. Katsylo, which is based on consideration of zero loci of sections in G-bundles over rational homogeneous manifolds, and for which there is no good reference as far as we know. As an application we present a proof of the rationality of the space of 7 unordered points in $\mathbb{P}^2$ modulo projectivities due to Katsylo, since the reference is not easily accessible.

In Chapter 3 we give an account of Katsylo's proof of the rationality of $\mathfrak{M}_3$, the moduli space of curves of genus 3, which is birationally the same as the moduli space of plane quartic curves under projectivities. This is inserted as an example of the difficulties that arise in dealing with moduli spaces of plane curves of small degree, and is not so much intended as a reproduction of Katsylo's proof, but rather as an attempt to geometrize as many as possible of Katsylo's purely algebraic and computational arguments. We hope that we have at least partially succeeded in clarifying the underlying geometry and structure of proof of this difficult result.

The main new result of Chapter 4 is Theorem 4.1.0.1, saying that the moduli space of plane curves of degree d is rational for all sufficiently large d. We obtain effective bounds for the degree d: for $d \equiv 0 \pmod 3$, $d \geq 210$ this is the result of [Kat89], and we prove that for $d \equiv 1 \pmod 3$, $d \geq 37$, and $d \equiv 2 \pmod 3$, $d \geq 65$, this space is rational, too. The proof uses the methods of covariants of [Shep] in slightly modified form, the method of writing a form as a sum of powers of linear forms, and some combinatorics and finite field techniques that may be interesting in other situations.

In Chapter 5 we first prove an analogue of the Clebsch-Gordan formula for $\mathrm{SL}_3(\mathbb{C})$ (Theorem 5.2.1.1) which allows one to describe all bilinear maps $U \times V \to W$ of finite-dimensional $\mathrm{SL}_3(\mathbb{C})$-representations explicitly, and to manipulate them in a very efficient way in computer algebra systems. In addition to this new result, we also introduce other computational tools for dealing with maps of this type in an algorithmically efficient way, based again on writing bihomogeneous polynomials as sums of powers of linear forms and on interpolation. As an example of these techniques, we prove the rationality of $\mathbb{P}(V(4,\,4))/\mathrm{SL}_3(\mathbb{C})$ (Theorem 5.4.2.2). The main application of these new computational methods, however, is the proof that the moduli space of plane curves of degree d is always rational, except possibly for 15 values of d which are given explicitly. This presents a substantial advancement over the state of knowledge on rationality of moduli of plane curves prior to the present work.

Finally I would like to thank Yuri Tschinkel for many useful discussions and for his proposal to work on the subject which turned out to be so rewarding; furthermore I am very grateful to Fedor Bogomolov for many stimulating discussions and shaping my view of the subject. Special thanks go to Hans-Christian Graf von Bothmer. Chapters 4 and 5 present joint work with him, and without his mathematical and computational skills the results in these two Chapters could not have been obtained.

Chapter 1

Fundamental structures in invariant theory (with an eye towards the rationality problem)

1.1 Introduction

In this chapter we introduce the basic notions involved in the rationality problem for invariant function fields, and discuss the action of $\mathrm{PGL}_n(\mathbb{C})$ on pairs of $n \times n$ matrices by simultaneous conjugation as a guiding example. We give various results for specific groups, tori, solvable groups, and special groups in the sense of Séminaire Chevalley (cf. [Se58]). We introduce the Hesselink stratification of the nullcone of a representation of a reductive group as a unifying concept for various methods for proving rationality of quotient spaces. Together with a new criterion for the stable rationality of certain quotients of Grassmannians by an SL_n-action (Proposition 1.3.2.10), we obtain the rationality of the moduli space of curves of degree 34 (Theorem 1.3.2.11).
In section 1.4 we give a short overview of unramified cohomology and the rationality problem over an arbitrary ground field. Apart from this section, we work over the field of complex numbers $\mathbb{C}$ throughout this text.

1.2 The rationality problem

1.2.1 Quotients and fields of invariants

Let G be a linear algebraic group over $\mathbb{C}$ acting (morphically) on an algebraic variety X.

Definition 1.2.1.1. A quotient of X by the action of G, denoted by X/G, is any model of the field $\mathbb{C}(X)^G$ of invariant rational functions; a quotient is thus uniquely determined up to birational equivalence, and since we are interested in birational properties of X/G here, we will also refer to it as *the* quotient of X by G.

Note that $\mathbb{C}(X)^G$ is certainly always finitely generated over $\mathbb{C}$, since it is a subfield of $\mathbb{C}(X)$ which is finitely generated over $\mathbb{C}$. In the context of fields there is no fourteenth problem of Hilbert ([Nag], [Stein], [Muk1])!
Of course one would like X/G to parametrize generic G-orbits in X to be able to apply geometry.

Definition 1.2.1.2. If V is a G-variety, then a variety W together with a morphism $\pi : V \to W$ is called a geometric quotient if

(1) π is open and surjective,

(2) the fibres of π are precisely the orbits of the action of G on V,

(3) for all open sets $U \subset W$, the map $\pi^* : \mathcal{O}_W(U) \to \mathcal{O}_V(\pi^{-1}(U))^G$ is an isomorphism.

One then has the following theorem due to Rosenlicht ([Ros], Thm. 2).

Theorem 1.2.1.3. *There exists a nonempty G-stable open subset $U \subset X$ in every G-variety X such that there is a geometric quotient for the action of G on U.*

For a modern proof, see [Po-Vi] or [Gross].

Definition 1.2.1.4. (1) An algebraic variety X is called rational if there exists a birational map $X \dashrightarrow \mathbb{P}^n$ for some n.

(2) X is called stably rational if there exists an integer n such that $X \times \mathbb{P}^n$ is rational.

(3) X is called unirational if there exists a dominant map $\mathbb{P}^n \dashrightarrow X$ for some n.

Clearly, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and the implications are known to be strict ([A-M], [B-CT-S-SwD]). Since (1)-(3) are properties of the function field $\mathbb{C}(X)$, we will also occasionally say that $\mathbb{C}(X)$ is rational, stably rational or unirational. There are other well-known notions capturing properties of varieties which are close to the rational varieties, notably retract rationality ([Sa2]) and rational connectedness ([Koll]), which we have no use for here. We can now state the main problem which we are concerned with in this work.

Problem 1.2.1.5. Let G be a connected linear algebraic group, and let V be a G-representation. V is always assumed to be finite-dimensional.

(1) Is V/G rational?

(2) Is $\mathbb{P}(V)/G$ rational?

Remark 1.2.1.6. (1) The existence of stably rational, non-rational varieties shows that the answer to the preceding problem is clearly no if V is replaced by an arbitrary rational variety; just take X non-rational such that $X \times \mathbb{C}^*$ is rational, and let the multiplicative group $\mathbb{G}_m$ act on the second factor of $X \times \mathbb{C}^*$ such that $(X \times \mathbb{C}^*)/\mathbb{G}_m$ is birational to X.

(2) By the results of Saltman ([Sal]), the answer to (1) is likewise no if G is not assumed to be connected; G can even be taken to be a finite solvable group acting on V via its regular representation.

(3) The rationality of $\mathbb{P}(V)/G$ implies the rationality of V/G. One uses the following theorem of Rosenlicht [Ros].

Theorem 1.2.1.7. *If G is a connected solvable group acting on a variety X, then the quotient map $X \dashrightarrow X/G$ has a rational section $\sigma : X/G \dashrightarrow X$.*

In our case we have the quotient map $V/G \dashrightarrow \mathbb{P}(V)/G$ for the action of the torus $T = \mathbb{C}^*$ by homotheties on V/G. If T_{ineff} is the ineffectivity kernel for the action of T on V/G, the action of T/T_{ineff} on V/G is generically free (T_{ineff} coincides with the so-called stabilizer in general position for a torus action, [Po-Vi], §7.2). Hence by Theorem 1.2.1.7, the preceding quotient map is a locally trivial T/T_{ineff}-principal bundle in the *Zariski* topology, so that V/G is birational to $\mathbb{P}(V)/G \times T/T_{\text{ineff}}$ which is rational if $\mathbb{P}(V)/G$ is.

(4) $\mathbb{P}(V \oplus \mathbb{C})/G$ (trivial G-action on $\mathbb{C}$) is birational to V/G: Map $v \in V$ to $[(v, 1)]$ in $\mathbb{P}(V \oplus \mathbb{C})$.

One (and my main) motivation for Problem 1.2.1.5 comes from the fact that many moduli spaces in algebraic geometry are of the form $\mathbb{P}(V)/G$. But a solution to Problem 1.2.1.5 or parts of it typically has diverse applications throughout algebra, representation theory and geometry. We discuss one famous and guiding example in detail to illustrate this.

Let n be positive integer, $G = \mathrm{GL}_n(\mathbb{C})$, and let $V = \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ be two copies of the adjoint representation of G so that V is the space of pairs of $n \times n$-matrices $(A,\ B)$ and $g \in G$ acts on V by simultaneous conjugation:

$$g \cdot (A,\ B) = (gAg^{-1},\ gBg^{-1})\,.$$

Let $K_n := \mathbb{C}(V)^G$. The question whether K_n is rational is a well-known open problem. K_n is known to be unirational for all n, stably rational if n is a divisor of 420, rational for $n = 2,\ 3,\ 4$. Excellent surveys are [For02], [LeBr]. Here we just want to show how the field K_n shows up in several areas of mathematics and discuss some approaches to Problem 1.2.1.5 for K_n.

- Let $\mathfrak{Bun}_{\mathbb{P}^2}(k, n)$ be the moduli space of stable rank k vector bundles on $\mathbb{P}^2$ with Chern classes $c_1 = 0$, $c_2 = n$. It is nonempty for $1 < k \le n$. Then

  $$\mathbb{C}(\mathfrak{Bun}_{\mathbb{P}^2}(k, n)) \simeq K_d(t_1, \dots, t_N)$$

  where the t_i are new indeterminates, $d = \gcd(k, n)$ and $N = 2nk - k^2 - d^2$. In particular, for $k = n$, the field K_n is the function field of the moduli space of stable rank n vector bundles on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = n$. See [Kat91].
  The above identification arises as follows: From the monad description of vector bundles on $\mathbb{P}^2$ one knows that $\mathbb{C}(\mathfrak{Bun}_{\mathbb{P}^2}(k, n)) \simeq \mathbb{C}(S_k)^G$ where S_k consists of pairs (A, B) of matrices such that the eigenvalues of A are pairwise distinct and the rank of the commutator of A and B is equal to k. One then uses sections and the no-name lemma (see chapter 2) to prove the above isomorphism ([Kat91]).

- For details on the following see [Pro67], [Pro76] and [Pro]. Let $X = (x_{ij})$ and $Y = (y_{ij})$ be two generic $n \times n$-matrices (the x_{ij} and y_{ij} are commuting indeterminates), and let R be the subring generated by X and Y inside the ring of $n \times n$ matrices with coefficients in $\mathbb{C}[x_{ij};\, y_{ij}]$. R is called a ring of generic matrices. Let D be its division ring of fractions, C the centre of D. An element in the center of R is a scalar matrix $p \cdot \mathrm{Id}$ with p a polynomial in x_{ij} and y_{ij} which is necessarily a polynomial invariant of pairs of matrices. C is the field of quotients of the center of R, thus it is a subfield of K_n. On the other hand, it is known that K_n is generated by elements

$$\mathrm{tr}(M_1 M_2 \dots M_{j-1} M_j)$$

with $M_1 M_2 \dots M_{j-1} M_j$ an arbitrary word in the matrices A and B (so each M_i is either equal to A or B). Since D is a central simple algebra of dimension n^2 over its centre C, the trace of every element of D lies in C. Thus K_n is contained in C, and thus equals the centre of the generic division ring D.
[Pro67] also shows that if $\bar{C}$ is the Galois extension of C obtained by adjoining the roots of the characteristic polynomial of X to C, then the Galois group is the symmetric group $\mathfrak{S}_n$, $[\bar{C} : C] = n!$ and $\bar{C}$ is a purely transcendental extension of $\mathbb{C}$. This was a stimulus to study the rationality properties of K_n as a fixed field of $\mathfrak{S}_n$ acting on a rational function field over C ([For79], [For80]).

- K_d is the function field of the relative degree $g-1$ Jacobian $\mathcal{J}ac_d^{g-1} \to |\mathcal{O}_{\mathbb{P}^2}(d)|_{\mathrm{smooth}}$ over the family $|\mathcal{O}_{\mathbb{P}^2}(d)|_{\mathrm{smooth}} \subset |\mathcal{O}_{\mathbb{P}^2}(d)|$ of smooth projective plane curves of degree d. Here $g = (1/2)(d-1)(d-2)$ is the genus of a smooth plane curve of degree d. $\mathcal{J}ac_d^{g-1}$ parametrizes pairs $(C, \mathcal{L})$ consisting of a smooth plane curve of degree d and a line bundle $\mathcal{L}$ of degree $g-1$ on C. See [Beau00], section 3.

We will discuss in a little more detail now how the description of K_d as the function field of a relative Jacobian over a family of plane curves arises, and show how this can be used to give a simple geometric proof of the rationality of K_3 due to Michel van den Bergh.
The field K_d is related to $\mathcal{J}ac_d^{g-1}$ via the following theorem on representations of degree d plane curves as linear determinants.

Theorem 1.2.1.8. *Let C be a smooth plane curve of degree d, Jac_C^{g-1} its degree $g-1$ Jacobian, $\Theta \subset Jac_C^{g-1}$ the theta-divisor corresponding to degree $g-1$ line bundles on C which have nonzero global sections.*
For each $\mathcal{L} \in Jac_C^{g-1} \backslash \Theta$, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1)^d \longrightarrow \mathcal{L} \longrightarrow 0$$

with A a matrix of linear forms such that $\det A = F$ where F is a defining equation of C, $I_C = (F)$.
Conversely, every matrix A of linear forms on $\mathbb{P}^2$ with $\det A = F$ gives rise to an exact sequence as before where $\mathcal{L}$ (the cokernel of A) is a line bundle on C with $\mathcal{L} \in Jac_C^{g-1} \backslash \Theta$.

Proof. As is well known, the following are equivalent for a coherent sheaf on $\mathbb{P}^n$:

- $\Gamma_*(\mathcal{F}) := \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(i))$ is a Cohen-Macaulay module over the homogeneous coordinate ring S of $\mathbb{P}^n$.
- The sheaf $\mathcal{F}$ is locally Cohen-Macaulay and has trivial intermediate cohomology: $H^j(\mathbb{P}^n, \mathcal{F}(t)) = 0 \ \forall 1 \le j \le \dim \mathrm{Supp}(\mathcal{F}) - 1$, $\forall t \in \mathbb{Z}$.

Such a sheaf is called *arithmetically Cohen Macaulay* (ACM).
Now let $\mathcal{L}$ be a degree $g-1$ line bundle on C with $H^0(C, \mathcal{L}) = 0$. Put $\mathcal{M} := \mathcal{L}(1)$. The ACM condition is vacuous for line bundles on C ($\mathcal{M}_x$, $x \in C$, is of course always Cohen Macaulay since C is a reduced hypersurface). Moreover, one has

$$H^0(\mathbb{P}^2, \mathcal{M}(-1)) = H^1(\mathbb{P}^2, \mathcal{M}(-1)) = 0. \tag{1.1}$$

The vanishing of H^1 comes from Riemann-Roch which yields $\chi(\mathcal{L}) = \chi(\mathcal{M}(-1)) = 0$.
Since $\mathcal{M}$ is ACM, $\dim \mathrm{Supp}(\mathcal{M}) + \mathrm{proj.dim}\mathcal{M} = \dim \mathbb{P}^2$ by the Auslander-Buchsbaum formula, whence by Hilbert's syzygy theorem, $\mathcal{M}$ has a minimal graded free resolution

$$0 \to \bigoplus_{i=1}^r \mathcal{O}(-f_i) \xrightarrow{A} \bigoplus_{i=1}^r \mathcal{O}(-e_i) \longrightarrow \mathcal{M} \to 0$$

where, moreover, one has $e_i \ge 0$ for all i since $H^0(\mathbb{P}^2, \mathcal{M}(-1)) = 0$. The support of $\mathcal{M}$, the curve C, is defined by $\det A = 0$ set-theoretically, whence

$\det A$ is a power of F. If one localizes A at the generic point of C, the above exact sequence together with the structure theorem of matrices over a principal ideal domain yields $\det A = F$.
Now condition 1.1 yields that one has $e_i = 0$, all i, and $f_j = 1$, all j. Namely, the condition $H^1(C, \mathcal{M}(-1)) = 0$ means that $\mathcal{M}$ is a 0-regular sheaf in the sense of Castelnuovo-Mumford regularity ([Mum2]) whence

> $\mathcal{M}$ is spanned by $H^0(\mathcal{M})$ and for all $j \geq 0$
>
> $$H^0(\mathbb{P}^2, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{M}(j)) \to H^0(\mathbb{P}^2, \mathcal{M}(j+1))$$
>
> is surjective.

Thus $\mathcal{M}$ has the minimal graded free resolution

$$0 \to \bigoplus_{i=1}^r \mathcal{O}(-f_i) \xrightarrow{\ A\ } \bigoplus_{i=1}^r \mathcal{O} \longrightarrow \mathcal{M} \to 0$$

with $r = h^0(\mathcal{M})$. We get the exact sequence

$$H^1(\mathbb{P}^2, \mathcal{M}(-1)) \longrightarrow \bigoplus_{i=1}^r H^0(\mathbb{P}^2, \mathcal{O}(f_i - 2)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(-2))^r$$

which together with the fact that the f_i must be positive (the map induced by $\bigoplus_{i=1}^r \mathcal{O} \to \mathcal{M}$ on H^0 is an isomorphism) implies that we must have $f_i = 1$ for all i. It also follows that $d = r$ since $\det A = F$.

Conversely, suppose that A is a d by d matrix of linear forms on $\mathbb{P}^2$ with $\det A = F$ where F is a defining equation of the smooth curve C. Then one has an exact sequence

$$0 \longrightarrow \mathcal{O}(-2)^d \longrightarrow \mathcal{O}(-1)^d \longrightarrow \mathcal{L} \longrightarrow 0$$

where $\mathcal{L}$ is an ACM sheaf on C of rank 1, thus a line bundle. By the exact sequence, $H^0(\mathcal{L}) = H^1(\mathcal{L}) = 0$ whence $\deg \mathcal{L} = g - 1$, by Riemann-Roch. □

Corollary 1.2.1.9. *Let V_d be the vector space of d by d matrices of linear forms on $\mathbb{P}^2$. $\mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$ acts on V_d by $(M_1, M_2) \cdot A := M_1 A M_2^{-1}$. Then $V_d / \mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$ is birational to $\mathcal{J}ac_d^{g-1}$, the relative Jacobian of degree $g-1$ line bundles over the space of smooth degree d curves C in $\mathbb{P}^2$.*

Writing A in V_d as

$$A = A_0x_0 + A_1x_1 + A_2x_2, \quad A_i \in \mathrm{Mat}_{d\times d}(\mathbb{C}),$$

we may identify V_d with the space of triples $(A_0,\ A_1,\ A_2)$ of $d\times d$ scalar matrices A_i where $(M_1, M_2) \in \mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$ acts as

$$(M_1, M_2)\cdot(A_0,\ A_1,\ A_2) = (M_1A_0M_2^{-1},\ M_1A_1M_2^{-1},\ M_1A_2M_2^{-1})\,.$$

The subvariety

$$\{(\mathrm{Id},\ B_1,\ B_2)\,|\,B_1,\ B_2 \in \mathrm{Mat}_{d\times d}(\mathbb{C})\} \subset \mathrm{Mat}_{d\times d}(\mathbb{C}) \times \mathrm{Mat}_{d\times d}(\mathbb{C}) \times \mathrm{Mat}_{d\times d}(\mathbb{C})$$

is a $(\mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C}),\ \mathrm{GL}_d(\mathbb{C}))$-section in the sense of Chapter 2, 2.2.1. Hence K_d, the field of invariants for the action of $\mathrm{GL}_d(\mathbb{C})$ by simultaneous conjugation on pairs of matrices, is the function field of $\mathcal{J}ac_d^{g-1}$.

Remark 1.2.1.10. Instead of $\mathcal{J}ac_d^{g-1}$ it is occasionally useful to work with other relative Jacobians with the same function field K_d: since a line in $\mathbb{P}^2$ cuts out a divisor of degree d on a smooth plane curve C of degree d, we have for d odd

$$\mathcal{J}ac_d^{g-1} \simeq \mathcal{J}ac_d^{0}$$

since $g-1 = \frac{1}{2}d(d-3)$; also, in general,

$$\mathcal{J}ac_d^{g-1} \simeq \mathcal{J}ac_d^{\binom{d}{2}}\,.$$

Also note that $\mathcal{J}ac_d^{g}$ is rational since it is birational to a (birationally trivial) projective bundle over $\mathrm{Sym}^g\mathbb{P}^2$ which is rational, but this yields no conclusion for $\mathcal{J}ac_d^{g-1}$.

Theorem 1.2.1.11. *The field K_3, i.e. the function field of $\mathcal{J}ac_3^0$, is rational.*

Proof. We follow [vdBer]. We have to prove that the variety $\mathcal{J}ac_3^0$, parametrizing pairs $(C,\mathcal{L})$, where C is a smooth plane cubic and $\mathcal{L}$ is a line bundle of degree 0 on C, is rational. Fix once and for all a line $l \subset \mathbb{P}^2$. Let $\mathcal{L}$ be represented by a divisor D of degree 0 on C. For a general curve C, l intersects C in three points $p_1,\ p_2,\ p_3$ (uniquely defined by C *up to order*), and since by Riemann Roch $h^0(C, \mathcal{O}(p_i + D)) = 1$, there are uniquely determined points

$q_1,\ q_2,\ q_3$ on C with $p_i + D \equiv q_i$, the symbol $\equiv$ denoting linear equivalence. Thus

$$q_i + p_j + D \equiv q_j + p_i + D \implies q_i + p_j \equiv q_j + p_i \ \forall i,\ j \in \{1,\ 2,\ 3\}\,,$$

and if r_{ij} denotes the third point of intersection of the line $\overline{p_i q_j}$ through p_i and q_j with C, clearly

$$p_i + q_j + r_{ij} \equiv p_j + q_i + r_{ji}\,,$$

whence $r_{ij} \equiv r_{ji}$, and by Riemann Roch in fact $r_{ij} = r_{ji}$. Thus $r_{ij} = \overline{p_i q_j} \cap \overline{p_j q_i}$ lies on C, and we get nine points: $p_1,\ p_2,\ p_3,\ q_1,\ q_2,\ q_3$, and $r_{12},\ r_{13},\ r_{23}$.
Conversely, given three arbitrary points $p_1,\ p_2,\ p_3$ on l, and three further points $q_1,\ q_2,\ q_3$ in $\mathbb{P}^2$, we may set for $i < j$ $r_{ij} := \overline{p_i q_j} \cap \overline{p_j q_i}$ and find a cubic curve C through all of the $p_i,\ q_j,\ r_{ij}$. Then $D := q_1 - p_1$ is a degree 0 divisor on C. Applying the preceding construction, we get back the points we started with.
However note that the cubic curve C through $p_i,\ q_j,\ r_{ij}$ as before is not unique: generally, there is a whole $\mathbb{P}^1$ of such curves C. This is because any cubic C passing through the eight points $p_1,\ p_2,\ p_3,\ q_1,\ q_2,\ q_3,\ r_{12},\ r_{13}$ passes through the ninth point r_{23} since

$$q_1 + p_2 + r_{12} \equiv p_1 + q_2 + r_{12} \quad \text{and} \quad q_1 + p_3 + r_{13} \equiv p_1 + q_3 + r_{13}$$
$$\text{implies} \quad p_2 + q_3 \equiv p_3 + q_2 \quad \text{on } C$$

whence the intersection point $r_{23} = \overline{p_2 q_3} \cap \overline{p_3 q_2}$ necessarily lies on C. By explicit computation one may check that for generic choice of the p_i and q_j one gets indeed a pencil of cubic curves C.
The above can be summarized as follows:

- Let $\mathcal{P}$ be the parameter space inside $l^3 \times (\mathbb{P}^2)^3 \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))$ consisting of triples $((p_1,\ p_2,\ p_3),\ (q_1,\ q_2,\ q_3),\ C)$ where the p_i are three points on l, the q_j are three further points in $\mathbb{P}^2$, and C is a cubic curve through the $p_i,\ q_j$, and $r_{ij} := \overline{p_i q_j} \cap \overline{p_j q_i}$. Let the symmetric group $\mathfrak{S}_3$ act on $\mathcal{P}$ via

$$\sigma \cdot ((p_1,\ p_2,\ p_3),\ (q_1,\ q_2,\ q_3),\ C)$$
$$:= ((p_{\sigma(1)},\ p_{\sigma(2)},\ p_{\sigma(3)}),\ (q_{\sigma(1)},\ q_{\sigma(2)},\ q_{\sigma(3)}),\ C)\,.$$

  Then $\mathcal{J}ac_3^0$ is birational to $\mathcal{P}/\mathfrak{S}_3$.

- Let $\mathcal{Q}$ be the parameter space $l^3 \times (\mathbb{P}^2)^3$ of three points p_i on l and three additional points q_i in $\mathbb{P}^2$. $\mathfrak{S}_3$ acts on $\mathcal{Q}$:

  $$\sigma \cdot ((p_1,\, p_2,\, p_3),\ (q_1,\, q_2,\, q_3)) := ((p_{\sigma(1)},\, p_{\sigma(2)},\, p_{\sigma(3)}),\ (q_{\sigma(1)},\, q_{\sigma(2)},\, q_{\sigma(3)})\,.$$

  We have a forgetful map $\mathcal{P} \to \mathcal{Q}$ (the field of rational functions $\mathbb{C}(\mathcal{P})$ is a purely transcendental extension of $\mathbb{C}(\mathcal{Q})$ given by adjoining the solutions of a set of linear equations). Passing to the quotients, we get a map $\pi : \mathcal{P}/\mathfrak{S}_3 \dashrightarrow \mathcal{Q}/\mathfrak{S}_3$. Since the action of $\mathfrak{S}_3$ on $\mathcal{Q}$ is generically free, π is generically a $\mathbb{P}^1$-bundle (in the classical topology, i.e. a conic bundle).

To conclude the proof, it suffices to remark that the conic bundle π has a rational section, hence is birationally trivial: indeed, one just has to fix one further point $x \in \mathbb{P}^2$, and assigns to points $((p_1,\, p_2,\, p_3),\ (q_1,\, q_2,\, q_3))$ the triple $((p_1,\, p_2,\, p_3),\ (q_1,\, q_2,\, q_3),\ C)$ where C is the unique cubic passing through p_i, q_j, $r_{ij} := \overline{p_i q_j} \cap \overline{p_j q_i}$ and the point x. Moreover, the base $\mathcal{Q}/\mathfrak{S}_3$ is clearly rational: since the action of $\mathfrak{S}_3$ on $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3$ by permuting the factors is generically free, one sees from the no-name lemma (cf. Chapter 2, subsection 2.2.2) and the existence of sections for torus actions 1.2.1.7, that $\mathbb{C}(l^3 \times (\mathbb{P}^2)^3)^{\mathfrak{S}_3}$ is a purely transcendental extension of $\mathbb{C}((\mathbb{P}^2)^3)^{\mathfrak{S}_3}$ which is rational. $\square$

This proof is more geometric (and easier from my point of view) than the one given in [For79]. Since the projective geometry of plane quartics is quite rich, we would like to ask whether one can also obtain the result of [For80] in this way.

Problem 1.2.1.12. Can one prove the rationality of K_4 using its identification with $\mathcal{J}ac_4^2$ and the classical projective geometry of plane quartics?

The following remark shows that the stable rationality of K_d does not follow from a straightforward argument that is close at hand.

Remark 1.2.1.13. Put $k = \binom{d}{2}$ and in $\mathrm{Sym}^k\mathbb{P}^2 \times \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d)))$ consider the incidence correspondence X given by the rule that k unordered points in $\mathrm{Sym}^k\mathbb{P}^2$ lie on a plane curve of degree d in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d)))$. X is generically the projectivisation of a vector bundle over $\mathrm{Sym}^k\mathbb{P}^2$ hence rational. On the other hand, one has also the natural map $X \to \mathcal{J}ac_d^k$, assigning to a pair $(D,\, C)$ the point $(|D|,\, C)$ in $\mathcal{J}ac_d^k$, which makes X a $\mathbb{P}^N$-bundle in the *classical* or *étale* topology over the dense open subset $U \subset \mathcal{J}ac_d^k$ consisting of

pairs ($|D|$, C) with $|D|$ a non-special divisor class.
However, unfortunately, this is not a projective bundle in the *Zariski* topology: if, to the contrary, this was the case, let $\sigma : \mathcal{J}ac_d^k \dashrightarrow X$ be a rational section. Then, if $u : U_d \to |\mathcal{O}(d)|_{\mathrm{smooth}}$ is the universal curve, the pull-back of $(|D|, \sigma(|D|))$ on $\mathcal{J}ac_d^k \times_{|\mathcal{O}(d)|_{\mathrm{smooth}}} X$ to $\mathcal{J}ac_d^k \times_{|\mathcal{O}(d)|_{\mathrm{smooth}}} U_d$ would give a universal divisor or Poincaré line bundle on $\Omega \times_V u^{-1}(V)$ where $V \subset |\mathcal{O}(d)|_{\mathrm{smooth}}$ is some dense open set, and Ω some dense open set in $\mathcal{J}ac_d^k|_V$. But by results of Mestrano and Ramanan ([Me-Ra], Lemma 2.1 and Corollary 2.8),

> a Poincare bundle on $\mathcal{J}ac_d^r \times_{|\mathcal{O}(d)|_{\mathrm{smooth}}} U_d$ (or on $\Omega \times_{|\mathcal{O}(d)|_{\mathrm{smooth}}} u^{-1}(V)$ as above) exists if and only if $1-g+r$ and d are coprime, where $g = \frac{1}{2}(d-1)(d-2)$ is the genus.

Hence in all the cases we are interested in, $X \to \mathcal{J}ac_d^k$ is generically a non-trivial Severi-Brauer scheme.

1.2.2 Results for specific groups

Here we collect some results in the direction of Problem 1.2.1.5 which express exclusively properties of the group G acting, for specific groups G, and are independent of the particular G-representation V.
The following theorem is due to Miyata [Mi].

Theorem 1.2.2.1. *Let G be isomorphic to a subgroup of the Borel group $B_n \subset \mathrm{GL}_n(\mathbb{C})$ of invertible upper triangular matrices. Then the field of invariant rational functions $\mathbb{C}(V)^G$ for the G-module $V = \mathbb{C}^n$ is a purely transcendental extension of $\mathbb{C}$.*

Proof. The proof is an immediate application of the following

> **Claim.**If k is a field and G a group of automorphisms of the polynomial ring $k[t]$ in one indeterminate which transforms k into itself , then there is an invariant $p \in k[t]^G$ such that $k(t)^G = k^G(p)$.

To prove the claim, note that $k(t)^G$ is the field of fractions of $k[t]^G$: write $f \in k(t)^G$ as $f = u/v$, u, $v \in k[t]$ without common factor. After passing to the reciprocal if necessary, we may assume $\deg(u) \geq \deg(v) > 0$ and apply the division algorithm in $k[t]$ to write

$$u = qv + r\,,$$

$q,\ r \in k[t]$, $\deg(r) < \deg(v)$ whence q and r are uniquely determined by these requirements. Since f is invariant, G acts on both u and v via a certain character $\chi : G \to \mathbb{C}^*$, and the uniqueness of r and q implies that r is a weight vector of G for the character χ, and q is an (absolute) G-invariant. Since

$$\frac{u}{v} = q + \frac{r}{v},\ \frac{r}{v} \in k(t)^G,$$

and $\deg(r)+\deg(v) < \deg(u)+\deg(v)$, one obtains the statement by induction on $\deg(u) + \deg(v)$, the case $\deg(u) + \deg(v) = 0$ being trivial.
Now if $k[t]^G \subset k$, the claim is obvious. Otherwise, we take $p \in k[t]^G \backslash k$ of minimal degree. Then if f is in $k[t]^G$, one writes $f = pq + r$ with $\deg(r) < \deg(p)$ as before, and by uniqueness of quotient and remainder, q and r are G-invariant polynomials in $k[t]$. Thus, by the choice of p, $r \in k^G$ and $\deg(q) < \deg(f)$. Again by induction on the degree of f we obtain $f \in k^G[p]$. This means $k[t]^G = k^G[p]$, and since we have seen that $k(t)^G$ is the field of fractions of $k[t]^G$ the assertion of the claim follows.
To prove the theorem, one applies the claim to $k = \mathbb{C}(x_1, \dots, x_{n-1})$, $t = x_n$ where $x_1, \dots, x_n$ are coordinates on $V = \mathbb{C}^n$, and concludes by induction on the number of variables. $\square$

Corollary 1.2.2.2. *If V is a finite dimensional linear representation of either*

- *an abelian group $G \subset \mathrm{GL}(V)$ consisting of semi-simple elements (e.g. if G is finite)*
- *or a connected solvable group G,*

then $\mathbb{C}(V)^G$ is a purely transcendental extension of $\mathbb{C}$.

Proof. Simultaneous diagonalizability of commuting semisimple elements, or Lie-Kolchin theorem, respectively. $\square$

Remark 1.2.2.3. Note that the statement and proof of Theorem 1.2.2.1 remain valid if one works, instead of over $\mathbb{C}$, over a possibly nonclosed ground field; the Corollary 1.2.2.2 becomes false in general, however, because one needs the algebraic closedness to make the actions triangular: for example, let G be a cyclic group of order $p = 47$, and let G act on $\mathbb{Q}(x_1, \dots, x_p)$ by permuting the variables cyclically. It is known (cf. [Swan]) that $\mathbb{Q}(x_1, \dots, x_p)^G$ is not rational *over* $\mathbb{Q}$.

For semi-simple groups the only truely complete rationality result is the following due to P. Katsylo and F. Bogomolov.

Theorem 1.2.2.4. *The moduli spaces* $\mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2)/\mathrm{SL}_2(\mathbb{C})$ *of* d *unordered points in* $\mathbb{P}^1$ *are rational for all* d.

See [Kat84], [Bo-Ka], [Bogo2] for a proof. It should be mentioned that this result is also used in the recent work of Kim and Pandharipande [Ki-Pa]. There they prove the rationality of the moduli space

$$\overline{M}_{0,\,n}(X,\ \beta)$$

of n-pointed genus 0 stable maps of class $\beta \in H_2(X,\ \mathbb{Z})$ into a rational homogeneous variety $X = G/P$. Here $\overline{M}_{g,\,n}(X,\ \beta)$ parametrizes data

$$[\mu\,:\,C \to X;\ p_1, \dots, p_n]$$

where C is a complex, projective, connected, reduced, nodal curve of arithmetic genus g, $p_1, \dots, p_n$ are distinct points in the smooth locus of C, the map μ has no infinitesimal automorphisms and $\mu_*[C] = \beta$.
Despite the very small number of general rationality results, one has some satisfactory information with regard to the important question of existence of rational sections. We have already seen Rosenlicht's theorem 1.2.1.7. Let us recall quickly the theory of *special groups* cf. [Se56], [Se58], [Groth58].

Definition 1.2.2.5. Let G be an algebraic group, and $\pi\,:\,P \to X$ a morphism of algebraic varieties. Let P be equipped with a *right* G-action and suppose π is constant on G orbits. Then P is called *a* G*-principal bundle in the étale topology* (or *locally isotrivial fibre space with typical fibre* G or G*-torsor*) if for every point $x \in X$ there is a Zariski open neighborhood $U \ni x$ and an étale cover $f\,:\,U' \to U$ such that the pull-back $f^*(P|_U) \to U'$ is G-isomorphic to the trivial fibering $U' \times G \to U'$.
P is called *a* G*-principal bundle in the Zariski topology* if furthermore every $x \in X$ has a Zariski open neighborhood U such that $P|_U$ is trivial.

Definition 1.2.2.6. An algebraic group G is called *special* if every G-principal bundle in the étale topology is Zariski locally trivial.

The main results of interest to us are summarized in the following

Theorem 1.2.2.7. *(a) The general linear group* $\mathrm{GL}_n(\mathbb{C})$ *is special.*

(b) A closed subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ is special if and only if the quotient map $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/G$ is a Zariski locally trivial G-principal bundle (equivalently, if and only if $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/G$ has a rational section).

(c) The groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ and all connected linear solvable groups are special.

(d) If G is a linear algebraic group, H a normal subgroup, and if H and G/H are special, then G is special. In particular, any connected linear algebraic group the semisimple part of which is a direct product of groups of types SL or Sp is special.

Proof. (a):The proof uses the method of taking averages of group cocycles. Let $P \to X$ be a G-principal bundle, and let $X' \to X$ be a finite étale cover, which we can assume to be Galois with Galois group Γ, such that P becomes trivial on X':

$$\begin{array}{ccc} X' \times G & \longrightarrow & P \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

We recall the bijective correspondence between the set of isomorphism classes of G-principal bundles (in the étale topology) on X which become trivial when pulled-back to X' and the elements of the nonabelian group cohomology set $H^1(\Gamma, \mathrm{Mor}(X', G))$ with marked point. Here $\mathrm{Mor}(X', G)$ is the group of morphisms of X' into G; and Γ, which we assume to operate on the right on X', acts on $\mathrm{Mor}(X', G)$ (on the left) by

$$(\sigma \cdot \varphi)(x') := \varphi(x' \cdot \sigma)\,.$$

Elements of $H^1(\Gamma, \mathrm{Mor}(X', G))$ are by definition 1-cocycles of Γ with values in $\mathrm{Mor}(X', G)$ modulo an equivalence relation; a 1-cocycle is a map $\sigma \mapsto \varphi_\sigma$ from Γ to $\mathrm{Mor}(X', G)$ satisfying

$$\varphi_{\sigma\tau} = (\varphi_\tau)^\sigma \varphi_\sigma$$

where $(\cdot)^\sigma$ denotes the action of σ. Two 1-cocycles (φ_σ), (φ'_σ) are cohomologous if there is an $a \in \mathrm{Mor}(X', G)$ such that

$$\varphi'_\sigma = a^\sigma \varphi_\sigma a^{-1}, \quad \text{all } \sigma\,.$$

Now in the above pull-back diagram $X' \times G$ is a Galois cover of P with Galois group Γ, and $P = (X' \times G)/\Gamma$. Γ acts on $X' \times G$ compatibly with the projection π' to X' and the operation of G whence

$$(x', g) \cdot \sigma = (x' \cdot \sigma,\ \varphi_\sigma(x') \cdot g)$$

and the associativity gives the required cocycle condition for (φ_σ). Conversely, the datum of a 1-cocycle (φ_σ) determines an operation of Γ on $X' \times G$ and one may define P on X as the quotient. The condition that two 1-cocycles are cohomologous means precisely that the G-principal bundles so obtained are isomorphic. Note that the condition that (φ_σ) and (φ'_σ) are cohomologous means precisely that the isomorphism $(x', g) \mapsto (x', a(x')g)$ between trivial G-principal bundles on X' descends to the G-principal bundles on X defined by (φ_σ) and (φ'_σ) on X, respectively.
We turn to the proof of (a) of Theorem 1.2.2.7. Thus let $P \to X$ be a $\mathrm{GL}_n(\mathbb{C})$-principal bundle in the étale topology, and let $U \ni x$ be an open neighborhood, $f : U' \to U$ a Galois cover with group Γ on which P is trivial. The above considerations show that, if $\mathcal{O}_{f^{-1}(x)}$ is the semi-local ring of the fibre over x, then the set of isomorphism classes of $\mathrm{GL}_n(\mathbb{C})$-principal bundles on a Zariski neighborhood of x which become trivial on a Zariski neighborhood of $f^{-1}(x)$ are identified with the cohomology set $H^1(\Gamma,\ \mathrm{GL}_n(\mathcal{O}_{f^{-1}(x)}))$. Let x' be a point of the fibre $f^{-1}(x)$ and choose a matrix $b \in \mathrm{Mat}_{n\times n}(\mathcal{O}_{f^{-1}(x)})$ which is the identity in x' and the zero matrix in the other points of $f^{-1}(x)$. If (φ_σ) is a 1-cocycle representing the germ of P in x one puts

$$a = \sum_{\tau\in\Gamma} \tau(b)\varphi_\tau\,.$$

By definition, this is invertible in each point of the fibre $f^{-1}(x)$, thus belongs to $\mathrm{GL}_n(\mathcal{O}_{f^{-1}(x)})$. Since

$$a^\sigma\varphi_\sigma = \sum_{\tau\in\Gamma} \sigma(\tau(b))(\varphi_\tau)^\sigma\varphi_\sigma = \sum_{\tau\in\Gamma}(\sigma\tau)(b)\varphi_{\sigma\tau} = a\,,$$

we have $a^\sigma\varphi_\sigma a^{-1} = 1$, so our $\mathrm{GL}_n(\mathbb{C})$-principal bundle is trivial in a Zariski neighborhood of x.
Using the correspondence between $\mathrm{GL}_n(\mathbb{C})$-principal bundles in the étale resp. Zariski topology and vector bundles in the étale resp. Zariski topology (given by passing to the associated fibre bundles with typical fibre $\mathbb{C}^n$, and

conversely associated frame bundles), we obtain the fact which is fundamental to many techniques for proving rationality, that every vector bundle in the étale topology is a vector bundle in the Zariski topology.

(b): The proof consists in the trick of extension and reduction of the structure group.
By definition, if $G \subset \mathrm{GL}_n(\mathbb{C})$ is special, then $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/G$ is Zariski locally trivial. Conversely, suppose that $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/G$ is Zariski locally trivial, and let $P \to X$ be a G-principal bundle in the étale topology. We have an associated fibre bundle $Q := P \times^G \mathrm{GL}_n(\mathbb{C})$ which is a $\mathrm{GL}_n(\mathbb{C})$-principal bundle *in the Zariski topology* by part (a).
Now Q is a G-principal bundle in the Zariski topology over $Q \times^{\mathrm{GL}_n(\mathbb{C})} (\mathrm{GL}_n(\mathbb{C})/G)$ because Q itself is Zariski locally trivial and $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/G$ has a rational section by assumption. Now

$$Q \times^{\mathrm{GL}_n(\mathbb{C})} (\mathrm{GL}_n(\mathbb{C})/G) = P \times^G (\mathrm{GL}_n(\mathbb{C})/G)$$

has a canonical section $\sigma : X \to P \times^G (\mathrm{GL}_n(\mathbb{C})/G)$ since G leaves the coset corresponding to the identity in $\mathrm{GL}_n(\mathbb{C})/G$ invariant. Then P is the pull-back of $Q \to P \times^G (\mathrm{GL}_n(\mathbb{C})/G)$ via σ:

$$\begin{array}{ccc} P & \longrightarrow & Q = P \times^G \mathrm{GL}_n(\mathbb{C}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & P \times^G (\mathrm{GL}_n(\mathbb{C})/G) \end{array}$$

Thus the fact that $P \to X$ is Zariski locally trivial follows from the fact that $Q \to P \times^G (\mathrm{GL}_n(\mathbb{C})/G)$ has this property.

(c): For connected linear solvable groups, this follows from part (d) to be proven below since a connected solvable group is a successive extension of groups of type $\mathbb{G}_m$ and $\mathbb{G}_a$. Remark that both $\mathbb{G}_m$ and $\mathbb{G}_a$ are special since $\mathbb{G}_m = \mathrm{GL}_1(\mathbb{C})$ and the natural map $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})/\mathbb{G}_a$ has a rational section. The projection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/\mathrm{SL}_n(\mathbb{C})$ has a section given by assigning to a coset $g\mathrm{SL}_n(\mathbb{C})$ the matrix

$$\mathrm{diag}(\det g,\ 1, \dots, 1)\,.$$

Finally, the projection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/\mathrm{Sp}_n(\mathbb{C})$, $n = 2m$, has a section since $\mathrm{GL}_n(\mathbb{C})/\mathrm{Sp}_n(\mathbb{C})$ is the space of nondegenerate skew-symmetric bilinear forms on $\mathbb{C}^{2m}$ and the generic skew-symmetric form

$$\sum_{1\leq i<j\leq n} t_{ij}(x_i y_j - y_i x_j)$$

with indeterminate coefficients t_{ij} can be reduced to the canonical form $\sum_{k=1}^{m}(x_{2k-1}y_{2k} - y_{2k-1}x_{2k})$ over the function field $\mathbb{C}(t_{ij})$. The usual linear algebra construction of a corresponding symplectic basis $u_1, v_1, u_2, v_2, \ldots, u_m, v_m \in \mathbb{C}(t_{ij})^{2m}$ goes through: start with any $u_1 \neq 0$, find v_1 with $\langle u_1,\ v_1\rangle = 1$, put $H := \mathrm{span}(u_1,\ v_1)$, decompose $\mathbb{C}(t_{ij})^{2m} = H \oplus H^{\perp}$, and continue with the symplectic form $\langle\cdot,\cdot\rangle_{H^{\perp}}$ in the same way. This fails for orthogonal groups since one cannot take square roots rationally.

(d): The assertion follows immediately by the application of the techniques in part (b). If $P \to X$ is a principal G-bundle, then the associated G/H-principal bundle $P\times^G (G/H) \to X$ has locally around any point of X a section because G/H is special. P is an H-principal bundle over $P\times^G (G/H)$ and pulling back via the section one obtains an H-principal bundle Q locally around any point of X which is Zariski locally trivial because H is special; and P is just $Q \times^H G$, thus is Zariski locally trivial, too.
The second assertion follows because a connected linear algebraic group is an extension of its reductive part by the unipotent radical (connected solvable), and the reductive part an extension of the semi-simple part by a torus. $\square$

Remark 1.2.2.8. Grothendieck [Groth58] has shown that the only special semi-simple groups are the products of the groups of type $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$. Serre [Se58] has shown that any special algebraic group is linear and connected.

If X is a G-variety, G a linear algebraic group, one needs a practically verifiable condition when $X \to X/G$ is generically a G-principal bundle.

Definition 1.2.2.9. (1) The action of G on X is called *free* if the morphism $G\times X \to X \times X$, $(g,\ x) \mapsto (gx,\ x)$ is a closed embedding.

(2) G is said to act on X with *trivial stabilizers* if for each point $x \in X$ the stabilizer G_x of x in G is reduced to the identity.

Unfortunately, (1) and (2) are not equivalent. Mumford ([Mum], Ex. 0.4) gives an example of an action of the group $\mathrm{SL}_2(\mathbb{C})$ on a quasi-projective

variety with trivial stabilizers, but which is not free. However, when for each $x \in U$, $U \subset X$ some open dense set, the stabilizers are trivial, we will nevertheless sometimes say that G acts *generically freely* since this has become standard terminology. We also say more accurately that G acts with *generically trivial stabilizers.*
Despite the presence of the subtlety which is displayed in Mumford's example, one has the following result.

Theorem 1.2.2.10. *Let G be a connected linear algebraic group acting on a variety X with trivial stabilizers and let $X \to X/G$ be a geometric quotient. Then there is an open dense G-invariant subset $U \subset X$ such that $U \to U/G$ is a G-principal bundle in the étale topology.*

Proof. We use a Seshadri cover ([Sesh72], [BB] §8.4): given a connected linear algebraic group which acts on a variety X with finite stabilizers, there exists a finite morphism $\kappa : X_1 \to X$ with the following properties:

- X_1 is a normal variety and κ a (ramified) Galois cover with Galois group Γ acting on X_1.
- There exists a free action of G on X_1, commuting with the action of Γ, such that κ is G-equivariant.
- There exists a good geometric quotient $\pi : X_1 \to X_1/G$ with X_1/G a prevariety (not necessarily separated), and π is a Zariski locally trivial G-principal bundle.

"Good" geometric quotient means that π is affine. Now assume that in our original situation $V \to V/G$ is a geometric quotient, $V \subset X$ open. Shrinking V/G (and V) we can find a G-invariant open set $U \subset X$ such that in the diagram

$$\begin{array}{ccc} \kappa^{-1}(U) & \xrightarrow{\kappa} & U \\ \pi\downarrow & & \downarrow \\ \kappa^{-1}(U)/G & \xrightarrow{\bar{\kappa}} & U/G \end{array}$$

all arrows are geometric quotients, $\kappa^{-1}(U)/G$ is a variety, and $\bar{\kappa}$ is *étale*. It follows that $U \to U/G$ is a G-principal bundle in the étale topology since it becomes one (even a Zariski locally trivial one) after the étale base change $\bar{\kappa}$ (the hypothesis that G acts with trivial stabilizers on X implies that the above diagram is a fibre product). $\square$

Corollary 1.2.2.11. *Let a connected linear algebraic group G whose semi-simple part is a direct product of groups of type* SL *or* Sp *act on a rational variety X with generically trivial stabilizers. Then X/G is stably rational.*

Proof. By Rosenlicht's Theorem 1.2.1.3 and the preceding Theorem 1.2.2.10, there is a nonempty open G-invariant subset $U \subset X$ such that a geometric quotient $U \to U/G$ exists and is a G-principal bundle in the étale topology. This is Zariski locally trivial by Theorem 1.2.2.7, (d). Thus to conclude the proof it suffices to remark that a connected linear algebraic group (over $\mathbb{C}$) is a rational variety: take a Borel subgroup B in G and consider the B-principal bundle $G \to G/B$ over the (rational) flag variety G/B. Note that B is rational since it is a successive extension of groups $\mathbb{G}_a$ and $\mathbb{G}_m$. $\square$

However, Corollary 1.2.2.11, as it stands, is not applicable when we consider for example the action of $\mathrm{PGL}_3(\mathbb{C})$ on the space $\mathbb{P}(\mathrm{Sym}^4(\mathbb{C}^3)^\vee)$ of plane quartics. One has the following easy extension.

Corollary 1.2.2.12. *Let V be a linear representation of a connected linear algebraic group with semi-simple part a direct product of groups* SL *and* Sp. *Suppose that the generic stabilizer of G in V is trivial. Then $\mathbb{P}(V) \dashrightarrow \mathbb{P}(V)/G$ has a rational section.*

Proof. By Rosenlicht's Theorem 1.2.1.7, we see that $V/G \dashrightarrow \mathbb{P}(V)/G$ has a rational section, and composing with a rational section of $V \dashrightarrow V/G$ and the projection $V \dashrightarrow \mathbb{P}(V)$, we obtain a rational section of $\mathbb{P}(V) \dashrightarrow \mathbb{P}(V)/G$. $\square$

1.3 Cones and homogeneous bundles

Over the last decades a variety of different techniques have been developed to make progress on the rationality problem 1.2.1.5 in certain special cases. These methods will be discussed in the next chapter. However, one may be left with the impression that this is a somewhat incoherent arsenal of tricks, and no conceptual framework has yet been found which Problem 1.2.1.5 fits into. The purpose of this section is therefore to discuss certain concepts which seem to have an overall relevance to Problem 1.2.1.5, and in particular, show how the Hesselink stratification of the nullcone and the desingularizations of the strata in terms of homogeneous vector bundles give a strategy for approaching Problem 1.2.1.5.

1.3.1 Torus orbits and the nullcone

G will be a reductive linear algebraic group throughout this section, $T \subset G$ a fixed maximal torus. $X^*(T)$ is the group of characters $\chi : T \to \mathbb{C}^*$ of T, and $X_*(T)$ the group of cocharacters or one-parameter subgroups (1-psg) $\lambda : \mathbb{C}^* \to T$ of T. There is the perfect pairing of lattices

$$\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$$

with $\langle \chi, \lambda \rangle$ defined by $\chi(\lambda(s)) = s^{\langle \chi, \lambda \rangle}$ for $s \in \mathbb{C}^*$. Let V be a T-module with weight space decomposition

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi \,.$$

Definition 1.3.1.1. Let $v = \sum_{\chi \in X^*(T)} v_\chi$ be the decomposition of a vector $v \in V$ with respect to the weight spaces of V. Let $\mathrm{supp}(v)$ denote the set of those χ for which $v_\chi \neq 0$ (the *support* of v), let $\mathrm{Wt}(v)$ be the convex hull of $\mathrm{supp}(v)$ in the vector space $X^*(T)_{\mathbb{Q}} := X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (the *weight polytope* of v), and $C(v)$ the closed convex cone generated by the vectors in $\mathrm{supp}(v)$ in $X^*(T)_{\mathbb{Q}}$.

The geometry of T-orbit closures in the affine or projective cases in V resp. $\mathbb{P}(V)$ is completely encoded in $\mathrm{supp}(v)$ and $C(v)$ resp. $\mathrm{Wt}(v)$ in the following way.

Theorem 1.3.1.2. *(1) If F is a face of the cone $C(v)$, put*

$$v_F := \sum_{\chi \in \mathrm{supp}(v) \cap F} v_\chi \,.$$

Then the map $F \mapsto T \cdot v_F$ is a bijection of the set of faces of $C(v)$ and the set of T-orbits in $\overline{T \cdot v}$. If F_1 and F_2 are faces of $C(v)$, then

$$F_1 \subset F_2 \iff T \cdot v_{F_1} \subset \overline{T \cdot v_{F_2}} \,.$$

(2) If $v \neq 0$ is in V, and $X_v := \overline{T \cdot [v]} \subset \mathbb{P}(V)$ is the torus orbit closure of $[v]$ in $\mathbb{P}(V)$, then the T-orbits on X_v are in bijection with the faces of the weight polytope $\mathrm{Wt}(v)$*: for any point $[w] \in X_v$,* $\mathrm{Wt}(w)$ *is a face of* $\mathrm{Wt}(v)$*. For $[w_1]$, $[w_2] \in X_v$ one has $T \cdot [w_1] \subset \overline{T \cdot [w_2]}$ if and only if* $\mathrm{Wt}(w_1) \subset \mathrm{Wt}(w_2)$.

See [B-S], Prop. 7, p. 104, and [GKZ], Chapter 5, Prop. 1.8, for a proof. Since G is reductive, it is well known that $\mathbb{C}[V]^G$ is finitely generated and closed orbits are separated by G-invariants (e.g. [Muk], Thm. 4.51, Thm. 5.3), thus for $v \in V$ there is a unique closed orbit in $\overline{G \cdot v}$. Thus we can state the following Hilbert-Mumford theorem.

Theorem 1.3.1.3. *Let G be a reductive group, V a (finite dimensional) G-representation, and pick $v \in V$. Let X be the unique closed orbit in $\overline{G \cdot v}$. Then there is a 1-psg $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v$ exists and is in X.*

Proof. It follows from part (1) of Theorem 1.3.1.2 that every torus orbit in the torus orbit closure of v can be reached as the limit of v under a suitable 1-psg (the 1-psg corresponds to an integral linear form in $\mathrm{Hom}(X^*(T), \mathbb{Z})$ defining the face F corresponding to the torus orbit we want to reach). Thus one just has to prove that there is a $g \in G$ and a torus $T \subset G$ with $\overline{T \cdot gv} \cap X \neq \emptyset$. One has the Cartan decomposition $G = K \cdot T \cdot K$ of G where K is a maximal compact subgroup of G, and T the complexification of a maximal torus in K.
Suppose that $\overline{T \cdot gv} \cap X = \emptyset$ *for all* g. Then, since T-invariants separate disjoint closed T-invariant subsets, one may find for each $w \in G \cdot v$ a function $I_w \in \mathbb{C}[V]^T$ which is identically 0 on X and takes the value 1 in w. The compact set $K \cdot v$ is -as it is a subset of $G \cdot v$- covered by the open sets U_w in V where I_w does not vanish, for w running through $G \cdot v$, hence it is covered by finitely many of them. The sum of the absolute values of the I_w's corresponding to this finite covering family is then a continuous T-invariant function s which is strictly positive on the compact $K \cdot v$, but 0 on X. But then s is still strictly positive on $\overline{TK \cdot v}$ which is a contradiction: since $\overline{G \cdot v} = K \cdot \overline{TK \cdot v}$ as K is compact, the fact that $\overline{TK \cdot v}$ does not meet X implies that $\overline{G \cdot v}$ does not meet X which is false. $\square$

Definition 1.3.1.4. Suppose $v \in V$ is a vector in the G-representation V.

(1) v is *unstable* if $0 \in \overline{G \cdot v}$. The set of these is denoted by $\mathfrak{N}_G(V)$ (the *nullcone*).

(2) v is *semistable* if v is not unstable.

(3) v is *stable* if $G \cdot v$ is closed in V and the stabilizer subgroup $G_v \subset G$ of v is finite.

(4) v is called *T-unstable* if $0 \in \overline{T \cdot v}$, and the set of T-unstable elements is denoted by $\mathfrak{N}_T(V)$ (the *canonical cone*).

Theorem 1.3.1.5. *(1) The set $\mathfrak{N}_G(V)$ is defined by the vanishing of all invariants in $\mathbb{C}[V]^G$ of positive degree, $\mathfrak{N}_T(V)$ is defined by the vanishing of all T-invariants in $\mathbb{C}[V]^T$ of positive degree; this gives $\mathfrak{N}_G(V)$ and $\mathfrak{N}_T(V)$ scheme structures.*

(2) A vector $v \in V$ is T-unstable if and only if $0 \notin \mathrm{Wt}(v)$. One has $G \cdot \mathfrak{N}_T(V) = \mathfrak{N}_G(V)$. Hence $\mathfrak{N}_G(V)$ consists of those vectors v such that the orbit $G \cdot v$ contains an element whose weight polytope does not contain 0.

Proof. (1) follows immediately from the fact that G- or T-invariants separate closed orbits. The first assertion of (2) is a consequence of Theorem 1.3.1.2, (1). The fact that $G \cdot \mathfrak{N}_T(V) = \mathfrak{N}_G(V)$ follows from Theorem 1.3.1.3. $\square$

Remark 1.3.1.6. In general, $\mathfrak{N}_G(V)$ need neither be irreducible nor reduced nor equidimensional: in [Po92], Chapter 2 §3, it is shown that for the representation of $\mathrm{SL}_2(\mathbb{C})$ in the space of binary sextics $\mathrm{Sym}^6(\mathbb{C}^2)^\vee$, the nullcone is not reduced; for the representation of $\mathrm{SL}_3(\mathbb{C})$ in the space of ternary quartics $\mathrm{Sym}^4(\mathbb{C}^3)^\vee$ the nullcone has two irreducible components of dimensions 10 and 11, respectively ([Hess79], p. 156).

Remark 1.3.1.7. Theorem 1.3.1.5 (2) gives a convenient graphical way for the determination of unstable vectors known to Hilbert ([Hil93], §18): for example, if $\mathrm{SL}_2(\mathbb{C})$ acts on binary forms of degree d, $\mathrm{Sym}^d(\mathbb{C}^2)^\vee$, in variables x and y, $T \subset \mathrm{SL}_2(\mathbb{C})$ is the standard torus

$$T = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \; t \in \mathbb{C}^*,$$

and ϵ_1 is the weight

$$\epsilon_1 \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) := t,$$

then the weight spaces are spanned by $x^k y^{d-k}$ which is of weight $(d-2k)\epsilon_1$, for $k = 0, \dots, d$. The support of a binary degree d form thus does not contain

the origin if and only if it is divisible by $x^{[d/2]+1}$ or $y^{[d/2]+1}$, so that the unstable binary degree d forms are just those which have a zero of multiplicity $\geq [d/2]+1$.
Turning to ternary forms, the representation of $\mathrm{SL}_3(\mathbb{C})$ in $\mathrm{Sym}^d(\mathbb{C}^3)^\vee$ (coordinates x, y, z), one can use "Hilbert's triangle": consider in the plane an equilateral triangle ABC with barycenter the origin 0. Let ϵ_1, ϵ_2, ϵ_3 be the vectors pointing from 0 to A, B, C. Points of the plane with integer coordinates with respect to the basis ϵ_1, ϵ_2 are thus identified with the character lattice of the standard maximal torus of $\mathrm{SL}_3(\mathbb{C})$. For the monomial $x^a y^b z^c$, $a+b+c=d$, the point in the plane $-a\epsilon_1 - b\epsilon_2 - c\epsilon_3$ then represents the associated weight; pick a line l in this plane not passing through zero, and let H_l be the corresponding closed half-space in the plane not containing 0. Then a ternary degree d form f is unstable if, after a coordinate change, it may be written as a sum of monomials whose weights lie entirely in H_l for some line l. In this way it is possible to obtain finitely many representatives for all possible types.

1.3.2 Stratification of the nullcone and rationality

The importance of the nullcone $\mathfrak{N}_G(V)$ for us derives from the fact that it contains a lot of rational subvarieties which are birational to homogeneous vector bundles over generalized flag varieties whose fibres are linearly embedded in V. We are now going to describe this.
Let G be reductive as before, $T \subset G$ a maximal torus. Let $W := N_G(T)/Z_G(T)$ be the Weyl group which acts on T by conjugation, hence on $X^*(T)$. Choose a W-invariant scalar product $\langle \cdot, \cdot \rangle$ on $X^*(T)_{\mathbb{Q}}$ which takes integral values on $X^*(T)$.
Denote by $R \subset X^*(T)$ the set of roots, the set of nonzero weights of T in the adjoint representation of G on its Lie algebra $\mathfrak{g}$. For $\alpha \in R$ denote by U_α the root subgroup corresponding to α, i.e. the unique connected T-invariant unipotent subgroup $U_\alpha \subset G$ with Lie algebra the one dimensional root subspace $\mathfrak{g}_\alpha$.

Definition 1.3.2.1. (1) For $c \in X^*(T)_{\mathbb{Q}}$ denote by P_c the subgroup of G generated by the torus T and the root groups U_α for $\alpha \in R$ with $\langle \alpha, c \rangle \geq 0$. Furthermore, U_c resp. L_c will be the subgroups generated by U_α with $\langle \alpha, c \rangle > 0$ resp. by the torus T and the root subgroups U_α with $\langle \alpha, c \rangle = 0$.

(2) For $c \in X^*(T)_{\mathbb{Q}}$, $c \neq 0$, one denotes by $H^+(c)$ the half space $\{x \in X^*(T)_{\mathbb{Q}} \mid \langle x - c,\, c\rangle \geq 0\}$ in $X^*(T)_{\mathbb{Q}}$ and by $H^0(c) = \{x \in X^*(T)_{\mathbb{Q}} \mid \langle x - c,\, c\rangle = 0\}$ its bounding hyperplane.

(3) For a subset $\Sigma \subset X^*(T)_{\mathbb{Q}}$ and a finite-dimensional G-module V, V_Σ denotes the subspace of V consisting of those $v \in V$ with $\mathrm{supp}(v) \subset \Sigma$. Obviously, there are only finitely many such subspaces since the number of weight spaces in V is finite.

Remark that $P_c \subset G$ is a parabolic subgroup containing T (because the subset of those $\alpha \in R$ with $\langle \alpha,\, c\rangle \geq 0$ contains some basis for the root system R and is closed with respect to addition); it follows from this observation that L_c is a reductive Levi subgroup of P_c containing T, and U_c is the unipotent radical of P_c, $P_c = L_c \ltimes U_c$. Moreover:

Lemma 1.3.2.2.

(1) For $c \in X^*(T)_{\mathbb{Q}}$, $c \neq 0$, the subspace $V_{H^+(c)}$ is stable under P_c, the subspace $V_{H^0(c)}$ is stable under the Levi subgroup L_c.

(2) $G \cdot V_{H^+(c)}$ is closed in V, and the image of the homogeneous vector bundle $G \times_{P_c} V_{H^+(c)} \to G/P_c$ under the natural G-map to V. Moreover, the nullcone can be expressed as a union (which is actually finite) of such images:

$$\mathfrak{N}_G(V) = \bigcup_{c \neq 0,\, c \in X^*(T)_{\mathbb{Q}}} G \cdot V_{H^+(c)}\,.$$

Proof. (1) follows from the following well-known fact from the representation theory of reductive groups: if $\chi \in X^*(T)$, $v \in V_\chi$, $\alpha \in R$ and $g \in U_\alpha$, then

$$g \cdot v - v \in \bigoplus_{l \geq 1} V_{\chi + l\alpha}\,.$$

It is seen as follows: let $x_\alpha : \mathbb{G}_a \to G$ be the root homomorphism which is an isomorphism onto its image U_α and $t \cdot x_\alpha(k) \cdot t^{-1} = x_\alpha(\alpha(t)k)\; \forall\, t \in T, \forall\, k \in \mathbb{G}_a$. Then $x_\alpha(k) \cdot v$ is a polynomial in k with coefficients in V:

$$x_\alpha(k) \cdot v = \sum_{l=0}^{N} v_l k^l,\ \forall\, k \in \mathbb{C}$$

and on the one hand, for $t \in T$

$$t \cdot (x_\alpha(k) \cdot v) = \sum_{l=0}^{N} k^l t \cdot v_l\,,$$

whereas on the other hand

$$\begin{aligned} t \cdot (x_\alpha(k) \cdot v) &= (t \cdot x_\alpha(k) t^{-1})(tv) \\ = x_\alpha(\alpha(t) \cdot k)\chi(t) v &= \sum_{l=0}^{N} \alpha(t)^l k^l \chi(t) \cdot v_l\,. \end{aligned}$$

Equating coefficients in the two polynomials in k yields $\chi(t)\alpha(t)^l \cdot v_l = t \cdot v_l$, all $t \in T$, which together with the fact that $v = v_0$ (put $k = 0$) gives the desired result.

(2): $G \cdot V_{H^+(c)}$ is the image of the natural G-map $G \times_{P_c} V_{H^+(c)} \to V$ which factors into the closed embedding $i : G \times_{P_c} V_{H^+(c)} \to G/P_c \times V$ given by $i([(g,\, v)]) := (gP_c,\, g \cdot v)$, followed by the projection $G/P_c \times V \to V$ onto the second factor which is proper since G/P_c is compact. Thus $G \cdot V_{H^+(c)}$ is closed in V. The last assertion about the nullcone is an immediate consequence of Theorem 1.3.1.5, (2). $\square$

We need a criterion for when the natural G-map $G \times_{P_c} V_{H^+(c)} \to V$ is birational onto its image $G \cdot V_{H^+(c)}$.

Theorem 1.3.2.3. *Let $v \in V$ be a T-unstable element. Then the norm $||\cdot|| := \sqrt{\langle \cdot, \cdot \rangle}$ induced by the W-invariant scalar product on $X^*(T)_{\mathbb{Q}}$ achieves its minimum in exactly one point c of $\mathrm{Wt}(v)$. Thus there exists a smallest positive integer n such that $n \cdot c$ is in $X^*(T)$. Since $n \cdot c$ is orthogonal to the roots of L_c, it extends to a character of P_c and L_c. Let $Z_c := (\ker(n \cdot c|_{L_c}))^\circ \subset L_c$ be the corresponding reductive subgroup of L_c with maximal torus $T' = (\ker(n \cdot c) \cap T)^\circ$.*
The space $V_{H^+(c)}$ decomposes as $V_{H^+(c)} = V_{H^0(c)} \oplus \bigoplus_{\chi \in X^*(T),\, \chi \in H^+(c) \setminus H^0(c)} V_\chi$. Let v_0 be the component of v in $V_{H^0(c)}$ with respect to this decomposition. Suppose that v_0 is not in the nullcone of Z_c in $V_{H^0(c)}$. Then the G-map

$$\begin{gathered} G \times_{P_c} V_{H^+(c)} \to G \cdot V_{H^+(c)} \subset V \\ ([(g, v')]) \mapsto g \cdot v' \end{gathered}$$

is birational onto $G \cdot V_{H^+(c)}$ and the fibres of the bundle $G \times_{P_c} V_{H^+(c)}$ are embedded as linear subspaces of V.

Proof. All faces of the polytope $\mathrm{Wt}(v)$ are described by rational linear equalities and inequalities, and $||\cdot||^2$ is a rational quadratic form on $X^*(T)_{\mathbb{Q}}$. By the differential criterion for extrema with boundary conditions, one gets a system of linear equations defined over $\mathbb{Q}$ which determine c.
We now use an argument in [Po-Vi] §5, due to Kirwan and Ness. Remark that the set of vectors u in $V_{H^+(c)}$ for which c is the point of $\mathrm{Wt}(u)$ closest to the origin and for which the projection u_0 onto $V_{H^0(c)}$ is not unstable for Z_c is open in $V_{H^+(c)}$ and not empty by assumption. We denote this set by $\Omega^+(c)$. Let $g_1,\, g_2 \in G,\, u_1,\, u_2 \in \Omega^+(c)$ be such that $g_1u_1 = g_2u_2$. So that $([(g_1,\, u_1)])$ and $([(g_2,\, u_2)])$ in the bundle $G \times_{P_c} V_{H^+(c)}$ map to the same image point in V. It thus suffices to show that for each $g \in G$, $gu_1 = u_2$ implies $g \in P_c$.
Look at the Bruhat decomposition $G = U_{P_c} W P_c$ where U_{P_c} is the unipotent radical of P_c. We may thus write $g = p_1 w p_2$ with $p_1,\, p_2 \in P_c$, w some (representative of an) element in W. We now make the following

> Claim: For each $p \in P_c$ and every $u \in \Omega^+(c)$, the weight polytope $\mathrm{Wt}(pu)$ contains the point c (one cannot "move $\mathrm{Wt}(u)$ away from c" with elements in P_c).

Assuming this claim for the moment, one can finish the proof as follows: rewriting $gu_1 = u_2$ as $w(p_2u_1) = p_1^{-1}u_2$ we see that $w \cdot c \in w \cdot \mathrm{Wt}(p_2u_1) = \mathrm{Wt}(wp_2u_1) \subset H^+(c)$. But since c is the only element of norm $||c||$ in $H^+(c)$ and W acts by isometries, we must have $w \cdot c = c$. Thus $P_c = P_{w\cdot c} = wP_cw^{-1}$, whence $w \in N_G(P_c) = P_c$.
We turn to the proof of the claim. In fact, the claim is just a reformulation of the property of the $u \in \Omega^+(c)$ to have Z_c-semistable projection u_0 onto $V_{H^0(c)}$ and the Hilbert-Mumford criterion in the form of Theorem 1.3.1.5, (2). Recall from above that if $\chi \in X^*(T)$, $v \in V_\chi$, $\alpha \in R$ and $g \in U_\alpha$, then $g \cdot v - v \in \bigoplus_{l \geq 1} V_{\chi + l\alpha}$, so if $\langle \alpha, c \rangle > 0$, then replacing u by $f \cdot u$ for $f \in U_\alpha$ gives $\mathrm{Wt}(u) \cap H^0(c) = \mathrm{Wt}(f \cdot u) \cap H^0(c)$. Thus the weight polytope of u can be moved away from c by an element of P_c if and only if it can be moved away from c by an element of L_c and L_c is in turn generated by Z_c and a one-dimensional central subtorus in L_c (central since all roots of L_c are trivial on it). But under the restriction of characters of T to the subtorus T' which

is the maximal torus in Z_c, $H^0(c)$ maps bijectively onto $X^*(T')_{\mathbb{Q}}$ and the point c gets identified with the origin in $X^*(T')_{\mathbb{Q}}$. Thus the weight polytope of u in $X^*(T)_{\mathbb{Q}}$ can be moved away from c by P_c if and only if the weight polytope of the projection u_0 of u onto $V_{H^0(c)}$ can be moved away from 0 in $X^*(T')_{\mathbb{Q}}$ by the action of Z_c. Our assumption that u_0 be Z_c-semistable exactly prevents this possibility. $\square$

Remark 1.3.2.4. Note that, though the way it was stated, Theorem 1.3.2.3 involves the choice of a T-unstable element v, and $c \in X^*(T)_{\mathbb{Q}}$ is afterwards determined as the element of $\mathrm{Wt}(v)$ of minimal length, the *only important requirement* is that $\mathfrak{N}_{Z_c}(V_{H^0(c)}) \neq V_{H^0(c)}$. In fact, if $c \in X^*(T)_{\mathbb{Q}}$, $c \neq 0$, is any element with this property, then the set $\Omega^+(c) \subset V_{H^+(c)}$ of vectors whose projection unto $V_{H^0(c)}$ is Z_c-semistable is nonempty, and the weight polytopes of all these vectors must automatically contain c then, and are contained in $H^+(c)$, hence c is the vector of minimal distance to the origin in all those weight polytopes.

Definition 1.3.2.5. An element $c \in X^*(T)_{\mathbb{Q}}$, $c \neq 0$, with $\mathfrak{N}_{Z_c}(V_{H^0(c)}) \neq V_{H^0(c)}$ is called *stratifying*.

The finer structure of the nullcone is described in

Theorem 1.3.2.6. *Let $c \in X^*(T)_{\mathbb{Q}}$ be a stratifying element. Then $G \times_{P_c} \Omega^+(c) \to G \cdot \Omega^+(c) =: \mathfrak{S}(c)$ is an isomorphism, and $G \cdot V_{H^+(c)} \subset \mathfrak{N}_G(V)$ is the closure of $\mathfrak{S}(c)$. We call $\mathfrak{S}(c)$ a nonzero stratum of $\mathfrak{N}_G(V)$. One has $\mathfrak{S}(c_1) = \mathfrak{S}(c_2)$ if and only if c_1 and c_2 are in the same W-orbit, and $\mathfrak{N}_G(V)$ is a finite disjoint union of $\{0\}$ and the nonzero strata.*

See [Po-Vi], §5, [Po03]. Through the paper [Po03], an algorithm -using only the configuration of weights with multiplicities of V and the roots of G in $X^*(T)_{\mathbb{Q}}$- is now available to determine completely the family of stratifying elements resp. strata. We pass over all this, since for our immediate purposes, Theorem 1.3.2.3 is sufficient.
We will now describe how Theorem 1.3.2.3 can be used to develop a general technique for approaching the rationality problem 1.2.1.5. This was suggested in [Shep89] and we will develop it in greater detail.

Theorem 1.3.2.7. *Let $\Gamma \subset G$ be connected reductive groups, V a G-module, and M a Γ-submodule of V which is contained in the nullcone $\mathfrak{N}_G(V)$ of G*

in V. Let $\mathfrak{S}(c)$, $0 \neq c \in X^*(T)_{\mathbb{Q}}$, $T \subset G$ a maximal torus, be a stratum of $\mathfrak{N}_G(V)$. Let

$$G \times_{P_c} V_{H^+(c)} \to G \cdot V_{H^+(c)} = \overline{\mathfrak{S}(c)} \subset V$$

be the associated desingularization of $\overline{\mathfrak{S}(c)}$ by the homogeneous vector bundle

$$G \times_{P_c} V_{H^+(c)} \xrightarrow{\pi} G/P_c\,.$$

Assume:

(a) $\mathfrak{S}(c) \cap M$ is dense in M and the rational map $\pi : \mathbb{P}(M) \dashrightarrow G/P_c$ induced by the bundle projection is dominant.

(b) $(G/P_c)/\Gamma$ is stably rational in the sense that $(G/P_c)/\Gamma \times \mathbb{P}^r$ is rational for some $r \leq \dim \mathbb{P}(M) - \dim G/P_c$.

(c) Let Z be the kernel of the action of Γ on G/P_c: assume Γ/Z acts generically freely on G/P_c, Z acts trivially on $\mathbb{P}(M)$, and there exists a Γ/Z-linearized line bundle $\mathcal{L}$ on the product $\mathbb{P}(M) \times G/P_c$ cutting out $\mathcal{O}(1)$ on the fibres of the projection to G/P_c.

Then $\mathbb{P}(M)/\Gamma$ is rational.

Proof. Let $Y := G/P_c$, $X :=$the (closure of) the graph of π, $p : X \to Y$ the restriction of the projection which (maybe after shrinking Y) we may assume (by (a)) to be a projective space bundle for which $\mathcal{L}$ is a relatively ample bundle cutting out $\mathcal{O}(1)$ on the fibres. The main technical point is the following result from descent theory ([Mum], §7.1): by Theorem 1.2.2.10 there are nonempty open subsets $X_0 \subset X$ and $Y_0 \subset Y$ such that we have a fibre product square with the bottom horizontal arrow a Γ/Z-principal bundle:

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0/(\Gamma/Z) \\ {\scriptstyle p}\downarrow & & {\scriptstyle \bar{p}}\downarrow \\ Y_0 & \longrightarrow & Y_0/(\Gamma/Z) \end{array}$$

and by [Mum], loc. cit., $\mathcal{L}$ descends to a line bundle $\bar{\mathcal{L}}$ on $X_0/(\Gamma/Z)$ cutting out $\mathcal{O}(1)$ on the fibres of $\bar{p}$. Hence $\bar{p}$ is also a Zariski locally trivial projective bundle (of the same rank as p). By (b), it now follows that $\mathbb{P}(M)/\Gamma$ is rational. $\square$

Example 1.3.2.8. Let E be a complex vector space of odd dimension n ($n \geq 3$) and consider the action of the group $G = \mathrm{SL}(E)$ on $V = \Lambda^2(E)$. We choose a basis $e_1, \ldots, e_n$ of E so that $\mathrm{SL}(E)$ is identified with the group $\mathrm{SL}_n(\mathbb{C})$ of $n \times n$ matrices of determinant one. Let

$$T = \mathrm{diag}(t_1, \ldots, t_n), \quad t_i \in \mathbb{C}, \quad \prod_{i=1}^{n} t_i = 1$$

be the standard diagonal torus, and denote by $\epsilon_i \in X^*(T)$ the ith coordinate function of T

$$\epsilon_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_i, \quad i = 1, \ldots, n\,.$$

In $\mathbb{R}^n$ with its standard scalar product consider the hyperplane $H := \langle (1, 1, \ldots, 1) \rangle^{\perp}$. We make the identifications

$$\begin{gathered}\epsilon_1 = (n-1,\ -1,\ -1, \ldots, -1), \quad \epsilon_2 = (-1,\ n-1,\ -1, \ldots, -1), \ldots, \\ \epsilon_n = (-1,\ -1,\ -1, \ldots, n-1) \quad (\text{then } \epsilon_i \in H \quad \forall\, i = 1, \ldots, n)\end{gathered}$$

whence $X^*(T) \otimes \mathbb{R}$ becomes identified with H, $X^*(T)$ being the subset of vectors $a_1\epsilon_1 + \cdots + a_n\epsilon_n$ with $a_i \in \mathbb{Z}$ for all i, and $X^*(T)_{\mathbb{Q}}$ is similarly defined by the condition $a_i \in \mathbb{Q}$. The restriction of the standard Euclidean scalar product on $\mathbb{R}^n$ to H is then a W-invariant scalar product, integral on $X^*(T)$. Denote it by $\langle \cdot, \cdot \rangle$. Note that the Weyl group W is the symmetric group $\mathfrak{S}_n$ on n letters and acts by permuting the ϵ_i. The roots of $(G,\, T)$ are then

$$\alpha_{ij} := \epsilon_i - \epsilon_j, \quad 1 \leq i,\ j \leq n,\ i \neq j$$

with corresponding root subgroups

$$U_{\alpha_{ij}} = \{ A \in \mathrm{SL}_n(\mathbb{C}) \,|\, A = \mathrm{Id} + r \cdot E_{ij} \}\,,$$

with E_{ij} the $n \times n$ elementary matrix with a single nonzero entry, namely 1, in the $(i,\, j)$-spot. Thus the ϵ_i form the vertices of a simplex in $H \simeq X^*(T) \otimes \mathbb{R}$ and the roots are the pairwise differences of the vectors leading from the origin to the vertices. The weights of T in $\Lambda^2(E)$ are obviously

$$\pi_{kl} = \epsilon_k + \epsilon_l, \quad 1 \leq k < l \leq n, \quad V_{\pi_{kl}} = \mathbb{C} \cdot (e_k \wedge e_l)\,.$$

Define an element $c \in X^*(T)_{\mathbb{Q}}$ by

$$c := \frac{2}{n-1}(1,\ 1,\ 1, \ldots,\ 1,\ -(n-1)) \in H\,.$$

We consider as above the affine hyperplane $H^0(c)$ perpendicular to c and passing through c, and the positive half space $H^+(c)$ it defines. The following facts concerning the relative position of $H^0(c)$ and the weights π_{kl} and roots α_{ij} are easily established by direct calculation:

- $\pi_{kl} \in H^+(c) \iff \pi_{kl} \in H^0(c) \iff 1 \le k < l \le n-1$,
- $\langle \alpha_{ij},\, c\rangle = 0 \iff 1 \le i,\, j \le n-1,\, i \ne j\,,$
- $\langle \alpha_{ij},\, c\rangle > 0 \iff 1 \le i < j = n\,.$

Note that then in the above notation one has for the group P_c

$$P_c = \left\{ \begin{pmatrix} M & a \\ 0 & b \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \,:\, M \in \mathbb{C}^{(n-1)\times(n-1)},\ a \in \mathbb{C}^{(n-1)\times 1},\right.$$
$$\left. b \in \mathbb{C},\ 0 \in \mathbb{C}^{1\times(n-1)} \right\}\,.$$

Similarly, the reductive group L_c is

$$L_c = \left\{ \begin{pmatrix} M & 0^t \\ 0 & b \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \,:\, M \in \mathrm{GL}_{n-1}(\mathbb{C}),\ b \in \mathbb{C}^*,\ 0 \in \mathbb{C}^{1\times(n-1)} \right\}\,.$$

Now $\frac{n-1}{2}c = \epsilon_1 + \cdots + \epsilon_{n-1}$. This extends to the character of L_c which maps an element of L_c to the determinant of M. Hence the group Z_c is

$$Z_c = \left\{ \begin{pmatrix} M & 0^t \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \,:\, M \in \mathrm{SL}_{n-1}(\mathbb{C}),\ 0 \in \mathbb{C}^{1\times(n-1)} \right\}\,.$$

The action of Z_c on $V_{H^0(c)}$ is equivalent to the standard action of $\mathrm{SL}_{n-1}(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^{n-1})$ and the nullcone for this action is not the whole space (there exists the Pfaffian). Hence c is stratifying.
The flag variety G/P_c can be identified with the Grassmannian $\mathrm{Grass}(n-1, E)$ of $n-1$-dimensional subspaces in E, or dually, $\mathbb{P}(E^\vee)$, the projective space of lines in $E^\vee$. The open set $G\cdot\Omega^+(c)$ in $\Lambda^2(E)$ can be identified with the two forms represented by skew-symmetric matrices of maximal rank $n-1$. Every vector in $\Lambda^2(E)$ is unstable. If we view $\Lambda^2(E)$ as $\Lambda^2(E^\vee)^\vee$, i.e. skew-forms on $E^\vee$, then the bundle projection

$$G \times_{P_c} \Omega^+(c) \to G/P_c$$

is identified with the map which assigns to a skew-form ω of maximal rank its image under the linear map

$$E^\vee \to E, \quad e \mapsto \omega(e, \cdot)\,,$$

(an element of Grass$(n-1, E)$), or dually, its kernel in $\mathbb{P}(E^\vee)$. The associated method for proving rationality is called *the 2-form trick* and appears in [Shep], Prop.8.

Example 1.3.2.9. Let E and F be complex vector spaces with $\dim E =: n > \dim F =: m$. As in the previous example, for the action of $G = \mathrm{SL}(E) \times \mathrm{SL}(F)$ on $V = \mathrm{Hom}(E, F)$, every vector is unstable (there is a dense orbit). Choose bases $e_1, \dots, e_n$ for E and $f_1, \dots, f_m$ for F, so that $\mathrm{SL}(E) \times \mathrm{SL}(F) \simeq \mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$. If T^E resp. T^F denote the standard maximal tori of diagonal matrices in $\mathrm{SL}_n(\mathbb{C})$ resp. $\mathrm{SL}_m(\mathbb{C})$, then

$$T = T^E \times T^F$$

is a maximal torus of G. For $(\mathrm{SL}_n(\mathbb{C}), T^E)$ and $(\mathrm{SL}_m(\mathbb{C}), T^F)$, we use the definitions and concrete realization of weight lattices as in Example 1.3.2.8, except that we endow now all objects with superscripts E and F to indicate which group we refer to: thus we write, for example, ϵ_i^E, ϵ_j^F, W^E, $X^*(T^E) \otimes \mathbb{R} \simeq H^E$, and so forth.
Then we have $X^*(T) = X^*(T^E) \times X^*(T^F)$ and we may realize $X^*(T) \otimes \mathbb{R}$ as

$$H := H^E \times H^F \subset \mathbb{R}^n \times \mathbb{R}^m \simeq \mathbb{R}^{n+m}$$

with scalar product

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, x_2 \rangle^E + \langle y_1, y_2 \rangle^F, \quad x_1, x_2 \in H^E, \, y_1, y_2 \in H^F,$$

which is invariant under the Weyl group $W = W^E \times W^F \simeq \mathfrak{S}_n \times \mathfrak{S}_m$ of G acting by permutations of the ϵ_i^E and ϵ_j^F separately. The weights of V with respect to T are given by

$$\pi_{kl} := (-\epsilon_k^E, \epsilon_l^F), \, 1 \le k \le n, \, 1 \le l \le m, \quad V_{\pi_{kl}} = \mathbb{C} \cdot (e_k^\vee \otimes f_l)$$

where under the isomorphism $E^\vee \otimes F \simeq \mathrm{Hom}(E, F)$, the vector $e_k^\vee \otimes f_l$ corresponds to a matrix with only one nonzero entry 1 in the (l, k) position. Note that the π_{kl} form the vertices of a polytope in H which is the product of two simplices. The roots of (G, T) in H are the vectors

$$(\alpha_{pq}^E, 0), \, 1 \le p, q \le n, \, p \ne q, \quad (0, \alpha_{rs}^F), \, 1 \le r, s \le m, \, r \ne s$$

(the disjoint union of the root systems of $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SL}_m(\mathbb{C})$ in the orthogonal subspaces H^E and H^F), and the root subgroups in G are then

$$U_{(\alpha^E_{pq},\,0)} = U_{\alpha^E_{pq}} \times \{\mathrm{Id}_m\}, \quad U_{(0,\,\alpha^F_{rs})} = \{\mathrm{Id}_n\} \times U_{\alpha^F_{rs}}\,.$$

Now define $c \in X^*(T)_{\mathbb{Q}}$ by

$$c := \left(-\frac{1}{m}(n-m,\, n-m, \dots,\, n-m,\, -m,\, -m, \dots,\, -m),\, 0 \right) \in H^E \times H^F$$

where there are m entries with value $n-m$ followed by another $n-m$ entries with value $-m$ in the row vector in the first component. In fact, $c = -\frac{1}{m}((\epsilon^E_1,\, 0) + \dots + (\epsilon^E_m,\, 0))$. The following facts are easily verified by direct computation:

- $\pi_{kl} \in H^0(c) \iff \pi_{kl} \in H^+(c) \iff 1 \le k \le m\,,$
- $\langle (0, \alpha^F_{rs}),\, c\rangle = 0 \,\forall\, r,\, s,\, 1 \le r,\, s \le m, r \neq s\,,$
- $\langle (\alpha^E_{pq}, 0),\, c\rangle = 0 \iff 1 \le p,\, q \le m,\, p \neq q$ or $m+1 \le p,\, q \le n,\, p \neq q\,,$
- $\langle (\alpha^E_{pq}, 0),\, c\rangle > 0 \iff 1 \le q \le m,\, m+1 \le p \le n\,.$

Hence we get

$$P_c = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \,:\, A \in \mathbb{C}^{m\times m},\, B \in \mathbb{C}^{(n-m)\times m},\, C \in \mathbb{C}^{(n-m)\times(n-m)} \right\} \times \mathrm{SL}_m(\mathbb{C})\,,$$

and

$$L_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \,:\, A \in \mathrm{GL}_m(\mathbb{C}),\, C \in \mathrm{GL}_{n-m}(\mathbb{C}) \right\} \times \mathrm{SL}_m(\mathbb{C})\,,$$
$$Z_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}) \,:\, A \in \mathrm{SL}_m(\mathbb{C}),\, C \in \mathrm{SL}_{n-m}(\mathbb{C}) \right\} \times \mathrm{SL}_m(\mathbb{C})\,,$$

since $-m\cdot c$ extends to the character of L_c which maps an element of L_c to the determinant of A. The action of Z_c on $V_{H^0(c)}$ is equivalent to the standard action of $\mathrm{SL}_m(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$ on $\mathrm{Hom}(\mathbb{C}^m,\, \mathbb{C}^m)$, whence $\mathfrak{N}_{Z_c}(V_{H^0(c)}) \neq V_{H^0(c)}$ since an endomorphism has a determinant. Thus c is stratifying.
The flag variety G/P_c is isomorphic to the Grassmannian $\mathrm{Grass}(n-m,\, E)$

of $(n-m)$-dimensional subspaces of E, and $G \cdot \Omega^+(c) \subset \mathrm{Hom}(E,\, F)$ is the open subset of homomorphisms of full rank m. The projection

$$G \times_{P_c} \Omega^+(c) \to G/P_c$$

is the map which associates to a homomorphism $\varphi \in \mathrm{Hom}(E,\, F)$ its kernel $\ker(\varphi) \in \mathrm{Grass}(n-m,\, E)$. In the case $n = m+1$, this is a projective space, and the associated method for proving rationality is called the *double bundle method*, which appeared first in [Bo-Ka]. We discuss this in more detail in Chapter 2.

The discussion in Example 1.3.2.9 shows that it will be very convenient to have results for the stable rationality of Grassmannians $\mathrm{Grass}(k,\, E)/G$ (where E is a representation of the reductive group G) analogous to Corollary 1.2.2.11. One has

Proposition 1.3.2.10. *Let E be a representation of $G = \mathrm{SL}_p(\mathbb{C})$, p prime. Let $X := \mathrm{Grass}(k,\, E)$ be the Grassmannian of k-dimensional subspaces of E. Assume:*

- *The kernel Z of the action of G on $\mathbb{P}(E)$ coincides with the center $\mathbb{Z}/p\mathbb{Z}$ of $\mathrm{SL}_p(\mathbb{C})$ and the action of G/Z on $\mathbb{P}(E)$ is almost free. Furthermore, the action of G on E is almost free and each element of Z not equal to the identity acts homothetically as multiplication by a primitive pth root of unity.*
- *$k \le \dim E - \dim G - 1$.*
- *p does not divide k.*

Then X/G is stably rational, in fact, $X/G \times \mathbb{P}^{\dim G+1}$ is rational.

Proof. Let $C_X \subset \Lambda^k(E)$ be the affine cone over X consisting of pure (complety decomposable) k-vectors. We will show that under the assumptions of the proposition, the action of G on C_X is almost free. This will accomplish the proof since C_X/G is generically a torus bundle over X/G hence Zariski-locally trivial; and the group $G = \mathrm{SL}_p(\mathbb{C})$ is special.
Let $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ be a general k-vector in $\Lambda^k(E)$. Since $k \le \dim E - \dim G - 1$ and, in $\mathbb{P}(E)$, $\dim(G \cdot [e_1]) = \dim G$ since Z is finite and G/Z acts almost freely on $\mathbb{P}(E)$, the $k-1$-dimensional projective linear subspace spanned by $e_1, \dots, e_k$ in $\mathbb{P}(E)$ will intersect the $\dim E - 1 - \dim G$ codimensional orbit

$G \cdot [e_1]$ only in $[e_1]$. Hence, if an element $g \in G$ stabilizes $e_1 \wedge \cdots \wedge e_k$, it must lie in Z. Thus $g \cdot (e_1 \wedge \cdots \wedge e_k) = \zeta^k (e_1 \wedge \cdots \wedge e_k)$ for a primitive p-th root of unity ζ if $g \neq 1$. But since p does not divide k, the case $g \neq 1$ cannot occur. $\square$

As an application we prove the following result which had not been obtained by different techniques so far.

Theorem 1.3.2.11. *The moduli space* $\mathbb{P}(\mathrm{Sym}^{34}(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ *of plane curves of degree* 34 *is rational.*

Proof. As usual, $V(a,\, b)$ denotes the irreducible $\mathrm{SL}_3(\mathbb{C})$-module whose highest weight has numerical labels a, b (we choose the standard diagonal torus and Borel subgroup of upper triangular matrices for definiteness). Then

$$V(0,\, 34) \subset \mathrm{Hom}(V(14,\, 1),\, V(0,\, 21))\,,$$

and $\dim V(14,\, 1) = 255$, $\dim V(0,\, 21) = 253$, so we get a map

$$\pi\, :\, \mathbb{P}(V(0,\, 34)) \dashrightarrow \mathrm{Grass}(2,\, V(14,\, 1))$$

$\dim \mathbb{P}(V(0,\, 34)) = 629$ and $\dim \mathrm{Grass}(2,\, V(14,\, 1)) = 506$. Moreover, Proposition 1.3.2.10 and its proof show that $\mathrm{Grass}(2,\, V(14,\, 1))/\mathrm{SL}_3(\mathbb{C}) \times \mathbb{P}^9$ is rational, and the action of $\mathrm{PGL}_3(\mathbb{C}) = \mathrm{SL}_3(\mathbb{C})/Z$, where Z is the center of $\mathrm{SL}_3(\mathbb{C})$, is almost free on $\mathrm{Grass}(2,\, V(14,\, 1))$. Moreover, let $\mathcal{O}_P(1)$ be the $\mathrm{SL}_3(\mathbb{C})$-linearized line bundle induced by the Plücker embedding on $\mathrm{Grass}(2,\, V(14,\, 1))$:

$$\mathrm{Grass}(2,\, V(14,\, 1)) \subset \mathbb{P}(\Lambda^2(V(14,\, 1)))\,.$$

If we choose on $\mathbb{P}(V(0,\, 34)) \times \mathrm{Grass}(2,\, V(14,\, 1))$ the bundle $\mathcal{L} := \mathcal{O}(1) \boxtimes \mathcal{O}_P(2)$, all the assumptions of Theorem 1.3.2.7 except the dominance of π have been checked (compare also Example 1.3.2.9). The latter dominance follows from an explicit computer calculation, where one has to check that for a random element x_0 in $V(0,\, 34)$ the corresponding homomorphism in $\mathrm{Hom}(V(14,\, 1),\, V(0,\, 21))$ has full rank, and the fibre of π over $\pi([x_0])$ has the expected dimension $\dim \mathbb{P}(V(0,\, 34)) - \dim \mathrm{Grass}(2,\, V(14,\, 1))$. $\square$

Remark 1.3.2.12. As far as we can see, the rationality of $\mathbb{P}(V(0,34))/\mathrm{SL}_3(\mathbb{C})$ cannot be obtained by direct application of the double bundle method, i.e. by applying Theorem 1.3.2.7 in the case discussed in Example 1.3.2.9 with base of the projection a projective space. In fact, a computer search yields that the inclusion $V(0,\,34) \subset \mathrm{Hom}(V(30,0),\,V(0,4) \oplus V(5,\,9))$ is the only candidate to be taken into consideration for dimension reasons: $\dim V(30,0) = \dim(V(0,\,4) \oplus V(5,\,9)) + 1$ and $\dim \mathbb{P}(V(0,\,34)) > \dim \mathbb{P}(V(30,\,0))$. However, on $\mathbb{P}(V(0,\,34)) \times \mathbb{P}(V(30,\,0))$ there does not exist a $\mathrm{PGL}_3(\mathbb{C})$-linearized line bundle cutting out $\mathcal{O}(1)$ on the fibres of the projection to $\mathbb{P}(V(30,0))$; for such a line bundle would have to be of the form $\mathcal{O}(1) \boxtimes \mathcal{O}(k)$, $k \in \mathbb{Z}$, and none of these is $\mathrm{PGL}_3(\mathbb{C})$-linearized: since $\mathcal{O} \boxtimes \mathcal{O}(1)$ is $\mathrm{PGL}_3(\mathbb{C})$-linearized it would follow that the $\mathrm{SL}_3(\mathbb{C})$ action on $H^0(\mathbb{P}(V(0,\,34)),\,\mathcal{O}(1)) \simeq V(34,\,0)$ factors through $\mathrm{PGL}_3(\mathbb{C})$ which is not the case.

1.4 Overview of some further topics

Here we give a brief description of additional topics connected with the rationality problem which are too important to be omitted altogether, but are outside the focus of the present text.

1.4.1 Cohomological obstructions to rationality and fields of definition

The first concerns *cohomological obstructions to rationality.* One of the first examples was given by Artin and Mumford [A-M] who showed

Proposition 1.4.1.1. *The torsion subgroup $T \subset H^3(X,\,\mathbb{Z})$ is a birational invariant of a smooth projective variety X. In particular, $T = 0$ if X is rational.*

They use this criterion to construct unirational irrational threefolds X, in fact X with 2-torsion in $H^3(X,\,\mathbb{Z})$. Later, David Saltman [Sa] proved that there are invariant function fields $\mathbb{C}(X)$ (where $X = V/G$) of the action of a finite group G which are not purely transcendental over the ground field $\mathbb{C}$ using as invariant the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(X)/\mathbb{C})$ which can be shown to equal the cohomological Brauer group $\mathrm{Br}(\tilde{X}) = H^2_{\text{ét}}(\tilde{X}, \mathbb{G}_m)$ of a smooth projective model $\tilde{X}$ of $\mathbb{C}(X)$. Here $\mathbb{G}_m$ denotes the sheaf (for the étale site on any scheme) defined by the standard multiplicative group

scheme. Moreover, in the sequel μ_n will, as usual, denote the subsheaf of $\mathbb{G}_m$ defined by $\mu_n(U) =$ group of nth roots of 1 in $\Gamma(U, \mathcal{O}_U)$. The unramified point of view was developed further in [Bogo3].
The notion of unramified cohomology generalizes the previous two examples. A particular feature of the unramified view point is that it bypasses the need to construct a smooth projective model for a given variety X, working directly with the function field of X, or rather with all smooth projective models of X at once. Below, k is an algebraically closed field of any characteristic.

Definition 1.4.1.2. Let X be a variety over k and $n > 0$ an integer prime to char(k). The i-th unramified cohomology group of X with coefficients in $\mu_n^{\otimes j}$ is by definition

$$H^i_{\mathrm{nr}}(k(X)/k,\, \mu_n^{\otimes j}) := \bigcap_{A \in \mathrm{DVR}(k(X))} \left(\mathrm{im}\left(H^i_{\text{ét}}(A,\, \mu_n^{\otimes j}) \to H^i_{\text{ét}}(k(X),\, \mu_n^{\otimes j})\right)\right)$$

where A runs over all rank one discrete valuation rings $k \subset A \subset k(X)$ with quotient field $k(X)$. The cohomology groups are to be interpreted as étale cohomology

$$H^i_{\text{ét}}(A,\, \mu_n^{\otimes j}) := H^i_{\text{ét}}(\mathrm{Spec}(A),\, \mu_n^{\otimes j}),$$
$$H^i_{\text{ét}}(k(X),\, \mu_n^{\otimes j}) := H^i_{\text{ét}}(\mathrm{Spec}(k(X)),\, \mu_n^{\otimes j})\,.$$

It is known that if $k(X)$ and $k(Y)$ are stably isomorphic over k, then

$$H^i_{\mathrm{nr}}(k(X)/k,\, \mu_n^{\otimes j}) \simeq H^i_{\mathrm{nr}}(k(Y)/k,\, \mu_n^{\otimes j})$$

and in particular, the higher unramified cohomology groups are trivial if X is stably rational (see [CT95]).

Clearly, if $G = \mathrm{Gal}(k(X)^s/k(X))$ is the absolute Galois group of $K := k(X)$

$$H^i_{\text{ét}}(\mathrm{Spec}(k(X)),\, \mu_n^{\otimes j}) = H^i(G,\, \mu_n^{\otimes j}) = H^i(K,\, \mu_n^{\otimes j})\,,$$

the latter being a Galois cohomology group [Se97] (μ_n the group of nth roots of 1 in K^s). There is an alternative description of unramified cohomology in terms of residue maps in Galois cohomology which is often useful. We would like to be as concrete as possible, so recall first that given a profinite group G and a discrete G-module M on which G acts continuously, and denoting by $C^n(G,\, M)$ the set of all continuous maps from G^n to M, we define the cohomology groups $H^q(G,\, M)$ as the cohomology of the complex $C^{\cdot}(G,\, M)$ with differential

$$\begin{aligned} d \,:\, C^n(G,\, M) \to C^{n+1}(G,\, M),& \\ (df)(g_1, \dots, g_{n+1}) &= g_1 \cdot f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n) \end{aligned}$$

and this can be reduced to the finite group case since

$$H^q(G,\, M) = \varinjlim H^q(G/U,\, M^U)\,,$$

the limit taken over all open normal subgroups U in G. We recall from [Se97], Appendix to Chapter II, the following:

Proposition 1.4.1.3. *If G is a profinite group, N a closed normal subgroup of G, Γ the quotient G/N, and M a discrete G-module with trivial action of N, then one has exact sequences for all $i \geq 0$*

$$0 \to H^i(\Gamma,\, M) \xrightarrow{\;\pi\;} H^i(G,\, M) \xrightarrow{\;r\;} H^{i-1}(\Gamma,\, \mathrm{Hom}(N,\, M)) \longrightarrow 0$$

provided the following two assumptions hold:

(a) The extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

splits.

(b) $H^i(N,\, M) = 0$ for all $i > 1$.

Here π is induced through the map $G \to \Gamma$ by functoriality and r is the residue map which has an explicit description as follows: an element $\alpha \in H^i(G,\, M)$ can be represented by a cocycle $f = f(g_1, \dots, g_i) \in C^i(G,\, M)$ which is normalized (i.e. equal to 0 if one g_j is 1) and which only depends on g_1 and the classes $\gamma_2, \dots, \gamma_i$ of $g_2, \dots, g_i$ in Γ. If then $\gamma_2, \dots, \gamma_i$ are elements in

Γ, one defines $r(f)(\gamma_2, \ldots, \gamma_i)$ to be the element of $\mathrm{Hom}(N,\, M)$ (continuous homomorphisms of N to M) given by

$$r(f)(\gamma_2, \ldots, \gamma_i)(n) := f(n,\, g_2, \ldots, g_i),\; n \in N\,.$$

The $(i-1)$-cochain $r(f)$ is checked to be an $(i-1)$-cocycle of Γ with values in $\mathrm{Hom}(N,\, M)$, and its class $r(\alpha)$ in $H^{i-1}(\Gamma,\, \mathrm{Hom}(N,\, M))$ is independent of f. The proof uses the spectral sequence of group extensions; item (b) is used to reduce the spectral sequence to a long exact sequence, and (a) to split the long exact sequence into short exact ones of the type given. Details may be found in [Se97].
Reverting to our original set-up we have the field extension $K/k = k(X)/k$ and a discrete valuation ring $A \subset K$ with $k \subset A$ and such that K is the field of fractions of A. To A there is an associated completion $\hat{K}_A$. Assume $\hat{K}_A$ of residue characteristic 0 for simplicity. The absolute Galois group of $\hat{K}_A$ splits as $\hat{\mathbb{Z}} \oplus G_A$ with G_A the absolute Galois group of the residue field of A. Since $\hat{\mathbb{Z}}$ has cohomological dimension 1 we can apply Proposition 1.4.1.3 to obtain a map

$$\varrho_A \,:\, H^i(K,\, \mu_n^{\otimes j}) \to H^i(\hat{\mathbb{Z}} \oplus G_A,\, \mu_n^{\otimes j}) \to H^{i-1}(G_A,\, \mu_n^{\otimes j-1})\,,$$

where the first map is restriction and the second one the residue map of Proposition 1.4.1.3. Then one has

$$H^i_{\mathrm{nr}}(k(X)/k,\, \mu_n^{\otimes j}) = \bigcap_{A \in \mathrm{DVR}(k(X))} (\ker(\varrho_A))$$

(cf. [CT95] or [CT-O], §1 for the proof). This is a purely Galois cohomological description of unramified cohomology.

To connect the notion of unramified cohomology with the classical work of Artin-Mumford and Saltman, we list a few results when unramified cohomology has been computed.

- Let X be a smooth projective variety over a field k, algebraically closed of characteristic 0. Then

  $$H^1_{\mathrm{nr}}(k(X)/k, \mu_n) \simeq (\mathbb{Z}/n\mathbb{Z})^{\oplus 2q} \oplus_n \mathrm{NS}(X)$$

  $q = \dim H^1(X,\, \mathcal{O}_X)$ is the dimension of the Picard variety of X and ${}_n\mathrm{NS}(X)$ is n-torsion in the Néron-Severi group of X. Cf. [CT95], Prop. 4.2.1.

- Let X be as in the previous example, suppose furthermore that X is unirational and let n be a power of a prime number l. Then

$$H^2_{\mathrm{nr}}(k(X)/k, \mu_n) \simeq_n H^3(X, \mathbb{Z}_l)$$

(where $H^3(X,\ \mathbb{Z}_l)$ is the third étale cohomology group of X with $\mathbb{Z}_l$-coefficients). Furthermore, $H^2_{\mathrm{nr}}(k(X)/k, \mu_n) \simeq_n \mathrm{Br}(X)$ which explains the relation to the examples of Artin-Mumford and Saltman. Cf. [CT95], Prop. 4.2.3.

Much less is known about higher unramified cohomology groups $H^i_{\mathrm{nr}}(k(X)/k, \mu_n^{\otimes j})$, $i \geq 3$, but see [CT-O], [Sa3] and [Pey], [Mer] for some computations and uses of unramified H^3. Let us mention at this point a problem which seems particularly attractive and which is apparently unsolved.

Problem 1.4.1.4. Since for any divisor n of 420 and any almost free representation V of $\mathrm{PGL}_n(\mathbb{C})$, the field $\mathbb{C}(V)^{\mathrm{PGL}_n(\mathbb{C})}$ is stably rational (see [CT-S], Prop. 4.17 and references there), it would be very interesting to compute some higher unramified cohomology groups of $\mathbb{C}(V)^{\mathrm{PGL}_8(\mathbb{C})}$, for some almost free representation V of $\mathrm{PGL}_8(\mathbb{C})$, or at least to detect nontrivial elements in some such group (higher should probably mean here degree at least 4). This would give examples of $\mathrm{PGL}_8(\mathbb{C})$-quotients which are not stably rational.

We conclude by some remarks how the rationality problem changes character if we allow our ground field k to be non-closed or of positive characteristic. First, Merkurjev [Mer] has shown that over nonclosed fields k there exist examples of connected simply connected semi-simple groups G with almost free action on a linear representation V such that $k(V)^G$ is non-rational (even not stably rational). On the other hand, over $k = \bar{\mathbb{F}}_p$, the recent article [BPT-2] proves stable rationality for many quotients V/G where G is a finite group G of Lie type, and V a faithful representation of G over $k = \bar{\mathbb{F}}_p$.

1.5 Stable rationality results for exceptional simple groups

Since the groups $G = \mathrm{SL}_n(\mathbb{C})$ and $G = \mathrm{Sp}_n(\mathbb{C})$ are special by Theorem 1.2.2.7, so any quotient V/G, where V is a generically free G-representation, is stably rational. It is interesting to investigate if V/G is always stably rational for the other simple simply connected groups G as well. For the series of Spin-groups this is not known, but for G one of the exceptional groups G_2, F_4, E_6 or E_7 the answer is positive by the results of [Bogo1] (the E_8 case is open). We will give a proof for the stable rationality of generically free G_2, F_4, E_6 or E_7-quotients below following [Bogo1]. Good references for the material on exceptional groups we will need are [Sp-Veld], [Post], [Adams] (the latter two, however, mostly deal with the real forms). There is also the very useful series of papers by Jacobson [Jac59], [Jac60], [Jac61].

1.5.1 The case G_2

We start with the G_2 case. Recall that a *composition algebra* over a field k is a (not necessarily associative) k-algebra C with an identity 1 and a nondegenerate quadratic form N on C such that $N(xy) = N(x)N(y)$ (all x and y in C). N is called the norm form, and the associated bilinear form will be denoted by $\langle \cdot, \cdot \rangle$. In any composition algebra C one may introduce an involutive anti-automorphism $C \to C$, $x \mapsto \bar{x}$, called the *conjugation*, defined by $\bar{x} := -s_1(x)$ where s_1 is the reflection in the subspace $1^\perp$ defined by $\langle \cdot, \cdot \rangle$. Then $x\bar{x} = \bar{x}x = N(x) \cdot 1$.
Now let $\mathbb{H} := \mathrm{Mat}_{2\times 2}(\mathbb{C})$ be the *split quaternion algebra* (over $\mathbb{C}$) of two by two complex matrices which is a composition algebra with norm form the determinant. If

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then consequently

$$x = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The *split octonion algebra* $\mathbb{O}$ is the composition algebra which is constructed from $\mathbb{H}$ by the process of *doubling*: as vector space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ and the product resp. norm form are given by

$$(x,\, y)(u,\, v) := (xu + \bar{v}y,\, vx + y\bar{u}), \quad x, y, u, v \in \mathbb{H}$$

resp.

$$N((x,\, y)) := \det(x) - \det(y), \quad x,\, y \in \mathbb{H}\,.$$

We view $\mathbb{H}$ as embedded into $\mathbb{O}$ as the first factor, and $1_{\mathbb{O}} = (1_{\mathbb{H}}, 0)$. The group of automorphisms $\mathrm{Aut}(\mathbb{O})$ of the composition algebra $\mathbb{O}$ (which means unital, norm-preserving algebra automorphisms) is the simple group of type G_2, $G_2 \subset O(N, \mathbb{O})$. Its dimension is 14.
The proof that almost free G_2 quotients are stably rational depends on the existence of *standard bases* in $\mathbb{O}$.

Theorem 1.5.1.1. *Let* $\mathrm{Im}(\mathbb{O}) := \{X \in \mathbb{O} \,|\, \bar{X} = -X\}$ *be the subspace of purely imaginary octonions. Let* $A, B, C \in \mathrm{Im}(\mathbb{O})$ *be nonzero elements with*

$$A \perp B,\ C \perp A, B, AB\,. \tag{1.2}$$

Then the elements

$$1,\ A,\ B,\ AB,\ C,\ AC,\ BC,\ (AB)C$$

form an orthogonal basis of $\mathbb{O}$ *and if* A', B', C' *are a different triple of purely imaginary octonions satisfying 1.2 with* $N(A) = N(A')$, $N(B) = N(B')$, $N(C) = N(C')$, *then there exists a unique automorphism of* $\mathbb{O}$ *(an element of* G_2*) which carries* A *into* A', B *into* B', C *into* C'.

If we normalize A, B, C so that $N(A) = N(B) = N(C) = 1$, then we call them a standard basis of $\mathbb{O}$. We postpone the proof of Theorem 1.5.1.1 to state the following:

Corollary 1.5.1.2. *Let* V *be a finite dimensional generically free* G_2 *representation, then* V/G_2 *is stably rational.*

Proof. By the no-name lemma 2.2.2.1 it suffices to find a tower of G_2-equivariant vector bundles X (starting from a G_2-representation as base) on which the action of G_2 is generically free and X/G_2 is stably rational. We take the 7-dimensional G_2-representation $\mathrm{Im}(\mathbb{O}) = \langle 1 \rangle^{\perp}$ and consider

$$X = \{(A,\ B,\ C) \,|\, A \perp B,\ C \perp A, B, AB,\ N(A) \neq 0,\ N(B) \neq 0,\ N(C) \neq 0\} \\ \subset \mathrm{Im}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O}) \oplus \mathrm{Im}(\mathbb{O})\,.$$

By Theorem 1.5.1.1 G_2 acts on X freely, and $G_2 \times (\mathbb{C}^*)^3$ transitively. Since X is birational to a tower of G_2-vector bundles on $\mathrm{Im}(\mathbb{O})$, and X/G_2 has a transitive torus action, X/G_2 is rational. $\square$

Since Theorem 1.5.1.1 is vital for the proof of the preceding result, we outline how one may obtain it. Computations in $\mathbb{O}$ are of course awkward since it is not associative, but, as is well known, it is at least (as any composition algebra) *alternative*, i.e. the trilinear function (the *associator*)

$$\{X, Y, Z\} := (XY)Z - X(YZ)$$

vanishes if two of its arguments are equal, cf. [Sp-Veld], Lemma 1.4.2. We will use this fact and the collection of formulas in the following Lemma.

Lemma 1.5.1.3. *Let X, Y, Z be in $\mathrm{Im}(\mathbb{O})$. Then:*

(1) $X^2 = -N(X) \cdot 1$.

(2) If $X \perp Y$ then $XY = -YX \in \mathrm{Im}(\mathbb{O})$ and $XY \perp X, Y$.

(3) One has $\langle XY, Z\rangle = \langle X, YZ\rangle$.

(4) If X, Y, Z are pairwise orthogonal and $Z \perp XY$, then $X \perp YZ$, $Y \perp ZX$, and

$$(XY)Z = (YZ)X = (ZX)Y \in \mathrm{Im}(\mathbb{O})\,.$$

Proof. In view of

$$N(X) = \bar{X}X = X\bar{X}, \quad \langle X,\, Y\rangle = \frac{1}{2}(X\bar{Y} + Y\bar{X}) = \frac{1}{2}(\bar{X}Y + \bar{Y}X)$$

(1) and (2) are immediate (for (2) one also uses alternativity). For (3) remark that for purely imaginary octonions U, V one has $\langle UV,\, U\rangle = 0$ by alternativity whence

$$\begin{aligned} 0 &= \langle (X+Z)Y,\, X+Z\rangle \\ &= \langle XY,\, Z\rangle + \langle ZY,\, X\rangle \\ &= \langle XY,\, Z\rangle + \langle \bar{X},\, \overline{ZY}\rangle \\ &= \langle XY,\, Z\rangle - \langle X,\, YZ\rangle\,. \end{aligned}$$

The orthogonality relations in (4) follow from (3), and the fact that $(XY)Z$, $(YZ)X$, $(ZX)Y$ are again purely imaginary follows from (2). Now since X and Y are orthogonal, using alternativity,

$$\begin{aligned} 0 \qquad &= N(X+Y)Z - N(X)Z - N(Y)Z \\ &= (X+Y)\left\{\overline{X+Y}Z\right\} - X(\bar{X}Z) - Y(\bar{Y}Z) \quad = X(\bar{Y}Z) + Y(\bar{X}Z) \end{aligned}$$

and conjugating the last equation, together with the fact that X, Y, Z are antiinvariant under conjugation and $ZY = -YZ$ by (2), gives $(ZX)Y = (YZ)X$. Then $(YZ)X = (XY)Z$ follows from symmetry. $\square$

Proof. (of Theorem 1.5.1.1) Using the preceding Lemma, one verifies the orthogonality relations for the eight elements in the statement of the Theorem. By the same Lemma, one sees by direct computation that the structural constants in the multiplication table for these eight elements depend only on $N(A)$, $N(B)$, $N(C)$. Hence the result. $\square$

1.5.2 The cases F_4 and E_6

Let $\mathfrak{A}$ be the set of 3×3 Hermitian matrices with entries octaves

$$\mathfrak{A} := \left\{ \xi = \begin{pmatrix} \lambda_1 & X & Y \\ \bar{X} & \lambda_2 & Z \\ \bar{Y} & \bar{Z} & \lambda_3 \end{pmatrix} \mid X, Y, Z \in \mathbb{O},\ \lambda_i \in \mathbb{C} \right\}$$

which under the product

$$\xi \circ \eta := \frac{1}{2}(\xi\eta + \eta\xi)$$

becomes a commutative, nonassociative unital $\mathbb{C}$-algebra in which the *Jordan identity* $\xi^2(\xi\eta) = \xi(\xi^2\eta)$ holds. $\mathfrak{A}$ is called the *exceptional Jordan algebra* or *Albert algebra* over $\mathbb{C}$. The group of algebra automorphisms of $\mathfrak{A}$ is the exceptional group of type F_4. In $\mathfrak{A}$, as for matrices over a field, every element ξ satisfies a Cayley-Hamilton cubic equation

$$p_\xi(t) = t^3 - l(\xi)t^2 + q(\xi)t - n(\xi)$$

where l (the trace) is a linear form, q a quadratic form, n a cubic form (the norm) on $\mathfrak{A}$ ([Sp-Veld], Prop. 5.1.5). Here

$$\begin{aligned} l(\xi) &= \lambda_1 + \lambda_2 + \lambda_3, \\ q(\xi) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 - X\bar{X} - Y\bar{Y} - Z\bar{Z}, \\ n(\xi) = \det(\xi) &= \lambda_1\lambda_2\lambda_3 - \lambda_1 Z\bar{Z} - \lambda_2 Y\bar{Y} - \lambda_3 X\bar{X} + (XZ)\bar{Y} + Y(\bar{Z}\bar{X})\,. \end{aligned}$$

l, q and n are invariant under F_4. The vector space automorphisms of $\mathfrak{A}$ which leave n invariant constitute the exceptional group E_6 (the simply connected form). As an easy reduction we have:

Theorem 1.5.2.1. *If generically free linear F_4-quotients are stably rational, the same holds for generically free linear E_6-quotients.*

Proof. By [Sp-Veld], Prop. 5.9.3, the group E_6 acts transitively on the set of lines in $\mathfrak{A}$ which are generated by vectors ξ with $n(\xi) \neq 0$. The stabilizer of a generic such ξ is the group F_4 by [Sp-Veld], Prop. 5.9.4. The center of E_6 is clearly $\mathbb{Z}/3\mathbb{Z}$ and hence, if an element in E_6 carries $\mathbb{C}\xi$ into itself, it must be in $F_4 \times \mathbb{Z}/3\mathbb{Z}$ (the product is clearly direct since F_4 has trivial center). In other words, $\mathbb{C}\xi$ is a $(E_6,\ F_4 \times \mathbb{Z}/3\mathbb{Z})$-section in the sense of Definition 2.2.1.1. Thus, if V is a generically free E_6-representation, then V/E_6 is stably equivalent to $(V \oplus \mathfrak{A})/E_6$ by Lemma 2.2.2.1, and $(V \oplus \mathfrak{A})$ has a $(E_6,\ F_4 \times \mathbb{Z}/3\mathbb{Z})$-section by the foregoing. Thus generically free E_6- and generically free $F_4 \times \mathbb{Z}/3\mathbb{Z}$-quotients are stably equivalent. Certainly, if W is a generically free $F_4 \times \mathbb{Z}/3\mathbb{Z}$-representation, $W/(F_4 \times \mathbb{Z}/3\mathbb{Z})$ is, again by Lemma 2.2.2.1 stably equivalent to $(W \oplus \mathbb{C})/(F_4 \times \mathbb{Z}/3\mathbb{Z})$, where the action of $F_4 \times \mathbb{Z}/3\mathbb{Z}$ on $\mathbb{C}$ is via a cube root of unity (and F_4 acts trivially). First dividing out the $\mathbb{Z}/3\mathbb{Z}$-action we see by Lemma 2.2.2.1 that this is $(W \oplus (\mathbb{C}/\mathbb{Z}/3\mathbb{Z}))/F_4 \simeq (W/F_4) \oplus \mathbb{C}$. Thus indeed generically free E_6- and F_4-quotients are stably equivalent. $\square$

The proof of the stably rationality for generically free F_4-quotients consists of a reduction to $\mathrm{Spin}_8(\mathbb{C})$ (more precisely the normalizer $N(\mathrm{Spin}_8(\mathbb{C}))$ in F_4).

In fact, from [Jac60] Theorem 8, one knows that the subgroup of F_4 leaving all the diagonal matrices in $\mathfrak{A}$ elementwise invariant is $\mathrm{Spin}_8(\mathbb{C})$ and the group leaving the subspace of diagonal matrices invariant is a semi-direct product of $\mathrm{Spin}_8(\mathbb{C})$ and the symmetric group $\mathfrak{S}_3$, the full group of outer automorphisms of $\mathrm{Spin}_8(\mathbb{C})$, permuting the diagonal entries. Thus it follows (since $\dim F_4 = 52$, $\dim \mathfrak{A} = 27$, $\dim \mathrm{Spin}_8(\mathbb{C}) = 28$) that the generic F_4-orbit in $\mathfrak{A}$ intersects the subspace of diagonal elements (reduction to normal form), and that $\mathrm{Spin}_8(\mathbb{C})$ is the stabilizer in general position of an element in $\mathfrak{A}$. Thus the subspace of $\mathrm{Spin}_8(\mathbb{C})$-invariants in $\mathfrak{A}$ is an $(F_4,\ N(\mathrm{Spin}_8(\mathbb{C})))$-section, and generically free F_4-quotients and $N(\mathrm{Spin}_8(\mathbb{C}))$-quotients are stably equivalent.

Let now $\mathrm{SO}(N)$ be the special orthogonal group of the norm form N of the split octonion algebra $\mathbb{O}$. Let the group $\mathrm{SO}(N)^3$ act in $\mathbb{O}^3$ componentwise. Inside $\mathrm{SO}(N)^3$ one has the subgroup of *related triples*

$$\{(r_1,\, r_2,\, r_3)\,|\, r_1(XY) = r_2(X)r_3(Y)\ \forall X,\, Y \in \mathbb{O}\}\,.$$

By [Sp-Veld], Proposition 3.6.3, this group is isomorphic to $\mathrm{Spin}_8(\mathbb{C}) = \mathrm{Spin}(N)$, and the three 8-dimensional representations of $\mathrm{Spin}_8(\mathbb{C})$ induced by $(r_1,\, r_2,\, r_3) \mapsto r_i$, $i = 1,\, 2,\, 3$, are irreducible and pairwise nonisomorphic ([Sp-Veld], Proposition 3.6.6), the standard representation of $\mathrm{SO}(N)$ and the two half-spin representations of $\mathrm{Spin}(N)$. This phenomenon is referred to as the principle of *triality*.
By [Sp-Veld], Proposition 3.6.4, the semidirect product $N(\mathrm{Spin}_8(\mathbb{C})) = \mathrm{Spin}_8(\mathbb{C}) \rtimes_\varphi \mathfrak{S}_3$ can be described as follows: recall for reference that for groups N, H and $\varphi : H \to \mathrm{Aut}(N)$, the semidirect product $N \rtimes_\varphi H$ is the set-theoretic product with multiplication

$$(n_1,\, h_1) \cdot (n_2,\, h_2) = (n_1 \varphi_{h_1}(n_2),\, h_1 h_2)\,.$$

For $r \in \mathrm{SO}(N)$ define $\hat{r}$ by $\hat{r}(X) := \overline{r(\bar{X})}$ (conjugation in $\mathbb{O}$). If $(r_1,\, r_2,\, r_3)$ is a related triple, so are the images

$$\begin{aligned} \tau_1 &: (r_1,\, r_2,\, r_3) \mapsto (\hat{r}_1,\, \hat{r}_3,\, \hat{r}_2), \\ \tau_2 &: (r_1,\, r_2,\, r_3) \mapsto (r_3,\, \hat{r}_2,\, r_1), \\ \tau_3 &: (r_1,\, r_2,\, r_3) \mapsto (r_2,\, r_1,\, \hat{r}_3) \end{aligned}$$

and the maps τ_1, τ_2, τ_3 generate a subgroup of the automorphisms of the group $\mathrm{Spin}_8(\mathbb{C})$ of related triples which is isomorphic to $\mathfrak{S}_3$. It gives us the homomorphism $\varphi : \mathfrak{S}_3 \to \mathrm{Aut}(\mathrm{Spin}_8(\mathbb{C}))$ and hence the group $N(\mathrm{Spin}_8(\mathbb{C}))$ in terms of the related triples picture we use. It follows that $N(\mathrm{Spin}_8(\mathbb{C}))$ acts in $\mathbb{O}^3$ where we must define

$$\begin{aligned} \tau_1(X_1,\, X_2,\, X_3) &= (\bar{X}_1,\, \bar{X}_3,\, \bar{X}_2), \\ \tau_2(X_1,\, X_2,\, X_3) &= (X_3,\, \bar{X}_2,\, X_1), \\ \tau_3(X_1,\, X_2,\, X_3) &= (X_2,\, X_1,\, \bar{X}_3) \end{aligned}$$

to get a well-defined action of the semidirect product. Then

Proposition 1.5.2.2. *The quotient* $\mathbb{O}^3/N(\mathrm{Spin}_8(\mathbb{C}))$ *is rational.*

Proof. First remark that for $(X,\, Y,\, Z) \in \mathbb{O}^3$ the functions $\langle X,\, X\rangle = N(X)$, $N(Y)$, $N(Z)$ and $\langle X,\, YZ\rangle$ are invariant under the group of related triples $\mathrm{Spin}_8(\mathbb{C})$: for the last of these this follows from

$$\langle r_1(X),\, r_2(Y)r_3(Z)\rangle = \langle r_1(X),\, r_1(YZ)\rangle = \langle X,\, YZ\rangle\,.$$

For $X,\, Y,\, Z \in \mathbb{O}$ the equations

$$\langle XY,\, Z\rangle = \langle Y,\, \bar{X}Z\rangle, \quad \langle XY,\, Z\rangle = \langle X,\, Z\bar{Y}\rangle$$

hold, see [Sp-Veld], Lemma 1.3.2. It follows that the function $\langle X,\, YZ\rangle$ is also invariant under the action of $\mathfrak{S}_3$ on $\mathbb{O}^3$ described above: indeed

$$\begin{aligned}\langle \tau_2(X),\, \tau_2(Y)\tau_2(Z)\rangle &= \langle Z,\, \bar{Y}X\rangle = \langle X,\, YZ\rangle,\\ \langle \tau_3(X),\, \tau_3(Y)\tau_3(Z)\rangle &= \langle Y,\, X\bar{Z}\rangle = \langle X,\, YZ\rangle\,.\end{aligned}$$

The claim is that the map

$$\begin{aligned} f\,:\, \mathbb{O}^3/N(\mathrm{Spin}_8(\mathbb{C})) &\dashrightarrow (\mathbb{C}^3)/\mathfrak{S}_3 \times \mathbb{C}\\ [(X,\, Y,\, Z)] &\mapsto (\{N(X),\, N(Y),\, N(Z)\},\, \langle X,\, YZ\rangle)\end{aligned}$$

is birational.
It is clear that f is dominant: one can choose $Y = Y_0$ and $Z = Z_0$ of any prescribed nonzero norms, and then we have to find X in the affine subspace $\langle X,\, Y_0Z_0\rangle = c_0 \neq 0$ with given norm $N(X) \neq 0$. But certainly we can find some X_0 with $\langle X_0,\, Y_0Z_0\rangle = c_0$ (e.g. $X_0 = c_0N(X_0)^{-1}N(Y_0)^{-1}Y_0Z_0$) and then some $\tilde{X}$ in the 7-dimensional (certainly not totally isotropic) complement $\langle Y_0Z_0\rangle^{\perp}$ with $N(\tilde{X}) \neq 0$ due to the nondegeneracy of the norm. One can then choose $X = X_0 + \lambda\tilde{X}$, some suitable $\lambda \in \mathbb{C}$.

To show that f is generically one-to-one we prove: if $(X,\, Y,\, Z)$ and $(X',\, Y',\, Z')$ are elements in $\mathbb{O}^3$ of nonzero norms with $N(X) = N(X')$, $N(Y) = N(Y')$, $N(Z) = N(Z')$ and $\langle X, YZ\rangle = \langle X',\, Y'Z'\rangle$, then the two triples are in the same $\mathrm{Spin}_8(\mathbb{C})$-orbit. It suffices to show that for each $(X,\, Y,\, Z)$ with nonzero norms one can find a related triple of rotations (r_1, r_2, r_3) carrying this into $(\lambda\cdot 1, \mu\cdot 1, Z')$ where λ is a square root of $N(X)$, μ a square root of $N(Y)$, $N(Z) = N(Z')$, and $\langle \bar{Y}X,\, Z\rangle = \langle \lambda\mu\cdot 1,\, Z'\rangle$. It is clear that we can find r_1 carrying X into $\lambda \cdot 1$, and the (somewhat surprising) fact that one can find a related triple $(r_1,\, r_2,\, r_3)$ with $r_1(X) = \lambda\cdot 1$, $r_2(Y) = \mu\cdot 1$ is then exactly [Sp-Veld], Lemma 3.4.2. Put $Z'' = r_3(Z)$. The subgroup of the group of related triples satisfying $r_1(1) = r_2(1) = 1$ is the group G_2: for $r_1(1 \cdot y) = r_2(1) r_3(y)$ implies $r_1 = r_3$ and $r_1(1 \cdot 1) = r_2(1) r_3(1)$ implies $r_3(1) = 1$ whence by $r_1(x \cdot 1) = r_2(x) r_3(1)$ one gets

$$r_1 = r_2 = r_3 \,.$$

Thus we only have to show that given $Z'' \in \mathbb{O}$ and $Z' \in \mathbb{O}$ of equal nonzero norms and with

$$\langle \lambda\mu \cdot 1,\ Z' \rangle = \langle \lambda\mu \cdot 1,\ Z'' \rangle$$

one can send Z'' into Z' by an element of G_2. But the previous relation means that Z' and Z'' have equal real parts, and the fact that $N(Z') = N(Z'')$ then gives that the imaginary parts of Z' and Z'' have equal norm. Thus the result follows from Theorem 1.5.1.1. $\square$

Lemma 1.5.2.3. *The stabilizer in general position in* $N(\mathrm{Spin}_8(\mathbb{C}))$ *of the representation* $\mathbb{O}^3$ *is isomorphic to* $\mathrm{SL}_3(\mathbb{C})$ *and as* $\mathrm{SL}_3(\mathbb{C})$*-representation* $\mathbb{O}^3$ *decomposes as*

$$\left(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^3 \oplus (\mathbb{C}^3)^\vee\right)^{\oplus 3} .$$

Proof. Note that the definition of the action of $N(\mathrm{Spin}_8(\mathbb{C}))$ on $\mathbb{O}^3$ shows that the stabilizer of a point in general position in this group coincides with the stabilizer in general position in $\mathrm{Spin}_8(\mathbb{C})$ (one cannot apply any permutation if the norms of X, Y, Z are pairwise distinct). Then the proof of Proposition 1.5.2.2 shows that the sought-for stabilizer in general position is equal to the stabilizer in general position of G_2 in $\mathbb{O}$. The determination of it, as well as the associated branching rule for $\mathbb{O}$, becomes most transparent if we use a model for $\mathbb{O}$ in terms of vector matrices (see [Sp-Veld], section 1.8): $\mathbb{O}$ can be realized as the algebra of vector matrices

$$X = \begin{pmatrix} a & v \\ w & b \end{pmatrix}, \quad a, b \in \mathbb{C},\ v \in \mathbb{C}^3, w \in \mathbb{C}^3$$

with multiplication

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + \langle v_1, w_2 \rangle & b_2 v_1 + a_1 v_2 - w_1 \times w_2 \\ a_2 w_1 + b_1 w_2 + v_1 \times v_2 & b_1 b_2 + \langle w_1, v_2 \rangle \end{pmatrix}$$

with conjugation and norm given by

$$\bar{X} = \begin{pmatrix} b & -v \\ -w & a \end{pmatrix}, \quad N(X) = ab - \langle v,\, w\rangle\,.$$

Here $\langle , \rangle$ denotes the standard scalar product on $\mathbb{C}^3$, and the vector product is defined by $\langle v_1 \times v_2,\, v_3\rangle = \det(v_1, v_2, v_3)$. An element of G_2 which preserves the diagonal matrices must leave the two copies of $\mathbb{C}^3$ stable: suppose for example that g maps

$$\begin{pmatrix} 0 & 0 \\ v^* & 0 \end{pmatrix}$$

to

$$\begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}, \; w \neq 0,$$

then the compatibility with the product would force

$$\begin{aligned} g\left(\begin{pmatrix} 0 & 0 \\ v^* & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) &= g\left(\begin{pmatrix} 0 & 0 \\ av^* & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & aw \\ aw^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & bw \\ aw^* & 0 \end{pmatrix} \end{aligned}$$

which is impossible. Thus if an element in G_2 preserves the subspace of diagonal matrices it must be of the form

$$\begin{pmatrix} a & v \\ v^* & b \end{pmatrix} \mapsto \begin{pmatrix} a & l(v) \\ m(v^*) & b \end{pmatrix}$$

$l : \mathbb{C}^3 \to \mathbb{C}^3$ and $m : \mathbb{C}^3 \to \mathbb{C}^3$ some invertible linear transformations which must satisfy $\langle l(v),\, m(v^*)\rangle = \langle v, v^*\rangle$ for all $v \in \mathbb{C}^3$, $v^* \in \mathbb{C}^3$. It follows that $m = (l^t)^{-1}$. On the other hand we also want l to be compatible with the product in $\mathbb{O}$ which means in particular

$$l(v_1) \times l(v_2) = (l^t)^{-1}(v_1 \times v_2)$$

whence $\det(l(v_1),\, l(v_2),\, l(v_3)) = \det(v_1,\, v_2,\, v_3)$ and $l \in \mathrm{SL}_3(\mathbb{C})$. This proves the lemma. $\square$

The subspace

$$\Delta := \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}\right\}$$

consisting of three copies of diagonal matrices in $\mathbb{O}^3$ (we use the description of $\mathbb{O}$ in terms of vector matrices from the proof of Lemma 1.5.2.3) is thus a $(N(\mathrm{Spin}_8(\mathbb{C})),\ N(\mathrm{SL}_3(\mathbb{C})))$-section for the action of $N(\mathrm{Spin}_8(\mathbb{C}))$-action on $\mathbb{O}^3$ whence

Corollary 1.5.2.4. *Linear generically free F_4-quotients and linear generically free $N(\mathrm{SL}_3(\mathbb{C}))$-quotients are stably equivalent.*

We have to determine $N(\mathrm{SL}_3(\mathbb{C}))$ explicitly:

Lemma 1.5.2.5. *The normalizer of the group $\mathrm{SL}_3(\mathbb{C})$ inside $N(\mathrm{Spin}_8(\mathbb{C})) = \mathrm{Spin}_8(\mathbb{C}) \rtimes_\varphi \mathfrak{S}_3$ (equivalently the stabilizer of the subspace Δ) is generated by the following three subgroups:*

(1) The subgroup $\mathfrak{S}_3$ in $N(\mathrm{Spin}_8(\mathbb{C}))$ acting on $\mathbb{O}^3$ through generators

$$\tau_2(X_1,\ X_2,\ X_3) = (X_3,\ \bar{X}_2,\ X_1),$$
$$\tau_3(X_1,\ X_2,\ X_3) = (X_2,\ X_1,\ \bar{X}_3)\,.$$

(2) The subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$ of the group $\mathrm{Spin}_8(\mathbb{C}) \subset N(\mathrm{Spin}_8(\mathbb{C}))$ of associated triples with generator ι

$$\iota\left(\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix}, \begin{pmatrix} a_3 & v_3 \\ w_3 & b_3 \end{pmatrix}\right)$$
$$:= \left(\begin{pmatrix} b_1 & w_1 \\ v_1 & a_1 \end{pmatrix}, \begin{pmatrix} b_2 & w_2 \\ v_2 & a_2 \end{pmatrix}, \begin{pmatrix} b_3 & w_3 \\ v_3 & a_3 \end{pmatrix}\right).$$

(3) A subgroup $\mathbb{C}^*$ of the group $\mathrm{Spin}_8(\mathbb{C}) \subset N(\mathrm{Spin}_8(\mathbb{C}))$ of associated triples which identifies $\lambda \in \mathbb{C}^*$ with the triple

$$(r_1,\ r_2,\ r_3) := \left(L_{X(\lambda)},\ L_{X(\lambda)}R_{X(\lambda)},\ L_{X(\lambda)^{-1}}\right)$$

where $X(\lambda)$ denotes the octonion $\mathrm{diag}(\lambda,\ \lambda^{-1})$ and L and R denote left resp. right multiplication with the respective subscript in $\mathbb{O}$.

(4) *A subgroup* $\mathbb{C}^*$ *of the* $\mathrm{Spin}_8(\mathbb{C}) \subset N(\mathrm{Spin}_8(\mathbb{C}))$ *of associated triples which identifies* $\mu \in \mathbb{C}^*$ *with the triple*

$$(r_1,\, r_2,\, r_3) := \left(R_{X(\mu)},\, R_{X(\mu)^{-1}},\, L_{X(\mu)}R_{X(\mu)}\right)$$

(5) *The group* $\mathrm{SL}_3(\mathbb{C})$ *itself.*

Proof. It is not hard to check that all the subroups given stabilize Δ and that (2), (3), (4) and (5) really define elements of $\mathrm{Spin}_8(\mathbb{C})$, i.e. they are associated triples of rotations $(r_1,\, r_2,\, r_3)$ with

$$r_1(XY) = r_2(X)r_3(Y),\ X,\ Y \in \mathbb{O}\,.$$

Let now an element in $N(\mathrm{Spin}_8(\mathbb{C}))$ be given which stabilizes Δ. After precomposing with an element in $\mathfrak{S}_3$ we may assume this element is given by an associated triple $(r_1,\, r_2,\, r_3)$ of rotations.
The element r_1 stabilizes the subspace of diagonal matrices $\mathrm{diag}(a,\, b) \in \mathbb{O}$ and its orthogonal complement, and it preserves the restriction of the norm to that subspace $\mathrm{diag}(a,\, b) \mapsto ab$. Thus the restriction of r_1 to the diagonal subspace in $\mathbb{O}$ is a composition of

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \lambda \in \mathbb{C}^*$$

and the involution

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Precomposing with elements in (2) and (3) we may thus suppose that (r_1, r_2, r_3) is an associated triple of rotations with r_1 leaving the diagonal matrices $\mathrm{diag}(a,\, b)$ in $\mathbb{O}$ pointwise fixed. We claim that the restriction of r_2 to diagonal matrices then preserves orientation. Indeed if

$$r_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}$$

then the compatibility requirement for associated triples

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = r_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} a' & 0 \\ 0 & b' \end{pmatrix}$$

gives a contradiction, and analogously for r_3. Hence we must have

$$r_2\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} \mu a & 0 \\ 0 & \mu^{-1}b \end{pmatrix}, \; r_3\left(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = \begin{pmatrix} \mu^{-1}c & 0 \\ 0 & \mu d \end{pmatrix}$$

and acting with suitable elements in (3) and (4) (corrsponding to $\lambda = \nu^{-1}$ a cube root of μ) we may assume $r_1(1) = r_2(1) = r_3(1) = 1$, whence we have an element in G_2, and since it stabilizes diagonal matrices pointwise, indeed in $\mathrm{SL}_3(\mathbb{C})$ which is the group in (5). □

Let $N := N(\mathrm{SL}_3(\mathbb{C}))$. We determine the abstract structure of this group.

Lemma 1.5.2.6. *The group N is a semidirect product*

$$(\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}) \ltimes (\mathrm{SL}_3(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*)\,.$$

The elements of $\mathfrak{S}_3$ and $\mathrm{SL}_3(\mathbb{C})$ commute, and the homomorphism $\mathfrak{S}_3 \to \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ is given in terms of the generators τ_2 and τ_3 by

$$\begin{aligned} \tau_2 \,:\, (\lambda,\, \mu) &\mapsto (\lambda^{-1}\mu,\, \mu), \\ \tau_3 \,:\, (\lambda,\, \mu) &\mapsto (\lambda,\, \lambda\mu^{-1})\,. \end{aligned}$$

The homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathrm{SL}_3(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*)$ is given by

$$\iota \,:\, (g,\, (\lambda, \mu)) \mapsto ((g^t)^{-1},\, (\lambda^{-1}, \mu^{-1}))\,.$$

Proof. A computation using the previous Lemma, and the definition of the action of $\mathfrak{S}_3$ on associated triples. □

The group N acts naturally on the space $\Delta \simeq \mathbb{C}^6$ consisting of three copies of diagonal matrices in $\mathbb{O}^3$, and according to Proposition 1.5.2.2, $\Delta/N \simeq \mathbb{O}^3/N(\mathrm{Spin}_8(\mathbb{C}))$ is rational.
We may consider $R = \mathbb{C}^3 \oplus (\mathbb{C}^3)^\vee$ as an N-representation, by letting $\mathbb{C}^* \times \mathbb{C}^*$ and $\mathfrak{S}_3$ act trivially, and $(\mathbb{Z}/2\mathbb{Z}) \ltimes \mathrm{SL}_3(\mathbb{C})$ in the natural way. The stabilizer in general position for the action of N in $\Delta \oplus R$ is $\mathrm{SL}_2(\mathbb{C})$ (since a general point in Δ is stabilized by $\mathrm{SL}_3(\mathbb{C})$, it coincides with the stabilizer in general position of the $\mathrm{SL}_3(\mathbb{C})$-representation R).
R decomposes as $\mathrm{SL}_2(\mathbb{C})$-representation as $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$. The subspace $\mathbb{C}^2 \oplus \mathbb{C}^2$ is invariant under the normalizer of $\mathrm{SL}_2(\mathbb{C})$ in N, whence we may construct (birationally) an N-vector bundle $\mathcal{E}$ over $\Delta \oplus R$ in the following way: it is a subbundle of the bundle $\Delta \oplus R \oplus R$ over $\Delta \oplus R$, whose fibre

over a point $x \in \Delta \oplus R$ with stabilizer $N_x \simeq \mathrm{SL}_2(\mathbb{C})$ is the subspace $\mathcal{E}_x \subset R$ which is isomorphic to $\mathbb{C}^2 \oplus \mathbb{C}^2$ as N_x-representation.
Thus we have a tower of N-equivariant bundles $\mathcal{E} \to \Delta \oplus R \to \Delta$, with a generically free action of N, and there is a torus action of $\mathbb{C}^* \times \mathbb{C}^*$ on the fibres of the two bundles in this tower via homotheties. The action of $\mathrm{SL}_3 \times \mathbb{C}^*$ resp. $\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^*$ is generically transitive on the fibres, so that

$$(\mathcal{E}/N)/(\mathbb{C}^* \times \mathbb{C}^*) \simeq \Delta/N$$

is rational, whence also $(\mathcal{E}/N$ is since it is a torus bundle over a rational space and thus Zariski locally trivial.

1.5.3 The case E_7

Consider the 56-dimensional $\mathbb{C}$-vector space

$$\mathfrak{Z} = \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_2$$

where $\mathfrak{A}_1 \simeq \mathfrak{A}_2 \simeq \mathfrak{A}$ is the 27-dimensional Albert algebra introduced in the previous subsection ($\mathfrak{Z}$ is sometimes called the space of 2×2-*Zorn matrices* and can be given the structure of an algebra, a so-called *Freudenthal algebra*; we will not need it). As we saw above, $\mathfrak{A}$ carries a cubic form $\xi \mapsto n(\xi)$ (the norm form) and a linear form $\xi \mapsto l(\xi)$ (the usual trace). We have a nondegenerate symmetric bilinear form on $\mathfrak{A}$ given by $(\xi, \eta) = l(\xi\eta)$ and by polarization from n we obtain a symmetric trilinear form $n(\xi,\, \eta,\, \zeta)$ on $\mathfrak{A}$ with $n(\xi) = n(\xi,\, \xi,\, \xi)$. We may then define a vector product on $\mathfrak{A}$ by

$$(\xi \times \eta,\, \zeta) := 3n(\xi,\, \eta,\, \zeta)\,.$$

Let us define, following [Brown], a nondegenerate skew-symmetric bilinear form $\{\cdot, \cdot\}$ and a quartic form Q on $\mathfrak{Z}$. Let

$$\Xi = (a,\, b,\, \xi_1,\, \xi_2), \quad H = (c,\, d,\, \eta_1,\, \eta_2)$$

be two elements of $\mathfrak{Z}$ and set

$$\{\Xi,\, H\} = ad - bc + (\xi_1,\, \eta_2) - (\eta_1,\, \xi_2),$$
$$Q(\Xi) = 8(\xi_2 \times \xi_2,\, \xi_1 \times \xi_1) - 8an(\xi_1) - 8bn(\xi_2) - 2\left[(\xi_1,\, \xi_2) - ab\right]^2\,.$$

Then the simply connected simple group of type E_7 is the group of linear automorphisms of $\mathfrak{Z}$ preserving $\{\cdot, \cdot\}$ and Q (see [Brown]).

Lemma 1.5.3.1. *A generic line $\mathbb{C}\Xi$ in $\mathfrak{Z}$ is a $E_6 \rtimes \mathbb{Z}/4\mathbb{Z}$-section of the action of E_7 on $\mathfrak{Z}$. Here $\mathbb{Z}/4\mathbb{Z}$ has a generator ϵ such that ϵ^2 generates the center $\mathbb{Z}/2\mathbb{Z}$ of E_7.*

Proof. By [Brown], §6, Theorem 3, E_7 acts transitively on the vectors Ξ in $\mathfrak{Z}$ with constant nonzero $Q(\Xi)$. The stabilizer in general position of E_7 in $\mathfrak{Z}$ is E_6: indeed, it suffices to calculate the stabilizer of

$$f = (1,\ 1,\ 0,\ 0)\,.$$

By polarization, the quartic form Q determines a symmetric four-linear form $Q(\cdot,\cdot,\cdot,\cdot)$, and for given Ξ, H, Z in $\mathfrak{Z}$ there is a unique element $T(\Xi, H, Z) \in \mathfrak{Z}$ with

$$\{T(\Xi, H, Z),\ \Theta\} = Q(\Xi, H, Z, \Theta), \quad \forall \Theta \in \mathfrak{Z}\,.$$

Then clearly every element stabilizing f also stabilizes $T(f, f, f)$ and by [Brown], §4, Theorem 1, $\mathfrak{Z}$ decomposes as

$$\mathfrak{Z} = \mathbb{C} \cdot f \oplus \mathbb{C}T(f, f, f) \oplus \mathfrak{A} \oplus \mathfrak{A}$$

and also putting $f_1 = (1/2)(f - T(f, f, f))$, $f_2 = (1/2)(f + T(f, f, f))$

$$\mathfrak{Z} = \mathbb{C} \cdot f_1 \oplus \mathbb{C}f_2 \oplus \mathfrak{A} \oplus \mathfrak{A}$$

and the skew-symmetric bilinear form and quartic form are given by the formulas above with respect to this decomposition. Thus if $g \in E_7$ stabilizes f, it stabilizes the whole subspace $\mathbb{C}^2 = \{(a, b, 0, 0) \,|\, a, b \in \mathbb{C}\} \subset \mathfrak{Z}$ pointwise. By [Brown], §5, Lemma 12, it follows that g is in E_6 acting as

$$g \cdot (a, b, \xi_1, \xi_2) = (a, b, g\xi_1, (g^t)^{-1}\xi_2)$$

whence also the decomposition

$$\mathfrak{Z} = \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{A} \oplus \mathfrak{A}^\vee$$

as an E_6-module. Suppose now $g \in E_7$ maps a point p on the line $\mathbb{C}f$ to another point on that line. Since g preserves Q it can only multiply p by a fourth root of unity. But the element

$$\epsilon\,:\,(a, b, \xi_1,\ \xi_2) \mapsto (ib,\ ia,\ (\sqrt[3]{i})^{-1}\xi_2,\ -\sqrt[3]{i}\xi_1)$$

is in E_7 and is multiplication by i on f. Hence the result, since $\epsilon^2 = -1$ generates the center. $\square$

Lemma 1.5.3.2. *Generically free linear quotients for* $E_6 \rtimes \mathbb{Z}/4\mathbb{Z}$ *are stably equivalent to generically free linear* $\mathrm{Out}(E_6) = \mathbb{Z}/2\mathbb{Z} \ltimes E_6$ *quotients.*

Proof. The central extensions

$$1 \to \mathbb{Z}/2\mathbb{Z} \simeq (\epsilon^2) \to \mathrm{Stab}_{E_7}(\mathbb{C}\Xi) \simeq \mathbb{Z}/4\mathbb{Z} \ltimes E_6 \to \mathrm{Out}(E_6) \to 1$$

and

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathrm{Out}(E_6) \to \mathrm{Out}(E_6) \to 1$$

give the same class in $H^2(\mathrm{Out}(E_6), \mathbb{C}^*)$, i.e. the corresponding $\mathbb{C}^*$-extensions are isomorphic. Since generically free linear $\mathrm{Out}(E_6)$ and $\mathrm{Out}(E_6) \times \mathbb{Z}/2\mathbb{Z}$-quotients are stably equivalent, it suffices to indicate a generically free $G_1 = \mathbb{Z}/4\mathbb{Z} \ltimes E_6$ representation V_1, a generically free $G_2 = \mathbb{Z}/2\mathbb{Z} \times \mathrm{Out}(E_6)$ representation V_2, and a $G = \mathbb{C}^* \times \mathrm{Out}(E_6)$ representation V (the group G being the common $\mathbb{C}^*$-extension induced by G_1 and G_2) such that V_1/G_1 and V_2/G_2 are both generically $\mathbb{C}^*$-bundles over V/G.
In fact we may start with some generically free E_6-representation W, and take the $\mathrm{Out}(E_6)$-representation $W \oplus W^\vee$ as space for V_1, V_2 and V. If E_6 acts in W via a matrix M, then $\mathrm{Out}(E_6) = E_6 \rtimes \mathbb{Z}/2\mathbb{Z}$ acts in $W \oplus W^\vee$ via

$$(M,\, \eta) \mapsto \begin{pmatrix} 0 & (M^t)^{-1} \\ M & 0 \end{pmatrix}$$

(η a generator of $\mathbb{Z}/2\mathbb{Z}$), and $E_6 \rtimes \mathbb{Z}/4\mathbb{Z}$ acts via

$$(M,\, \xi) \mapsto \begin{pmatrix} 0 & i(M^t)^{-1} \\ iM & 0 \end{pmatrix}$$

($\mathbb{Z}/4\mathbb{Z} = \langle \xi \rangle$), whereas $\mathrm{Out}(E_6) \times \mathbb{Z}/2\mathbb{Z}$ acts via

$$(M,\, \eta,\, \epsilon) \mapsto \begin{pmatrix} 0 & -(M^t)^{-1} \\ -M & 0 \end{pmatrix}$$

(ϵ a generator of $\mathbb{Z}/2\mathbb{Z}$). Finally $\mathrm{Out}(E_6) \times \mathbb{C}^*$ acts via

$$(M,\, \eta,\, \lambda) \mapsto \begin{pmatrix} 0 & \lambda(M^t)^{-1} \\ \lambda M & 0 \end{pmatrix}.$$

These actions have the properties required which concludes the proof. $\square$

We reduce the group $\mathrm{Out}(E_6)$ further via the following Lemma.

Lemma 1.5.3.3. *Consider the* $\mathrm{Out}(E_6)$ *representation* $\mathfrak{A} \oplus \mathfrak{A}^\vee$. *Then:*

(1) The stabilizer of a generic point $(x,\, y^*) \in \mathfrak{A} \oplus \mathfrak{A}^\vee$ *is* $\mathrm{Spin}_8(\mathbb{C})$. *The* $\mathrm{Spin}_8(\mathbb{C})$*-invariants in* $\mathfrak{A} \oplus \mathfrak{A}^\vee$ *consist of the six-dimensional space of pairs of diagonal matrices in* $\mathfrak{A}$.

(2) The normalizer N_1 *of* $\mathrm{Spin}_8(\mathbb{C})$ *in* $\mathrm{Out}(E_6)$ *is the group*

$$N_1 = (C_1 \times \mathfrak{S}_3) \ltimes ((\mathrm{Spin}_8(\mathbb{C}) \times T_1)/(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}))$$

where $C_1 \simeq \mathbb{Z}/2\mathbb{Z}$ *and* $T_1 \simeq (\mathbb{C}^*)^2$. *Here*

- C_1 *acts on* T_1 *by mapping* (λ, μ) *to* (λ^{-1}, μ^{-1})*, and the action of* C_1 *commutes with that of* $\mathrm{Spin}_8(\mathbb{C})$.
- $\mathfrak{S}_3$ *acts on* $\mathrm{Spin}_8(\mathbb{C})$ *as the group of outer automorphisms, and it acts on* T_1 *as in Lemma 1.5.2.6, i.e.*

$$\begin{aligned} \tau_2 &: (\lambda,\, \mu) \mapsto (\lambda^{-1}\mu,\, \mu), \\ \tau_3 &: (\lambda,\, \mu) \mapsto (\lambda,\, \lambda\mu^{-1})\,. \end{aligned}$$

The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *is embedded diagonally in* $\mathrm{Spin}_8(\mathbb{C}) \times T_1$*: into* $\mathrm{Spin}_8(\mathbb{C})$ *as the center, into* T_1 *as the subgroup consisting of pairs* $(\pm 1,\, \pm 1)$.

Proof. (1) : Note that the element ϵ maps $(\xi_1, \xi_2) \in \mathfrak{A} \oplus \mathfrak{A}$ (the second summand with the dual action of E_6) into $((\sqrt[3]{i})^{-1}\xi_2,\, -\sqrt[3]{i}\xi_1)$, so if the norms $n(\xi_1)$ and $n(\xi_2)$ are general, since E_6 acts by norm preserving maps, we see that the stabilizer of a point $(x,\, y^*)$ in general position in $\mathrm{Out}(E_6)$ coincides with the stabilizer in E_6.
As we have seen above ([Sp-Veld], Proposition 5.9.4), the stabilizer of a general $x \in \mathfrak{A}$ in E_6 is F_4. But for F_4 we have a nondegenerate symmetric bilinear invariant form on $\mathfrak{A}$, namely the trace form $(\cdot, \cdot)$, and $\mathfrak{A}$ and $\mathfrak{A}^\vee$ are canonically identified as F_4 representations. Therefore the stabilizer in general position in question is the stabilizer in general position in F_4 of a point $y \in \mathfrak{A}$, and we know from the previous subsection that it is $\mathrm{Spin}_8(\mathbb{C})$. We also know the decomposition of $\mathfrak{A}^2$ with respect to $\mathrm{Spin}_8(\mathbb{C})$, namely it is

$$\left(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \Sigma^+ \oplus \Sigma^- \oplus \mathbb{S}\right)^2,$$

$\Sigma^{\pm}$ the two half spin representations, $\mathbb{S}$ the standard representation of $\mathrm{SO}_8(\mathbb{C})$. Thus the subspace $\mathbb{C}^6$ of pairs of diagonal matrices in $\mathfrak{A}^2$ are the $\mathrm{Spin}_8(\mathbb{C})$-invariants.

(2): The group N_1 in $\mathrm{Out}(E_6)$ consists of the transformations carrying the previous $\mathbb{C}^6$ into itself. Note that there is a two-torus T_1 in E_6 preserving that subspace: indeed, the restriction of the norm form n to the diagonal matrices in $\mathfrak{A}$ is $\lambda_1\lambda_2\lambda_3$ for

$$n(\xi) = \det(\xi) = \lambda_1\lambda_2\lambda_3 - \lambda_1 Z\bar{Z} - \lambda_2 Y\bar{Y} - \lambda_3 X\bar{X} + (XZ)\bar{Y} + Y(\bar{Z}\bar{X}),$$

so the transformations (for $(\lambda, \mu) \in (\mathbb{C}^*)^2$)

$$\lambda_1 \mapsto \lambda^2\lambda_1,\ \lambda_2 \mapsto \lambda^{-2}\mu^2\lambda_2,\ \lambda_3 \mapsto \mu^{-2}\lambda_3,$$
$$Z \mapsto \lambda^{-1}Z,\ Y \mapsto \lambda\mu^{-1}Y,\ X \mapsto \mu X$$

are in E_6, and they map pairs of diagonal matrices in $\mathfrak{A}\oplus\mathfrak{A}^\vee$ into themselves. By [Jac60] (see the discussion after Theorem 7), the group $\mathfrak{S}_3$ of outer automorphisms of $\mathrm{Spin}_8(\mathbb{C})$ is realized inside F_4 (and thus in E_6) as follows: a permutation σ of $\{1,2,3\}$ acts on

$$\xi = \begin{pmatrix} \lambda_1 & X_{12} & X_{13} \\ \bar{X}_{12} & \lambda_2 & X_{23} \\ \bar{X}_{13} & \bar{X}_{23} & \lambda_3 \end{pmatrix}$$

in F_4 by changing λ_i to $\lambda_{\sigma(i)}$ and X_{ij} to $X_{\sigma(i)\sigma(j)}$ (note that $X_{ij} = \bar{X}_{ji}$). Note that the generators τ_2 and τ_3 above then may be viewed to correspond to the transpositions exchanging 1 and 2, and 2 and 3, respectively. Thus if $g \in N_1$, we may, after precomposing with the element ϵ effecting the outer automorphism of E_6, assume that $g \in E_6$. The restriction of g to the diagonal $\mathbb{C}^3$ in $\mathfrak{A}$ preserves the form $\lambda_1\lambda_2\lambda_3$, hence, as it is linear, it is a composite of restrictions of an element in $\mathfrak{S}_3$ and T_1. Thus we see that the groups given above generate N_1. The remaining statements on the structure of N_1 and the actions in the semidirect product are now verified by direct computation. Note that the intersection of T_1 and $\mathrm{Spin}_8(\mathbb{C})$ inside E_6 consists exactly of the center of $\mathrm{Spin}_8(\mathbb{C})$. $\square$

Maybe the rest has to be modified after this?
By taking another section, we reduce the group N_1 further.

Lemma 1.5.3.4. *The question of stable rationality for linear generically free E_7 quotients is reduced to the same question for the group*

$$N_2 = (C_1 \times C_2 \times \mathfrak{S}_3) \ltimes (\mathrm{SL}_3(\mathbb{C}) \times T_1 \times T_2)$$

where $C_2 \simeq \mathbb{Z}/2\mathbb{Z}$ and $T_2 \simeq (\mathbb{C}^*)^2$. Here the actions are as follows:

- *The actions of C_1 and $\mathrm{SL}_3(\mathbb{C})$ and T_2 commute, and C_1 acts on T_1 as in Lemma 1.5.3.3.*
- *C_2 acts on $\mathrm{SL}_3(\mathbb{C})$ via an exterior automorphism and on T_2 in the same way as C_1 acts on T_1. The actions of C_2 and T_1 commute.*
- *$\mathfrak{S}_3$ acts on T_1 and on T_2 as in Lemma 1.5.2.6. The $\mathfrak{S}_3$-action commutes with the $\mathrm{SL}_3(\mathbb{C})$-action.*

Proof. The subgroup $C_1 \ltimes T_1$ is normal in N_1, and thus we may view the triality $\mathbb{O}^3$ as in the previous subsection as a natural representation of the group N_1 induced from the quotient

$$N_1/(C_1 \ltimes T_1) \simeq \mathfrak{S}_3 \ltimes \mathrm{Spin}_8(\mathbb{C})\,.$$

But then the subspace Δ from the previous subsection gives a section for this action, and by Lemma 1.5.2.6 we obtain the result. □

We may eliminate the $\mathrm{SL}_3(\mathbb{C})$-part from N_2 in the following way.

Lemma 1.5.3.5. *The question of stable rationality for linear generically free E_7 quotients is reduced to the same question for the group*

$$H = \mathfrak{S}_3 \ltimes (\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{C}^*)^2)^2\,.$$

The actions of $\mathfrak{S}_3$ and $\mathbb{Z}/2\mathbb{Z}$ commute, and they both act on the two-dimensional torus as in the previous Lemma 1.5.3.4.

Proof. We have

$$N_2/(C_1 \ltimes T_1) = N \simeq (C_2 \times \mathfrak{S}_3) \ltimes (\mathrm{SL}_3(\mathbb{C}) \times T_2)$$

(the same N as in Lemma 1.5.2.6). We may therefore consider, as we have done above, the representation $\mathbb{C}^3 \oplus (\mathbb{C}^3)^\vee$ of the quotient $C_2 \ltimes \mathrm{SL}_3(\mathbb{C})$ of N as an N_2-representation. A generic line in $\mathbb{C}^3 \oplus (\mathbb{C}^3)^\vee$ gives a section for this action which replaces the group $\mathrm{SL}_3(\mathbb{C})$ in the formula for N_2 by a group $\mathbb{Z}/2\mathbb{Z} \times \mathrm{SL}_2(\mathbb{C})$. Considering the representation $\mathbb{C}^2 \oplus (\mathbb{C}^2)^\vee$ for the section group and a generic line in this representation, we see that we may reduce N_2 to the group

$$N_2 = (C_1 \times C_2 \times \mathfrak{S}_3) \ltimes ((\mathbb{Z}/2\mathbb{Z})^2 \times T_1 \times T_2)$$

where $(\mathbb{Z}/2\mathbb{Z})^2$ is central. Thus the Lemma follows. $\square$

The way it was constructed the group H has a natural representation on the space $\Delta \oplus \Delta$. It is almost free. The following Lemma concludes the proof for E_7.

Lemma 1.5.3.6. *The quotient $Q = \Delta/(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{C}^*)^2)$ is rational. $\mathfrak{S}_3$ acts on the space $Q \times Q$ of $(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{C}^*)^2)^2$-orbits in $\Delta \oplus \Delta$ and the $\mathfrak{S}_3$-action on Q is birational to a linear action. More precisely, there is an $\mathfrak{S}_3$-equivariant birational map*

$$Q \dashrightarrow \mathbb{C} \oplus \mathbb{C}^3$$

where $\mathfrak{S}_3$ acts trivially on $\mathbb{C}$ and by permutation of coordinates on $\mathbb{C}^3$. Hence $(\Delta \oplus \Delta)/H$ is rational.

Proof. The proof is more or less already contained in the proof of Proposition 1.5.2.2 but we give an independent argument here. Recall that a typical element of Δ can be written

$$\delta = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \right)$$

and $(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{C}^*)^2)$ acts via

$$\iota \cdot \delta = \left(\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} f & 0 \\ 0 & e \end{pmatrix} \right)$$

(where $\mathbb{Z}/2\mathbb{Z} = \langle \iota \rangle$) and $(\lambda, \mu) \in (\mathbb{C}^*)^2$ acts via

$$(\lambda, \mu) \cdot \delta =$$

$$\left(\begin{pmatrix} \lambda\mu a & 0 \\ 0 & \lambda^{-1}\mu^{-1} b \end{pmatrix}, \begin{pmatrix} \lambda^2\mu^{-1} c & 0 \\ 0 & \lambda^{-2}\mu d \end{pmatrix}, \begin{pmatrix} \lambda^{-1}\mu^2 e & 0 \\ 0 & \lambda\mu^{-2} f \end{pmatrix} \right)$$

The group $\mathfrak{S}_3$ with generators τ_2, τ_3 also acts via

$$\tau_2 \cdot \delta = \left(\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}, \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right),$$
$$\tau_3 \cdot \delta = \left(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} f & 0 \\ 0 & e \end{pmatrix} \right).$$

Interpreting Δ as the times the diagonal subspace in octonions, we know that the functions

$$f_1 = \langle X,\, X\rangle,\ f_2 = \langle Y,\, Y\rangle,\ f_3 = \langle Z,\, Z\rangle,\ F = \langle X,\, YZ\rangle$$

are invariant under $(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{C}^*)^2)$, concretely,

$$f_1 = ab,\ f_2 = cd,\ f_3 = ef,\ F = adf + bce\,,$$

so F is $\mathfrak{S}_3$-invariant, and $\mathfrak{S}_3$ permutes $f_1,\ f_2,\ f_3$. To conclude the proof it remains to notice that

$$\begin{gathered} \Delta/(\mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{C}^*)^2) \to \mathbb{C} \oplus \mathbb{C}^3, \\ [\mathrm{diag}(a,b),\ \mathrm{diag}(c,d),\ \mathrm{diag}(e,f)] \mapsto (F,\ f_1,\ f_2,\ f_3) \end{gathered}$$

is birational: one needs only check injectivity because dominance follows from dimensional reasons. If two elements δ_1 and δ_2 in Δ with the same invariants are given, one may apply torus elements to both of them to put them in the forms

$$\delta_1' = \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} e_1 & 0 \\ 0 & f_1 \end{pmatrix} \right),$$
$$\delta_2' = \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} e_2 & 0 \\ 0 & f_2 \end{pmatrix} \right),$$

but then the equality of the invariants f_3 and F for these amounts to the equality of the symmetric functions in e_i and f_i, $i = 1, 2$. Thus one can conclude by applying ι if necessary. □

1.5.4 Concluding remarks

The investigation of the stable birational type of generically free linear quotients for the exceptiona group E_8 is not readily amenable to the techniques of the previous subsections since it does not have representations of small dimension with nontrivial stabilizer in general position which would furnish an initial reduction step.

Let us mention

Proposition 1.5.4.1. *Generically free linear quotients for the groups* $\mathrm{SO}_n(\mathbb{C})$ *and* $O_n(\mathbb{C})$ *are stably rational.*

Proof. In the sum $V \oplus \cdots \oplus V$ of n standard representations we consider the subvariety X which consists of n-tuples of mutual orthogonal nonzero vectors in V. It is birational to a tower of vector bundles over V, and the $\mathrm{SO}_n(\mathbb{C})$-(resp. $\mathrm{O}_n(\mathbb{C})$)-action on X has a linear section, namely a product of generic lines in the factors V. Thus we reduce the question to the group $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ for $\mathrm{SO}_n(\mathbb{C})$ and to $\mathbb{Z}/2\mathbb{Z}^n$ for $\mathrm{O}_n(\mathbb{C})$. The assertion follows. $\square$

The groups $\mathrm{Spin}_n(\mathbb{C})$ also act on the variety X from the previous proof, and considering the section given there, one sees that the question of stable rationality of generically free $\mathrm{Spin}_n(\mathbb{C})$-quotients reduces to a group H_n which is the preimage of $(\mathbb{Z}/2\mathbb{Z})^{n-1} \subset \mathrm{SO}_n(\mathbb{C})$ in $\mathrm{Spin}_n(\mathbb{C})$. But the general question remains open for this group.
For $n \leq 6$ the group $\mathrm{Spin}_n(\mathbb{C})$ is special due to exceptional isomorphisms in the Dynkin diagrams.
In [Kord], Kordonskii proves the stable rationality for $\mathrm{Spin}_7(\mathbb{C})$ and $\mathrm{Spin}_{10}(\mathbb{C})$.

Chapter 2

Techniques for proving rationality and some recent results for moduli spaces of plane curves

2.1 Introduction

In this chapter we collect, in all brevity, the main methods for approaching the rationality problem (Problem 1.2.1.5). In each case we list some illustrative results obtained by the respective method as a guide to the literature, or develop applications in the text itself.

2.2 Methods

2.2.1 Slice method and 2-form trick

Given a linear algebraic group G and G-variety X, the study of birational properties of the quotient X/G can be reduced to a smaller variety and smaller group $H < G$ in the following way.

Definition 2.2.1.1. A $(G,\, H)$-section of X is an irreducible H-stable subvariety Y in X whose translates by elements of G are dense in X, and with the property that if $g \in G$ carries two points in Y into one another, then g is already in H.

Then Y/H is birational to X/G as can be seen by restricting rational functions on X to Y which induces an isomorphism of invariant function fields. The *slice method*, consisting in finding a $(G,\, H)$-section as above, is not so much a direct method for proving rationality as rather a preliminary or intermediate simplification step applied in the course of the study of the birational properties of a given space. As such it corresponds to the simple idea of *reduction to normal form*.
The so called *2-form trick* (see [Shep], Prop. 8) has already been mentioned above, and is contained in Theorem 1.3.2.7 in conjunction with Example 1.3.2.8. We phrase it here again in more explicit form for reference.

Theorem 2.2.1.2. *Let E be a finite dimensional representation of odd dimension of a reductive group G, and let V be a subrepresentation of $\Lambda^2(E)^\vee$. Let Z be the kernel of the action of G on $\mathbb{P}(E)$, and suppose Z acts trivially on $\mathbb{P}(V)$. Assume that the action of G/Z on $\mathbb{P}(E)$ is almost free and that there exists a G/Z-linearized line bundle $\mathcal{L}$ on $\mathbb{P}(V)\times\mathbb{P}(E)$ such that $\mathcal{L}$ cuts out $\mathcal{O}(1)$ on the fibres of the projection to $\mathbb{P}(E)$.*
Suppose that for some $v_0\in V$ the associated 2-form in $\Lambda^2(E)^\vee$, viewed as a skew-symmetric map $E\to E^\vee$, has maximal rank $\dim E-1$. Then the rational map

$$\varphi\,:\,\mathbb{P}(V)\dashrightarrow\mathbb{P}(E)\,,$$

associating to a 2-form its kernel, is well-defined, and if $\dim V>\dim E$, $\dim\varphi^{-1}(\varphi([v_0]))=\dim V-\dim E$, then φ is dominant. Hence, if $\mathbb{P}(E)/G$ is stably rational of level $\le\dim V-\dim E$, then $\mathbb{P}(V)/G$ is rational.

This method was used in [Shep] to prove the rationality of the moduli spaces $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ of plane curves of degrees $d\equiv 1 \pmod 4$.
To give an illustration of both the slice method and 2-form trick we will study the invariant function field

$$\mathbb{K}_d=\mathbb{C}\left(\mathrm{Sym}^2(\mathbb{C}^d)^\vee\otimes\mathbb{C}^3\right)^{\mathrm{SL}_d(\mathbb{C})\times\mathrm{SL}_3(\mathbb{C})\times\mathbb{C}^*}\,.$$

Recall that a *theta-characteristic* θ on a smooth plane curve C of degree d is a line bundle which is a square root of the canonical line bundle ω_C. Then the above function field is the field of rational functions on the moduli space of pairs $(C,\,\theta)$ where C is a smooth plane curve of degree d as above and θ a theta-characteristic on C with $h^0(C,\,\theta)=0$, see [Beau00]. So C is the discriminant curve of a net of quadrics in $\mathbb{P}^{d-1}$; the above invariant function field may also be interpreted as the moduli space of nets of quadrics in $\mathbb{P}^{d-1}$. At the 2008 Geometric Invariant Theory Conference in Göttingen, F. Catanese asked whether $\mathbb{K}_d$ was rational; this may in general be quite tricky to decide, in particular for d even (note that $\mathbb{K}_4$ is the function field of $\mathfrak{M}_3$, and the proof of rationality of $\mathfrak{M}_3$ presented great difficulties, see [Kat92/2], [Kat96]). We assume d odd in the sequel and show how the problem may be reduced to the question of rationality for a simpler invariant function field (this approach suggests that the problems of rationality of $\mathbb{K}_d$ for d odd and even are interrelated, and there could be an inductive procedure for proving rationality for all d). For $d = 5$ this approach was worked out in [Kat92/1]. Note that canonically $\mathbb{C}^3 \simeq \Lambda^2(\mathbb{C}^3)^\vee$, so that

$$\mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \mathbb{C}^3 \simeq \mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee \subset \Lambda^2((\mathbb{C}^d)^\vee \otimes (\mathbb{C}^3)^\vee)$$

so that by the 2-form trick we may obtain a rational map

$$\varphi : \mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee \dashrightarrow \mathbb{P}(\mathbb{C}^d \otimes \mathbb{C}^3)\,.$$

We need

Lemma 2.2.1.3. *The inclusion*

$$\iota : \mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee \subset \Lambda^2((\mathbb{C}^d)^\vee \otimes (\mathbb{C}^3)^\vee)$$

is given in terms of coordinates $x_1, \dots, x_d$ in $(\mathbb{C}^d)^\vee$ and $y_1,\ y_2,\ y_3$ in $(\mathbb{C}^3)^\vee$ by

$$\iota((x_i x_j) \otimes (y_k \wedge y_l)) = (x_i \otimes y_k) \wedge (x_j \otimes y_l) + (x_j \otimes y_k) \wedge (x_i \otimes y_l)\,.$$

The map $\varphi : \mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee \dashrightarrow \mathbb{P}(\mathbb{C}^d \otimes \mathbb{C}^3)$ is dominant for odd $d \geq 5$.

Proof. We just have to check the well-definedness and dominance of φ. If $d = 5$ then the element

$$\begin{aligned}\omega = (x_5^2 - 2x_1x_2) \otimes (y_2 \wedge y_3) + (x_1^2 + x_3^2 + x_4^2) \otimes (y_1 \wedge y_3) \\ +(2x_4x_5 - 2x_2x_3) \otimes (y_1 \wedge y_2)\end{aligned}$$

when viewed as an element of $\Lambda^2((\mathbb{C}^5)^\vee \otimes (\mathbb{C}^3)^\vee)$ or alternatively an anti-symmetric 15×15-matrix, has rank exactly 14. Thus, since $d - 5$ is even, it suffices to indicate an element in

$$\mathrm{Sym}^2(\mathbb{C}^2)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee \subset \Lambda^2((\mathbb{C}^2)^\vee \otimes (\mathbb{C}^3)^\vee)$$

whose associated 6×6 antisymmetric matrix has maximal rank: we decompose $\mathbb{C}^{d-5} \otimes \mathbb{C}^3 = (\mathbb{C}^2 \otimes \mathbb{C}^3) \oplus \cdots \oplus (\mathbb{C}^2 \otimes \mathbb{C}^3)$ ($(d-5)/2$ times) and consider an antisymmetric matrix of block diagonal form with one 15×15 block and $(d-5)/2$ blocks of size 6×6. An element of the required form is

$$\pi_j = x_j^2 \otimes (y_1 \wedge y_2) + (x_j x_{j+1}) \otimes (y_1 \wedge y_3) + x_{j+1}^2 \otimes (y_2 \wedge y_3)$$

(here j runs over the even numbers between 6 and $d-1$). Thus we have found an element in the image of ι with one-dimensional kernel for every d, namely

$$\kappa = \omega + \sum_j \pi_j \, .$$

Thus φ is well-defined for $d \geq 5$, d odd. Moreover, the kernel of $\iota(\kappa)$ is spanned exactly by the matrix in $\mathbb{C}^d \otimes \mathbb{C}^3$ which in terms of coordinates $e_1, \dots, e_d$ in $\mathbb{C}^d$ and $f_1,\ f_2,\ f_3$ in $\mathbb{C}^3$ dual to the x_i and y_j is

$$m = e_1 \otimes f_1 + e_2 \otimes f_2 + e_3 \otimes f_3$$

(note that it suffices to check this for $d = 5$, since the π_j vanish on m by construction). Since $\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$ has a dense orbit on $\mathbb{P}(\mathbb{C}^d \otimes \mathbb{C}^3)$ (the matrices of maximal rank), and m has maximal rank, this checks dominance of φ. $\square$

Thus we see that if we put

$$L := \overline{\varphi^{-1}([m])} \subset \mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \mathbb{C}^3$$

then L is a linear subspace which is a $(\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}),\ H)$-section of $\mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \mathbb{C}^3$ where H is the stabilizer of $[m] \in \mathbb{P}(\mathbb{C}^d \otimes \mathbb{C}^3)$ in $\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$. Moreover,

$$\mathbb{K}_d \simeq \mathbb{C}(L)^{H \times \mathbb{C}^*} \, .$$

We would like to describe the H-representation L more explicitly. We note that $\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$ acts on $\mathbb{C}^d \otimes \mathbb{C}^3$, viewed as $d \times 3$-matrices, as

$$(A,\ B) \cdot M = AMB^t, \quad (A,\ B) \in \mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}),\ M \in \mathrm{Mat}(d \times 3,\ \mathbb{C})$$

and consequently

$$H = \left\{ \left(\left(\left(\begin{array}{cc} \lambda^{d-3}s & 0 \\ * & \lambda^{-3}S \end{array} \right)^t \right)^{-1}, s \right) \mid S \in \mathrm{SL}_{d-3}(\mathbb{C}),\ s \in \mathrm{SL}_3(\mathbb{C}),\ \lambda \in \mathbb{C}^* \right\}.$$

We introduce some further notation. We denote by

$$P = \left\{ \left(\begin{array}{cc} s & 0 \\ * & S \end{array} \right) \mid S \in \mathrm{GL}_{d-3}(\mathbb{C}),\ s \in \mathrm{GL}_3(\mathbb{C}) \right\} \subset \mathrm{GL}_d(\mathbb{C})$$

the indicated parabolic subgroup of $\mathrm{GL}_d(\mathbb{C})$ and put

$$P' = \left\{ \left(\begin{array}{cc} s & 0 \\ * & S \end{array} \right) \mid S \in \mathrm{SL}_{d-3}(\mathbb{C}),\ s \in \mathrm{SL}_3(\mathbb{C}) \right\}.$$

P' is a subgroup of H in the natural way; we will investigate the structure of L as P'-module first and afterwards do the bookkeeping for the various torus actions. Associated to the standard representation of $\mathrm{GL}_d(\mathbb{C})$ on $\mathbb{C}^d$ and the parabolic P we have the P-invariant subspace F below and complement E:

$$F := \langle e_4, \dots, e_d \rangle \subset \mathbb{C}^d,\ E := \langle e_1,\ e_2,\ e_3 \rangle \subset \mathbb{C}^d,$$

so that we have a filtration

$$\begin{gathered} \mathrm{Sym}^3(F) \subset \mathrm{Sym}^3(F) \oplus \mathrm{Sym}^2(F) \otimes E \subset \\ \mathrm{Sym}^3(F) \oplus \mathrm{Sym}^2(F) \otimes E \oplus F \otimes \mathrm{Sym}^2(E) \subset \mathrm{Sym}^3(\mathbb{C}^d). \end{gathered}$$

Then we claim

Lemma 2.2.1.4. *There is an isomorphism of P'-modules (for $d \geq 5$ odd)*

$$L \simeq \mathrm{Sym}^3(\mathbb{C}^d)/\mathrm{Sym}^3(F).$$

Proof. We first remark that the dimensions are right: in fact

$$\begin{aligned} \dim L &= \dim \mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \mathbb{C}^3 - \dim \mathbb{P}(\mathbb{C}^d \otimes \mathbb{C}^3) \\ &= 3\binom{d+1}{2} - (3d-1) = \frac{3}{2}d^2 - \frac{3}{2}d + 1 \end{aligned}$$

whereas

$$
\begin{aligned}
\dim(\mathrm{Sym}^3(\mathbb{C}^d)/\mathrm{Sym}^3(F)) &= \dim(\mathrm{Sym}^3(\mathbb{C}^d)) - \dim(\mathrm{Sym}^3(F)) \\
&= \binom{d+2}{3} - \binom{d-1}{3} \\
&= \frac{(d+2)(d+1)d}{6} - \frac{(d-1)(d-2)(d-3)}{6} \\
&= \frac{3}{2}d^2 - \frac{3}{2}d + 1\,.
\end{aligned}
$$

We will construct a P'-isomorphism $\mathrm{Sym}^3(\mathbb{C}^d)/\mathrm{Sym}^3(F) \to L$. The representation of the stabilizer group $H \subset \mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$ in $\mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee$ induces a representation of the subgroup $P' \subset H$ which is isomorphic to

$$
P' \to \mathrm{Aut}\left(\mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^3\right)
$$

where the action of P' on $\mathrm{Sym}^2(\mathbb{C}^d)$ is obtained by restricting the usual action of $\mathrm{SL}_d(\mathbb{C})$ to the subgroup P', and the action of P' on $\mathbb{C}^3$ is given by

$$
\begin{pmatrix} s & 0 \\ * & S \end{pmatrix} \cdot v = s \cdot v\,.
$$

We identify the representation L in this picture by showing that there is a unique subspace of dimension

$$
\frac{3}{2}d^2 - \frac{3}{2}d + 1
$$

in $\mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^3$ which is invariant under the semisimple subgroup $\mathrm{SL}_{d-3}(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \subset P'$. One has

$$
\begin{aligned}
\mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^3 &\simeq \mathrm{Sym}^2(F \oplus E) \otimes E \\
&\simeq \big((\mathrm{Sym}^2(F)) \oplus (F \otimes E) \oplus (\mathrm{Sym}^2(E))\big) \otimes E \\
&\simeq (\mathrm{Sym}^2(F) \otimes E) \oplus (F \otimes \Lambda^2(E)) \oplus (F \otimes \mathrm{Sym}^2(E)) \\
&\oplus \mathrm{Sym}^3(E) \oplus \Sigma^{2,1}(E)
\end{aligned}
$$

where the last two lines give the decomposition into irreducible $\mathrm{SL}_{d-3}(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$-modules. The dimensions of these, listed in the order in which they occur in the last two lines of the previous formula, are

$$
3 \cdot \binom{d-2}{2},\ 3(d-3),\ 6(d-3),\ 10,\ 8\,.
$$

Now we always have

$$\frac{3}{2}d^2 - \frac{3}{2}d + 1 - (3(d-3) + 6(d-3) + 10 + 8) = \frac{3}{2}d^2 - \frac{21}{2}d + 10 > 0$$

as soon as $d \geq 7$. Thus then $(\mathrm{Sym}^2(F) \otimes E) \subset L$ and

$$\dim L - \dim(\mathrm{Sym}^2(F) \otimes E) = 6d - 8\,.$$

Certainly, $3(d-3) + 10 + 8 = 3d + 9 < 6d - 8$ for $d \geq 7$, so we find

$$L = (\mathrm{Sym}^2(F) \otimes E) \oplus (F \otimes \mathrm{Sym}^2(E)) \oplus \mathrm{Sym}^3(E)\,.$$

This is also true for $d = 5$: here $\dim L = 31$ and the dimensions of the previous representations are 9, 6, 12, 10, 8. Experimenting a little shows that we have to take again the 9, 12 and 10 dimensional representations to get 31. Now

$$\mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^3 \simeq (\mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^d)/(\mathrm{Sym}^2(\mathbb{C}^d) \otimes F)$$

as P'-representations. The composition of the inclusion with the projection:

$$\mathrm{Sym}^3(\mathbb{C}^d) \subset \mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^d \to (\mathrm{Sym}^2(\mathbb{C}^d) \otimes \mathbb{C}^d)/(\mathrm{Sym}^2(\mathbb{C}^d) \otimes F)$$

is P'-equivariant, and again viewing this as a map of $\mathrm{SL}_{d-3}(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$-modules or, equivalently, using the splitting $\mathbb{C}^d = F \oplus E$, we find that this map induces the desired P'-isomorphism

$$\mathrm{Sym}^3(\mathbb{C}^d)/\mathrm{Sym}^3(F) \to L\,.$$

Remark that $\mathrm{Sym}^3(E) \subset \mathrm{Sym}^3(\mathbb{C}^d)$ maps to L nontrivially, whence also the copies of $\mathrm{Sym}^2(F) \otimes E$ and $F \otimes \mathrm{Sym}^2(E)$ contained in $\mathrm{Sym}^3(\mathbb{C}^d)$ map to L nontrivially by P'-invariance. $\square$

We can now easily obtain

Proposition 2.2.1.5. *For the field*

$$\mathbb{K}_d = \mathbb{C}\left(\mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \mathbb{C}^3\right)^{\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \times \mathbb{C}^*}$$

one has the isomorphism

$$\mathbb{K}_d \simeq \mathbb{C}(\mathrm{Sym}^3(\mathbb{C}^d)/\mathrm{Sym}^3(F))^P\,.$$

Proof. We have seen $\mathbb{K}_d \simeq \mathbb{C}(L)^{H\times\mathbb{C}^*}$, $\mathbb{C}^*$ acting by homotheties. The group P is generated by the group P' and the two dimensional torus

$$T = \left\{ \begin{pmatrix} t_1\mathrm{Id}_3 & 0 \\ 0 & t_2\mathrm{Id}_{d-3} \end{pmatrix} \mid t_1,\, t_2 \in \mathbb{C}^* \right\}.$$

On the other hand, one may view P' as a subgroup of $H \times \mathbb{C}^*$ via the assignment

$$\begin{pmatrix} s & 0 \\ * & S \end{pmatrix} \mapsto \left(\left(\left(\begin{pmatrix} s & 0 \\ * & S \end{pmatrix}^t \right)^{-1}, s \right), 1 \right)$$

and there is also a two dimensional torus $\mathbb{C}^* \times \mathbb{C}^*$ embedded into $H \times \mathbb{C}^*$ via

$$(\lambda,\, \mu) \mapsto \left(\left(\left(\begin{pmatrix} \lambda^{d-3}\mathrm{Id}_3 & 0 \\ * & \lambda^{-3}\mathrm{Id}_{d-3} \end{pmatrix}^t \right)^{-1}, \mathrm{Id}_3 \right), \mu \right)$$

and $H \times \mathbb{C}^*$ is generated by P' and $\mathbb{C}^* \times \mathbb{C}^*$. We also know that there is the isomorphism

$$\mathrm{Sym}^3(\mathbb{C}^d)/\mathrm{Sym}^3(F) \to L$$

of P'-modules. Thus to prove the Proposition, it is sufficient to show that under this isomorphism T-orbits transform into $\mathbb{C}^*\times\mathbb{C}^*$-orbits. This is straightforward to check: we have seen that as $\mathrm{SL}_{d-3}(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$-representations

$$L = (\mathrm{Sym}^2(F) \otimes E) \oplus (F \otimes \mathrm{Sym}^2(E)) \oplus \mathrm{Sym}^3(E)$$

and then

$$\begin{pmatrix} t_1\mathrm{Id}_3 & 0 \\ 0 & t_2\mathrm{Id}_{d-3} \end{pmatrix} \in T$$

acts as a homothety on $\mathrm{Sym}^{2-i}(F) \otimes \mathrm{Sym}^{i+1}(E)$, $i = 0,\, 1,\, 2$, namely as multiplication by $t_2^{2-i}t_1^{i+1}$; but $(\lambda,\, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$ likewise acts via homotheties on the irreducible summands of L (as $\mathrm{SL}_{d-3}(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C})$-representation) viewed as a subspace in $\mathrm{Sym}^2(\mathbb{C}^d)^\vee \otimes \Lambda^2(\mathbb{C}^3)^\vee$, namely by multiplication by $\lambda^{Ai+B}\mu$, some A and B in $\mathbb{Z}$. This concludes the proof of the Proposition. $\square$

As we pointed out above, Proposition 2.2.1.5 does not solve the initial rationality problem for plane curves of odd degree with theta-characteristic, but we thought it useful to record this important reduction step as an illustration of the methods introduced in this section.
The decomposition

$$L = (\mathrm{Sym}^2(F) \otimes E) \oplus (F \otimes \mathrm{Sym}^2(E)) \oplus \mathrm{Sym}^3(E)\,.$$

suggests that there may be an inductive procedure to reduce rationality of $\mathbb{K}_d$ to rationality of $\mathbb{K}_{d-3}$.

2.2.2 Double bundle method

The main technical point is the so called "no-name lemma".

Lemma 2.2.2.1. *Let G be a linear algebraic group with an almost free action on a variety X. Let $\pi : \mathcal{E} \to X$ be a G-vector bundle of rank r on X. Then one has the following commutative diagram of G-varieties*

$$\begin{tikzcd} \mathcal{E} \arrow[r, dashed, "f"] \arrow[dr, "\pi"'] & X \times \mathbb{A}^r \arrow[d, "\mathrm{pr}_1"] \\ & X \end{tikzcd}$$

where G acts trivially on $\mathbb{A}^r$, pr_1 is the projection onto X, and the rational map f is birational.

If X embeds G-equivariantly in $\mathbb{P}(V)$, V a G-module, G is reductive and X contains stable points of $\mathbb{P}(V)$, then this is an immediate application of descent theory and the fact that a vector bundle in the étale topology is a vector bundle in the Zariski topology. The result appears in [Bo-Ka]. A proof without the previous technical restrictions is given in [Ch-G-R], §4.3.
The following result ([Bo-Ka], [Kat89]) is the form in which Lemma 2.2.2.1 is most often applied since it allows one to extend its scope to irreducible representations.

Theorem 2.2.2.2. *Let G be a linear algebraic group, and let U, V and W, K be (finite-dimensional) G-representations. Assume that the stabilizer in general position of G in U, V and K is equal to one and the same subgroup H in G which is also assumed to equal the ineffectiveness kernel in these*

representations (so that the action of G/H on U, V, K is almost free).
The relations $\dim U - \dim W = 1$ *and* $\dim V - \dim U > \dim K$ *are required to hold.*
Suppose moreover that there is a G-equivariant bilinear map

$$\psi : V \times U \to W$$

and a point $(x_0,\, y_0) \in V \times U$ *with* $\psi(x_0,\, y_0) = 0$ *and* $\psi(x_0,\, U) = W$, $\psi(V,\, y_0) = W$.
Then if K/G is rational, the same holds for $\mathbb{P}(V)/G$.

Proof. We abbreviate $\Gamma := G/H$ and let pr_U and pr_V be the projections of $V \times U$ to U and V. By the genericity assumption on ψ, there is a unique irreducible component X of $\psi^{-1}(0)$ passing through $(x_0,\, y_0)$, and there are non-empty open Γ-invariant sets $V_0 \subset V$ resp. $U_0 \subset U$ where Γ acts with trivial stabilizer and the fibres $X \cap \mathrm{pr}_V^{-1}(v)$ resp. $X \cap \mathrm{pr}_U^{-1}(u)$ have the expected dimensions $\dim U - \dim W = 1$ resp. $\dim V - \dim W$. Thus

$$\mathrm{pr}_V^{-1}(V_0) \cap X \to V_0, \quad \mathrm{pr}_U^{-1}(U_0) \cap X \to U_0$$

are Γ-equivariant bundles, and by Lemma 2.2.2.1 one obtains vector bundles

$$(\mathrm{pr}_V^{-1}(V_0) \cap X)/\Gamma \to V_0/\Gamma, \quad (\mathrm{pr}_U^{-1}(U_0) \cap X)/\Gamma \to U_0/\Gamma$$

of rank 1 and $\dim V - \dim W$ and there is still a homothetic $T := \mathbb{C}^* \times \mathbb{C}^*$-action on these bundles. By Theorem 1.2.1.7, the action of the torus T on the respective base spaces of these bundles has a section over which the bundles are trivial; thus we get

$$\mathbb{P}(V)/\Gamma \sim (\mathbb{P}(U)/\Gamma) \times \mathbb{P}^{\dim V - \dim W - 1} = (\mathbb{P}(U)/\Gamma) \times \mathbb{P}^{\dim V - \dim U}\,.$$

On the other hand, one may view $U \oplus K$ as a Γ-vector bundle over both U and K; hence, again by Lemma 2.2.2.1,

$$U/\Gamma \times \mathbb{P}^{\dim K} \sim K/\Gamma \times \mathbb{P}^{\dim U}\,.$$

Since U/Γ is certainly stably rationally equivalent to $\mathbb{P}(U)/\Gamma$ of level at most one, the inequality $\dim V - \dim U > \dim K$ insures that $\mathbb{P}(V)/\Gamma$ is rational as K/Γ is rational. □

In [Kat89] this is used to prove the rationality of the moduli spaces $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ of plane curves of degree $d \equiv 0 \pmod 3$ and $d \geq 210$. A clever inductive procedure is used there to reduce the genericity requirement for the occurring bilinear maps ψ to a purely numerical condition on the labels of highest weights of irreducible summands in V, U, W. This method is only applicable if d is large.
Likewise, in [Bo-Ka], the double bundle method is used to prove the rationality of $\mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2)/\mathrm{SL}_2(\mathbb{C})$, the moduli space of d points in $\mathbb{P}^1$, if d is even.

2.2.3 Method of covariants

Virtually all the methods for addressing the rationality problem (Problem 1.2.1.5) are based on introducing some fibration structure over a stably rational base in the space for which one wants to prove rationality; with the Double Bundle Method, the fibres are linear, but it turns out that fibrations with nonlinear fibres can also be useful if rationality of the generic fibre of the fibration over the function field of the base can be proven. The *Method of Covariants* (see [Shep]) accomplishes this by inner linear projection of the generic fibre from a very singular centre.

Definition 2.2.3.1. If V and W are G-modules for a linear algebraic group G, then a covariant φ of degree d from V with values in W is simply a G-equivariant polynomial map of degree d

$$\varphi \,:\, V \to W\,.$$

In other words, φ is an element of $\mathrm{Sym}^d(V^\vee) \otimes W$.

The method of covariants, phrased in a way that we find quite useful, is contained in the following theorem.

Theorem 2.2.3.2. *Let G be a connected linear algebraic group the semi-simple part of which is a direct product of groups of type* SL *or* Sp. *Let V and W be G-modules, and suppose that the action of G on W is generically free. Let Z be the ineffectivity kernel of the action of G on $\mathbb{P}(W)$, and assume that the action of $\bar{G} := G/Z$ is generically free on $\mathbb{P}(W)$, and Z acts trivially on $\mathbb{P}(V)$.*
Let

$$\varphi \,:\, V \to W$$

be a (non-zero) covariant of degree d. Suppose the following assumptions hold:

(a) $\mathbb{P}(W)/G$ *is stably rational of level* $\leq \dim \mathbb{P}(V) - \dim \mathbb{P}(W)$.

(b) If we view φ *as a map* $\varphi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$ *and denote by* B *the base* scheme *of* φ*, then there is a linear subspace* $L \subset V$ *such that* $\mathbb{P}(L)$ *is contained in* B *together with its full infinitesimal neighbourhood of order* $(d-2)$*, i.e.*

$$\mathcal{I}_B \subset \mathcal{I}_{\mathbb{P}(L)}^{d-1}.$$

Denote by π_L *the projection* $\pi_L : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/L)$ *away from* $\mathbb{P}(L)$ *to* $\mathbb{P}(V/L)$.

(c) Consider the diagram

$$\begin{tikzcd} \mathbb{P}(V) \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\pi_L"] & \mathbb{P}(W) \\ \mathbb{P}(V/L) & \end{tikzcd}$$

and assume that one can find a point $[\bar{p}] \in \mathbb{P}(V/L)$ *such that*

$$\varphi|_{\mathbb{P}(L+\mathbb{C}p)} : \mathbb{P}(L+\mathbb{C}p) \dashrightarrow \mathbb{P}(W)$$

is dominant.

Then $\mathbb{P}(V)/G$ *is rational.*

Proof. By Corollary 1.2.2.12, the projection $\mathbb{P}(W) \dashrightarrow \mathbb{P}(W)/G$ has a rational section σ. Remark that property (c) implies that the generic fibre of π_L maps dominantly to $\mathbb{P}(W)$ under φ, which means that the generic fibre of φ maps dominantly to $\mathbb{P}(V/L)$ under π_L, too. Note also that the map φ becomes linear on a fibre $\mathbb{P}(L+\mathbb{C}g)$ because of property (b) and that thus the generic fibre of φ is birationally a vector bundle via π_L over the base $\mathbb{P}(V/L)$. Thus, if we introduce the graph

$$\Gamma = \overline{\{([q],[\bar{q}],[f]) \mid \pi_L([q]) = [\bar{q}], \varphi([q]) = [f]\}} \subset \mathbb{P}(V) \times \mathbb{P}\big(V/L\big) \times \mathbb{P}(W)$$

and look at the diagram

$$
\begin{tikzcd}
\Gamma \arrow[d, "\mathrm{pr}_{23}"'] \arrow[r, dashed, leftrightarrow, "1:1", "\mathrm{pr}_1"'] & \mathbb{P}(V) \arrow[r, dashed] & \mathbb{P}(V)/\bar{G} \arrow[dd, dashed, "\bar{\varphi}"] \\
\mathbb{P}\big(V/L\big) \times \mathbb{P}(W) \arrow[d] & & \\
\mathbb{P}(W) \arrow[rr, dashed] & & \mathbb{P}(W)/\bar{G}. \arrow[ll, dashed, bend left, "\sigma"]
\end{tikzcd}
$$

we find that the projection pr_{23} is dominant and makes Γ birationally into a vector bundle over $\mathbb{P}(V/L) \times \mathbb{P}(W)$. Hence Γ is birational to a succession of vector bundles over $\mathbb{P}(W)$ or has a *ruled structure* over $\mathbb{P}(W)$. Since $\bar{G}$ acts generically freely on $\mathbb{P}(W)$, the generic fibres of φ and $\bar{\varphi}$ can be identified and we can pull back this ruled structure via σ (possibly replacing σ by a suitable translate). Hence $\mathbb{P}(V)/\bar{G}$ is birational to $\mathbb{P}(W)/\bar{G} \times \mathbb{P}^N$ with $N = \dim \mathbb{P}(V) - \dim \mathbb{P}(W)$. Thus by property (a), $\mathbb{P}(V)/G$ is rational. $\square$

In [Shep] essentially this method is used to prove the rationality of the moduli spaces of plane curves of degrees $d \equiv 1 \pmod 9$, $d \geq 19$.
It should be noted that covariants are also used in the proof of rationality for $\mathbb{P}(\mathrm{Sym}^d\mathbb{C}^2)/\mathrm{SL}_2(\mathbb{C})$, the moduli space of d points in $\mathbb{P}^1$, if d is odd and sufficiently large (see [Kat83]).

2.2.4 Zero loci of sections in G-bundles and configuration spaces of points

The technique exposed below was explained to me by P. Katsylo whom I thank for his explanations. The proof is an immediate application of Lemma 2.2.2.1.

Theorem 2.2.4.1. *Let G be a linear algebraic group and let $\mathcal{E}$ be a rank n G-vector bundle over a smooth projective G-variety X of the same dimension $\dim X = n$. Suppose that $\mathcal{E}$ is spanned by its global sections $V := H^0(X,\, \mathcal{E})$. Let $N := c_n(\mathcal{E})$ be the n-th Chern class of $\mathcal{E}$. Suppose that the rational map*

$$
\begin{aligned}
\alpha\,:\, V \dashrightarrow X^{(N)} &= \left(\prod_{i=1}^{N} X\right)/\mathfrak{S}_N \\
s &\mapsto Z(s)
\end{aligned}
$$

assigning to a general global section of $\mathcal{E}$ *its zeroes, is dominant (thus* $\dim H^0(\mathcal{E}) \geq \dim X \cdot c_n(\mathcal{E})$*). If the action of* G *on the symmetric product* $X^{(N)}$ *is almost free, then* V/G *is birational to* $(X^{(N)}/G) \times \mathbb{C}^d$ *where* $d = \dim V - N \cdot \dim X$*.*

This result can be applied in two ways: if we know stable rationality of level $\leq \dim V - N \cdot \dim X$ of $X^{(N)}/G$, the configuration space of N unordered points in X, then we can prove rationality of V/G. On the other hand, if rationality of V/G is already known, stable rationality of $X^{(N)}/G$ follows.
As an example, we consider the space

$$(\mathbb{P}^2)^{(7)}/\mathrm{SL}_3(\mathbb{C}),$$

the configuration space of 7 points in $\mathbb{P}^2$. Rationality of it is proven in the MPI preprint [Kat94].

Theorem 2.2.4.2. *The space* $(\mathbb{P}^2)^{(7)}/\mathrm{SL}_3(\mathbb{C})$ *is rational.*

Proof. If $\mathcal{T}_{\mathbb{P}^2}$ denotes the tangent bundle of $\mathbb{P}^2$, then we have $c_2(\mathcal{T}_{\mathbb{P}^2}(1)) = 7$, a general global section of $\mathcal{T}_{\mathbb{P}^2}(1)$ has as zero locus seven points in $\mathbb{P}^2$, and the map

$$H^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(1)) \dashrightarrow (\mathbb{P}^2)^{(7)}$$

is dominant. Moreover, since $\mathcal{T}_{\mathbb{P}^2}(1) \simeq \mathcal{R}^\vee(2)$, where $\mathcal{R}$ is the tautological subbundle on $\mathbb{P}^2$ (viewed as the Grassmannian of 2-dimensional subspaces in a three-dimensional vector space), we have by the theorem of Borel-Bott-Weil

$$H^0(\mathbb{P}^2, \mathcal{T}(1)) \simeq V(1,\ 2)$$

as $\mathrm{SL}_3(\mathbb{C})$-representations. Since $\dim V(1,\ 2) = 15$ the map

$$\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(1))) \dashrightarrow (\mathbb{P}^2)^{(7)}$$

is birational, and

$$(\mathbb{P}^2)^{(7)}/\mathrm{SL}_3(\mathbb{C}) \simeq \mathbb{P}(V(1,\ 2))/\mathrm{SL}_3(\mathbb{C}).$$

We prove rationality of the latter quotient by a variant of the double bundle method as follows: consider the $\mathrm{SL}_3(\mathbb{C})$-representation

$$V = V(1,\ 2) \oplus (V(0,\ 2) \oplus V(1,\ 0)) \oplus V(1,\ 0).$$

The three-dimensional torus $T = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ acts in V via

$$(t_1,\, t_2,\, t_3) \cdot (f,\, (g_1,\, g_2),\, h) = (t_1 f,\, (t_2 g_1,\, t_2 g_2),\, t_3 h)\,.$$

We define two SL_3-equivariant maps

$$\begin{aligned} \beta \,:\, V(1,\,2) \times (V(0,\,2) \oplus V(1,\,0)) &\to V(1,\,1), \\ \psi \,:\, V(1,\,2) \times V(1,\,0) &\to V(1,\,0)\,. \end{aligned}$$

Recall that $V(a,\,b)$ is the kernel of

$$\Delta = \sum_{i=1}^{3} \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} \,:\, \mathrm{Sym}^a(\mathbb{C}^3) \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee \to \mathrm{Sym}^{a-1}(\mathbb{C}^3) \otimes \mathrm{Sym}^{b-1}(\mathbb{C}^3)^\vee$$

where e_i and x_j are dual coordinates in $\mathbb{C}^3$ and $(\mathbb{C}^3)^\vee$. In addition there is an $\mathrm{SL}_3(\mathbb{C})$-equivariant map

$$\begin{aligned} \omega \,:\, V(a,\,b) \times V(c,\,d) &\to \mathrm{Sym}^{a+c+1}(\mathbb{C}^3) \otimes \mathrm{Sym}^{b+d-2}(\mathbb{C}^3)^\vee \\ \omega(r,\,s) &= \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) e_{\sigma(1)} \frac{\partial r}{\partial x_{\sigma(2)}} \frac{\partial s}{\partial x_{\sigma(3)}}\,. \end{aligned}$$

Then

$$\beta(f,\, (g_1,\, g_2)) := \Delta(\omega(f,\, g_1)) + \Delta(f g_2),\; \psi(f, g_2) := \Delta^2(f g_2^2)$$

(followed by the suitable equivariant projection if necessary). Thus β is bilinear, ψ is linear in the first and quadratic in the second argument. One sets

$$\begin{aligned} X := \{(f,\, (g_1, g_2),\, h) \,:\, \beta(f,\, (g_1, g_2)) = 0 \text{ and } h \wedge \psi(f,\, g_2) = 0\} \\ \subset V(1,\,2) \oplus (V(0,\,2) \oplus V(1,\,0)) \oplus V(1,\,0) \end{aligned}$$

which is an $\mathrm{SL}_3(\mathbb{C})$ and T-invariant subvariety (note that the condition $h \wedge \psi(f,\, g_2) = 0$ means that h and $\psi(f,\, g_2)$ are linearly dependent in $V(1,\,0) = \mathbb{C}^3$). For the special points

$$\begin{aligned} F = 3e_2 x_1 x_3 - 2e_1 x_1 x_2 + 6e_3 x_3 x_2 - 2e_2 x_2^2,\; G_1 = x_1 x_3 - x_2^2, \\ G_2 = 2e_2,\; H = \psi(F, G_2) = -32e_2 \end{aligned}$$

one checks that $(F,\ (G_1, G_2),\ H) \in X$ and that

$$\dim\ker(\beta(F,\cdot)) = 1,\ \dim\ker(\beta(\cdot,(G_1,G_2))) = 7,$$
$$\dim(\ker(\beta(\cdot,(G_1,G_2))) \cap \ker(\psi(\cdot,G_2))) = 4\,.$$

So there is a unique irreducible $\mathrm{SL}_3(\mathbb{C})$ and T-invariant component X_0 of X passing through $(F,\ (G_1, G_2),\ H)$; we consider the two fibration structures on X_0 via the projections

$$\pi_1\ :\ X_0 \to V(1,\ 2),\ \pi_2\ :\ X_0 \to (V(0,\ 2) \oplus V(1,\ 0)) \oplus V(1,\ 0)\,.$$

The fibres of π_1 are generically two-dimensional linear spaces in which the subtorus $\{1\} \times \mathbb{C}^* \times \mathbb{C}^*$ still acts via rescaling. Hence

$$\mathbb{P}(V(1,\ 2))/\mathrm{SL}_3(\mathbb{C}) \simeq X_0/(\mathrm{SL}_3(\mathbb{C}) \times T)\,.$$

On the other hand, via π_2, X_0 is generically a vector bundle over $(V(0,\ 2) \oplus V(1,\ 0)) \oplus V(1,\ 0)$ of rank 5: in fact, $\dim\ker(\beta(\cdot,(G_1,G_2))) = 7$ and $\dim(\ker(\beta(\cdot,(G_1,G_2))) \cap \ker(\psi(\cdot,G_2))) = 4$, so that the preimage of the line $\mathbb{C}H$ in $V(1,\ 0)$ under $\psi(\cdot, G_2)$ restricted to $\ker(\beta(\cdot,(G_1,G_2)))$ will be a 5-dimensional subspace of the 7-dimensional subspace $\ker(\beta(\cdot,(G_1,G_2)))$ of $V(1,\ 2)$. Thus

$$\mathbb{P}(V(1,\ 2))/\mathrm{SL}_3(\mathbb{C}) \simeq$$
$$[((V(0,\ 2) \oplus V(1,\ 0)) \oplus V(1,\ 0))/(\mathrm{SL}_3(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*)] \times \mathbb{C}^4\,.$$

But $((V(0,\ 2) \oplus V(1,\ 0)) \oplus V(1,\ 0))/(\mathrm{SL}_3(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*)$ has dimension 2. But a unirational surface is rational by Castelnuovo's solution of the Lüroth problem for surfaces. □

We conclude by remarking that Theorem 2.2.4.1 makes it obvious, in view of the theorem of Borel-Bott-Weil, that there is an intimate connection of the rationality problem 1.2.1.5 for a reductive group G and the problem of stable rationality/rationality of configuration spaces of (unordered) points in generalized flag varieties G/P. Since Chern classes of homogeneous bundles $\mathcal{E}$ on G/P arising from a representation $\varrho\ :\ P \to \mathrm{Aut}(W)$ can easily be calculated via the splitting principle in terms of the weights of ϱ, and $H^0(G/P,\ \mathcal{E})$ is also quickly determined by Borel-Bott-Weil, it should not be too difficult to test the range of applicability of Theorem 2.2.4.1, but this remains to be done.

Chapter 3

The rationality of the moduli space of curves of genus 3

3.1 Introduction

The question whether $\mathfrak{M}_3$ is a rational variety or not had been open for a long time until an affirmative answer was finally given by P. Katsylo in 1996. There is a well known transition in the behaviour of the moduli spaces $\mathfrak{M}_g$ of smooth projective complex curves of genus g from unirational for small g to general type for larger values of g; the moral reason that $\mathfrak{M}_3$ should have a good chance to be rational is that it is birational to a quotient of a projective space by a *connected* linear algebraic group. No variety of this form has been proved irrational up to now. More precisely, $\mathfrak{M}_3$ is birational to the moduli space of plane quartic curves for $\mathrm{PGL}_3\,\mathbb{C}$-equivalence. All the moduli spaces $C(d)$ of plane curves of given degree d are conjectured to be rational (see [Dolg2], p.162; in fact, there it is conjectured that all the moduli spaces of hypersurfaces of given degree d in $\mathbb{P}^n$ for the $\mathrm{PGL}_{n+1}\,\mathbb{C}$-action are rational. I do not know if this conjecture should be attributed to Dolgachev or someone else).
Katsylo's proof is long and computational, and, due to the importance of the result, it seems desirable to give a more accessible and geometric treatment of the argument.
This chapter is divided into two main sections (sections 2 and 3) which are further divided into subsections. Section 2 treats roughly the contents of Katsylo's first paper [Kat92/2] and section 3 deals with his second paper

[Kat96].

3.2 A remarkable $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-section

3.2.1 (G,H)-sections and covariants

A general, i.e. nonhyperelliptic, smooth projective curve C of genus 3 is realized as a smooth plane quartic curve via the canonical embedding, whence $\mathfrak{M}_3$ is birational to the orbit space $C(4) := \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(4))/\mathrm{SL}_3\,\mathbb{C}$. We remark that whenever one has an affine algebraic group G acting on an irreducible variety X, then, according to a result of Rosenlicht, there exists a nonempty invariant open subset $X_0 \subset X$ such that there is a geometric quotient for the action of G on X_0 (cf. [Po-Vi], thm. 4.4). In the following we denote by X/G any birational model of this quotient, i.e. any model of the field $\mathbb{C}(X)^G$ of invariant rational functions.

The number of methods to prove rationality of quotients of projective spaces by connected reductive groups is quite limited (cf. [Dolg1] for an excellent survey). The only approach which our problem is immediately amenable to seems to be the method of (G,H)-sections. (There are two other points of view I know of: The first is based on the remark that if we have a nonsingular plane quartic curve C, the double cover of $\mathbb{P}^2$ branched along C is a Del Pezzo surface of degree 2, and conversely, given a Del Pezzo surface S of degree 2, then $|-K_S|$ is a regular map which exhibits S as a double cover of $\mathbb{P}^2$ branched along a plane quartic C; this sets up a birational isomorphism between $\mathfrak{M}_3$ and $\mathfrak{DP}(2)$, the moduli space of Del Pezzo surfaces of degree 2. We can obtain such an S by blowing up 7 points in $\mathbb{P}^2$, and one can prove that $\mathfrak{DP}(2)$ is birational to the quotient of an open subset of $P_2^7 := (\mathbb{P}^2)^7/\mathrm{PGL}_3\,\mathbb{C}$, the configuration space of 7 points in $\mathbb{P}^2$ (which is visibly rational), modulo an action of the Weyl group $W(E_7)$ of the root system of type E_7 by Cremona transformations (note that $W(E_7)$ coincides with the permutation group of the (-1)-curves on S that preserves the incidence relations between them). This group is a rather large finite group, in fact, it has order $2^{10}\cdot 3^4\cdot 5\cdot 7$. This approach does not seem to have led to anything definite in the direction of proving rationality of $\mathfrak{M}_3$ by now, but see [D-O] for more information.

The second alternative, pointed out by I. Dolgachev, is to remark that $\mathfrak{M}_3$ is birational to $\mathfrak{M}_3^{\mathrm{ev}}$, the moduli space of genus 3 curves together with an even theta-characteristic; this is the content of the classical theorem due to

G. Scorza. The latter space is birational to the space of nets of quadrics in $\mathbb{P}^3$ modulo the action of $\mathrm{SL}_4\,\mathbb{C}$, i.e. $\mathrm{Grass}(3, \mathrm{Sym}^2\,(\mathbb{C}^4)^\vee)/\mathrm{SL}_4\,\mathbb{C}$. See [Dolg3], 6.4.2, for more on this. Compare also [Kat92/1], where the rationality of the related space
$\mathrm{Grass}(3, \mathrm{Sym}^2\,(\mathbb{C}^5)^\vee)/\mathrm{SL}_5\,\mathbb{C}$ is proven; this proof, however, cannot be readily adapted to our situation, the difficulty seems to come down to that 4, in contrast to 5, is even).

Definition 3.2.1.1. Let X be an irreducible variety with an action of a linear algebraic group G, $H < G$ a subgroup. An irreducible subvariety $Y \subset X$ is called a (G, H)*-section* of the action of G on X if

(1) $\overline{G \cdot Y} = X$;

(2) $H \cdot Y \subset Y$;

(3) $g \in G,\ gY \cap Y \neq \emptyset \implies g \in H$.

In this situation H is the normalizer $N_G(Y) := \{g \in G \mid gY \subset Y\}$ of Y in G. The following proposition collects some properties of (G, H)-sections.

Proposition 3.2.1.2. *(1) The field $\mathbb{C}(X)^G$ is isomorphic to the field $\mathbb{C}(Y)^H$ via restriction of functions to Y.*

(2) Let Z and X be G-varieties, $f : Z \to X$ a dominant G-morphism, Y a (G, H)-section of X, and Y' an irreducible component of $f^{-1}(Y)$ that is H-invariant and dominates Y. Then Y' is a (G, H)-section of Z.

Part (2) of the proposition suggests that, to simplify our problem of proving rationality of $C(4)$, we should look at covariants $\mathrm{Sym}^4\,(\mathbb{C}^3)^\vee \to \mathrm{Sym}^2\,(\mathbb{C}^3)^\vee$ of low degree ($\mathbb{C}^3$ is the standard representation of $\mathrm{SL}_3\,\mathbb{C}$). The highest weight theory of Cartan-Killing allows us to decompose $\mathrm{Sym}^i(\mathrm{Sym}^4\,(\mathbb{C}^3)^\vee)$, $i \in \mathbb{N}$, into irreducible subrepresentations (this is best done by a computer algebra system, e.g. `Magma`) and pick the smallest i such that $\mathrm{Sym}^2\,(\mathbb{C}^3)^\vee$ occurs as an irreducible summand. This turns out to be 5 and $\mathrm{Sym}^2\,(\mathbb{C}^3)^\vee$ occurs with multiplicity 2.
For nonnegative integers a, b we denote by $V(a, b)$ the irreducible $\mathrm{SL}_3\,\mathbb{C}$-module whose highest weight has numerical labels a, b.
Let us now describe the two resulting independent covariants

$$\alpha_1,\ \alpha_2 : V(0, 4) \to V(0, 2)$$

of order 2 and degree 5 geometrically. We follow a classical geometric method of Clebsch to pass from invariants of binary forms to contravariants of ternary forms (see [G-Y], §215). The covariants α_1, α_2 are described in Salmon's treatise [Sal], p. 261, and p. 259, cf. also [Dix], p. 280-282. We start by recalling the structure of the ring of $\mathrm{SL}_2\,\mathbb{C}$-invariants of binary quartics ([Muk], section 1.3, [Po-Vi], section 0.12).

3.2.2 Binary quartics

Let

$$f_4 = \xi_0 x_0^4 + 4\xi_1 x_0^3 x_1 + 6\xi_2 x_0^2 x_1^2 + 4\xi_3 x_0 x_1^3 + \xi_4 x_1^4 \tag{3.1}$$

be a general binary quartic form. The invariant algebra $R = \mathbb{C}[\xi_0, \dots, \xi_4]^{\mathrm{SL}_2\,\mathbb{C}}$ is freely generated by two homogeneous invariants g_2 and g_3 (where subscripts indicate degrees):

$$g_2(\xi) = \det\begin{pmatrix} \xi_0 & \xi_2 \\ \xi_2 & \xi_4 \end{pmatrix} - 4\det\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_3 \end{pmatrix}, \tag{3.2}$$

$$g_3(\xi) = \det\begin{pmatrix} \xi_0 & \xi_1 & \xi_2 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_2 & \xi_3 & \xi_4 \end{pmatrix}. \tag{3.3}$$

If we identify f_4 with its zeroes $z_1, \dots, z_4 \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and write

$$\lambda = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

for the cross-ratio, then

$$g_3 = 0 \iff \lambda = -1,\ 2,\ \text{or}\ \frac{1}{2},$$
$$g_2 = 0 \iff \lambda = -\omega \text{ or } -\omega^2 \text{ with } \omega = e^{\frac{2\pi i}{3}},$$

the first case being commonly referred to as *harmonic cross-ratio*, the second as *equi-anharmonic cross-ratio* (see [Cl], p. 171; the terminology varies a lot among different authors, however).
Clebsch's construction is as follows: Let x, y, z be coordinates in $\mathbb{P}^2$, and let u, v, w be coordinates in the dual projective plane $(\mathbb{P}^2)^\vee$. Let $\varphi =$

$ax^4 + 4bx^3y + \ldots$ be a general ternary quartic. We want to consider those lines in $\mathbb{P}^2$ such that their intersection with the associated quartic curve C_φ is a set of points whose cross-ratio is harmonic resp. equi-anharmonic. Writing a line as $ux + vy + wz = 0$ and substituting in (2) resp. (3), we see that in the equi-anharmonic case we get a quartic in $(\mathbb{P}^2)^\vee$, and in the harmonic case a sextic. More precisely this gives us two $\mathrm{SL}_3\,\mathbb{C}$-equivariant polynomial maps

$$\sigma : V(0,4) \to V(0,4)^\vee\,, \tag{3.4}$$

$$\psi : V(0,4) \to V(0,6)^\vee\,, \tag{3.5}$$

and σ is homogeneous of degree 2 in the coefficients of φ whereas ψ is homogeneous of degree 3 in the coefficients of φ (we say σ is a *contravariant* of degree 2 on $V(0,4)$ with values in $V(0,4)$, and analogously for ψ). Finally we have the *Hessian covariant* of φ:

$$\mathrm{Hess} : V(0,4) \to V(0,6) \tag{3.6}$$

which associates to φ the determinant of the matrix of second partial derivatives of φ. It is of degree 3 in the coefficients of φ.
We will now cook up α_1, α_2 from φ, σ, ψ, Hess: Let φ operate on ψ; by this we mean that if $\varphi = ax^4 + 4bx^3y + \ldots$ then we act on ψ by the differential operator

$$a\frac{\partial^4}{\partial u^4} + 4b\frac{\partial^4}{\partial u^3 \partial v} + \ldots$$

(i.e. we replace a coordinate by partial differentiation with respect to the dual coordinate). In this way we get a contravariant ρ of degree 4 on $V(0,4)$ with values in $V(0,2)$. If we operate with ρ on φ we get α_1.
We obtain α_2 if we operate with σ on Hess.
This is a geometric way to describe α_1, α_2. For every $c = [c_1 : c_2] \in \mathbb{P}^1$ we get in this way a rational map

$$f_c = c_1\alpha_1 + c_2\alpha_2 : \mathbb{P}(V(0,4)) \dashrightarrow \mathbb{P}(V(0,2))\,. \tag{3.7}$$

For the special quartics

$$\varphi = ax^4 + by^4 + cz^4 + 6fy^2z^2 + 6gz^2x^2 + 6hx^2y^2 \tag{3.8}$$

the quantities α_1 and α_2 were calculated by Salmon in [Sal], p. 257 ff. We reproduce the results here for the reader's convenience. Put

$$L := abc\,,\ P := af^2 + bg^2 + ch^2\,, \qquad (3.9)$$
$$R := fgh\,;$$

Then

$$\alpha_1 = (3L + 9P + 10R)(afx^2 + bgy^2 + chz^2) + \qquad (3.10)$$
$$(10L + 2P + 4R)(ghx^2 + hfy^2 + fgz^2)$$
$$-12(a^2f^3x^2 + b^2g^3y^2 + c^2h^3z^2)\,;$$
$$\alpha_2 = (L + 3P + 30R)(afx^2 + bgy^2 + chz^2) + \qquad (3.11)$$
$$(10L - 6P - 12R)(ghx^2 + hfy^2 + fgz^2)$$
$$-4(a^2f^3x^2 + b^2g^3y^2 + c^2h^3z^2)\,.$$

Note that the covariant conic $-\frac{1}{20}(\alpha_1 - 3\alpha_2)$ looks a little simpler.
Let us see explicitly, using (8)-(11), that f_c is dominant for every $c \in \mathbb{P}^1$; for $a = b = c = f = g = h = 1$ we get $\alpha_1 = 48(x^2+y^2+z^2)$, $\alpha_2 = 16(x^2+y^2+z^2)$, so the image of φ under f_c in this case is a nonsingular conic unless $c = [-1 : 3]$. But for $a = 1$, $b = c = 0$, $f = g = h = 1$ we obtain $\alpha_1 = 13x^2 + 6y^2 + 6z^2$, $\alpha_2 = 11x^2 - 18y^2 - 18z^2$, and for these values $-\alpha_1 + 3\alpha_2$ defines a nonsingular conic.
Let $\mathcal{L}_c$ be the linear system generated by 6 quintics which defines f_c and let B_c be its base locus; thus $U_c := \mathbb{P}(V(0,4))\backslash B$ is an $\mathrm{SL}_3\,\mathbb{C}$-invariant open set, and if $f_{c,0} := f_c|_{U_c}$, then $X_c := f_{c,0}^{-1}(\mathbb{C}h_0)$, where h_0 defines a non-singular conic, is a good candidate for an $(\mathrm{SL}_3\,\mathbb{C},\ \mathrm{SO}_3\,\mathbb{C})$-section of U_c. We choose $h_0 = xz - y^2$.

Proposition 3.2.2.1. *X_c is a smooth irreducible $\mathrm{SO}_3\,\mathbb{C}$-invariant variety, $\overline{\mathrm{SL}_3\,\mathbb{C}\cdot X} = \mathbb{P}(V(0,4))$, and the normalizer of X_c in $\mathrm{SL}_3\,\mathbb{C}$ is exactly $\mathrm{SO}_3\,\mathbb{C}$. X_c is an $(\mathrm{SL}_3\mathbb{C},\ \mathrm{SO}_3\,\mathbb{C})$-section of U_c.*

Proof. The $\mathrm{SO}_3\,\mathbb{C}$-invariance of X_c follows from its construction. We show that the differential $d(f_{c,0})_x$ is surjective for all $x \in X_c$: In fact,

$$d(f_{c,0})_x(T_xU_c) \supset d(f_{c,0})_x(\mathfrak{sl}_3(x)) = \mathfrak{sl}_3(f_{c,0}(x)) = T_{\mathbb{C}h_0}\,\mathbb{P}V(0,2)$$

Here $\mathfrak{sl}_3(x)$ denotes the tangent space to the $\mathrm{SL}_3\,\mathbb{C}$-orbit of x in U_c, i.e. if $O_x\ :\ \mathrm{SL}_3\,\mathbb{C} \to U_c$ is the map with $O_x(g) = gx$, then we get a map

$d(O_x)_e : \mathfrak{sl}_3 \to T_x U_c$, and $\mathfrak{sl}_3(x) := \{d(O_x)_e(\xi) \,|\, \xi \in \mathfrak{sl}_3\}$. Hence X_c is smooth.
Assume X_c were reducible, let X_1 and X_2 be two irreducible components. By prop. 2.1.2 (2) and the irreducibility of the group $\mathrm{SO}_3\,\mathbb{C}$, X_1 and X_2 are $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-sections of U_c, so we can find $g \in \mathrm{SL}_3\,\mathbb{C}$, $x_1 \in X_1$, $x_2 \in X_2$, such that $gx_1 = x_2$. But then, by the $\mathrm{SL}_3\,\mathbb{C}$-equivariance of $f_{c,0}$, g stabilizes $\mathbb{C}h_0$ and is thus in $\mathrm{SO}_3\,\mathbb{C}$. But, again by the irreducibility of $\mathrm{SO}_3\,\mathbb{C}$, x_2 is also a point of X_1, i.e. X_1 and X_2 meet. This contradicts the smoothness of X_c. □

The trouble is that, if $\overline{X}_c$ is the closure of X_c in $\mathbb{P}(V(0,4))$, then $\overline{X}_c$ is an irreducible component of the intersection of 5 quintics. To eventually prove rationality, however, we would like to have some equations of lower degree. This can be done for special c.

3.2.3 From quintic to cubic equations

If $\Gamma_{f_c} \subset \mathbb{P}V(0,4) \times \mathbb{P}V(0,2)$ is the graph of f_c, it is natural to look for $\mathrm{SL}_3\,\mathbb{C}$-equivariant maps

$$\vartheta : V(0,4) \times V(0,2) \to V'$$

where V' is another $\mathrm{SL}_3\,\mathbb{C}$-representation, ϑ is a homogeneous polynomial map in both factors $V(0,4)$, $V(0,2)$, *of low degree*, say d, in the first factor, *linear* in the second, and such that Γ_{f_c} is an irreducible component of $\{(x,y) \in \mathbb{P}V(0,4) \times \mathbb{P}V(0,2) \,|\, \vartheta(x,y) = 0\}$. If V' is irreducible, there is an easy way to tell if ϑ vanishes on Γ_{f_c} *for some* $c \in \mathbb{P}^1$: This will be the case if V' occurs with multiplicity one in $\mathrm{Sym}^{d+5} V(0,4)$. Here is the result.

Definition 3.2.3.1. Let $\Psi : V(0,4) \to V(2,2)$ be the up to factor unique $\mathrm{SL}_3\,\mathbb{C}$-equivariant, homogeneous of degree 3 polynomial map with the indicated source and target spaces, and let $\Phi : V(2,2) \times V(0,2) \to V(2,1)$ be the up to factor unique bilinear $\mathrm{SL}_3\,\mathbb{C}$-equivariant map. Define $\Theta : V(0,4) \times V(0,2) \to V(2,1)$ by $\Theta(x,y) := \Phi(\Psi(x), y)$.

Remark 3.2.3.2. The existence and essential uniqueness of the maps of definition 2.3.1 can be easily deduced from known (and implemented in `Magma`) decomposition laws for $\mathrm{SL}_3\,\mathbb{C}$-representations. That they are only determined up to a nonzero constant factor will never bother us, and we admit this ambiguity in notation. The explicit form of Ψ, Φ, Θ will be needed later for checking certain non-degeneracy conditions through explicit computation. They can be found in Appendix A, formulas (64), (65).

Theorem 3.2.3.3. *(1) The linear map $\Theta(f,\cdot) : V(0,2) \to V(2,1)$ has one-dimensional kernel for f in an open dense subset V_0 of $V(0,4)$, and, in particular,* $\ker\Theta(h_0^2,\cdot) = \mathbb{C}h_0$.

(2) For some $c_0 \in \mathbb{P}^1$, $\Gamma_{f_{c_0}}$ is an irreducible component of $\{\Theta(x,y) = 0\} \subset \mathbb{P}V(0,4) \times \mathbb{P}V(0,2)$.

(3) $\overline{X_{c_0}} \subset \mathbb{P}V(0,4)$ coincides with the closure $\overline{X}$ in $\mathbb{P}V(0,4)$ of the preimage X of h_0 under the morphism from $\mathbb{P}V_0 \to \mathbb{P}V(0,2)$ given by $f \mapsto \ker\Theta(f,\cdot)$, and is thus an irreducible component of the algebraic set $\{\mathbb{C}f \mid \Phi(\Psi(f), h_0) = 0\} \subset \mathbb{P}V(0,4)$ defined by 15 cubic equations.

(4) The rational map $\Psi : \mathbb{P}V(0,4) \dashrightarrow \overline{\Psi\mathbb{P}V(0,4)} \subset \mathbb{P}V(2,2)$ as well as its restriction to X are birational isomorphisms unto their images.

Proof. (1): One checks that $V(2,1)$ occurs with multiplicity one in the decomposition of $\mathrm{Sym}^8 V(0,4)$. Thus for some $c_0 \in \mathbb{P}^1$, we have $\Theta(f,(c_{0,1}\alpha_1 + c_{0,2}\alpha_2)(f)) = 0$ for all $f \in V(0,4)$. The fact that $\ker\Theta(h_0^2,\cdot) = \mathbb{C}h_0$ follows from a direct computation using the explicit form of Θ. Thus, by upper-semicontinuity, (1) follows.
(2): We have seen in (1) that $\Gamma_{f_{c_0}}$ is contained in $\{\Theta(x,y) = 0\}$. Again by (1),

$$\Gamma_{f_{c_0}} \cap ((U_{c_0} \cap \mathbb{P}V_0) \times \mathbb{P}V(0,2)) = \\ \{\Theta(x,y) = 0\} \cap ((U_{c_0} \cap \mathbb{P}V_0) \times \mathbb{P}V(0,2)) \ ,$$

and (2) follows.
(3) follows from to (2) and the definition of X_{c_0}.
(4): Since X is an $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-section of $\mathbb{P}V_0$, it suffices to prove that the $\mathrm{SL}_3\,\mathbb{C}$-equivariant rational map $\Psi : \mathbb{P}V(0,4) \dashrightarrow \overline{\Psi\mathbb{P}V(0,4)}$ (defined e.g. in the point $\mathbb{C}h_0^2$) is birational. We will do this by writing down an explicit rational inverse. To do this, remark that $V(a,b)$ sits as an $\mathrm{SL}_3\,\mathbb{C}$-invariant linear subspace inside $\mathrm{Sym}^a\mathbb{C}^3 \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee$ (it has multiplicity one in the decomposition into irreducibles), thus elements of $V(a,b)$ may be viewed as tensors $x = (x^{i_1,\dots,i_b}_{j_1,\dots,j_a}) \in T^b_a\,\mathbb{C}^3$, covariant of order b and contravariant of order

a, or of type $\binom{b}{a}$. The inverse of the determinant tensor $\det^{-1}$ is thus in $T^0_3\mathbb{C}^3$. For $f \in V(0,4)$ and $g \in V(2,2)$ one defines a bilinear $\mathrm{SL}_3\,\mathbb{C}$-equivariant map $\alpha : V(0,4) \times V(2,2) \to \mathrm{Sym}^2\mathbb{C}^3 \otimes \mathrm{Sym}^3(\mathbb{C}^3)^\vee$, $(f,g) \mapsto \alpha(f,g)$, as the contraction

$$s^{i_1\,i_2\,i_3}_{j_1\,j_2} := f^{i_1\,i_2\,i_4\,i_5} g^{i_6\,i_3}_{i_5\,j_1} \det{}^{-1}_{j_2\,i_4\,i_6}\,,$$

followed by the symmetrization map. One checks that $\mathrm{Sym}^2\mathbb{C}^3 \otimes \mathrm{Sym}^3(\mathbb{C}^3)^\vee$ decomposes as $V(2,3) \oplus V(1,2) \oplus V(0,1)$, but $\mathrm{Sym}^4 V(0,4)$ does not contain these as subrepresentations (use `Magma`), so $\alpha(f, \Psi(f)) = 0$ for all $f \in V(0,4)$. But the explicit forms of Ψ and α show that $\ker\alpha(\cdot, \Psi(h_0^2)) = \mathbb{C}h_0^2$, whence, by upper-semicontinuity, the dimension of the kernel of $\alpha(\cdot, \Psi(f))$ is one for all f in a dense open subset of $V(0,4)$, and the rational map $\Psi : \mathbb{P}V(0,4) \dashrightarrow \overline{\Psi\mathbb{P}V(0,4)} \subset \mathbb{P}V(2,2)$ has the rational inverse $\Psi(f) \mapsto \ker\alpha(\cdot, \Psi(f))$. $\square$

Remark 3.2.3.4. It would probably be illuminating to have a geometric interpretation of the covariant $\Psi : V(0,4) \to V(2,2)$ given above similar to the one for α_1, α_2 in subsection 2.2. Though there is a huge amount of classical projective geometry attached to plane quartics, I have been unable to find such a geometric description.

Clearly, Ψ vanishes on the cone of dominant vectors in $V(0,4)$, and one may check, using the explicit formula for Ψ in Appendix A (64), that Ψ also vanishes on the $\mathrm{SL}_3\,\mathbb{C}$-orbit of the degree 4 forms in two variables, x and y, say. However, this, together with the fact that Ψ is of degree 3, is not enough to characterize Ψ since the same holds also for e.g. the Hessian covariant.

Let us mention a different approach to obtaining a geometric interpretation of Ψ suggested by Igor Dolgachev whom we would like to thank for this. We start by describing the *Poncelet invariant* of pairs of conics in $\mathbb{P}^2$ (cf. [Dolg3], section 2.2).

Let Q_1 and Q_2 are conics in $\mathbb{P}^2$ defined by symmetric matrices A and B respectively, then

$$\mathcal{P}(A,\ B) := \mathrm{tr}(B \cdot \mathrm{Adj}(A))\,,$$

where $\mathrm{Adj}(A)$ denotes the adjugate matrix of cofactors of A, defines an invariant of pairs of conics whose vanishing has the following geometric meaning: if Q_1 and Q_2 are nonsingular, then $\mathcal{P}$ is zero if and only if there exists a self-polar triangle of Q_1 inscribed in Q_2; the notion of self-polarity is defined as follows. Given a nonsingular conic Q in $\mathbb{P}^2$ and a point $p \in \mathbb{P}^2$ the polar

line of p with respect to Q is the line through the the two points of Q whose tangent lines contain p, and a set of three noncollinear lines $l_1,\ l_2,\ l_3$ form a self-polar triangle with respect to Q if l_i is the polar line with respect to Q of the point of intersection of the other two lines.
Moreover, the degree 2 covariant

$$\begin{aligned} \mathcal{D}\,:\, V(2,0) &\to V(0,2)\\ M &\mapsto \mathrm{Adj}(M)\,, \end{aligned}$$

(where M is a symmetric 3 by 3 matrix) describes the passage from a conic to the dual conic. Thus we may view $\mathcal{P}(\mathcal{D}(\cdot),\cdot)$ as an element of

$$\mathrm{Sym}^2(V(2,\ 0)\oplus V(0,\ 2)) = \\ \big(\mathrm{Sym}^2V(2,0)\otimes\mathrm{Sym}^2V(0,2)\big)\oplus\big(\Lambda^2V(2,0)\otimes\Lambda^2V(0,2)\big)$$

in other words, we get a degree 2 invariant

$$\tilde{\mathcal{P}}\,:\, V(2,\ 2)\to\mathbb{C}\,,$$

which we want to call the *Poncelet invariant* here. In the following, we use symbolic notation as in [G-Y], and denote temporarily by $x=(x_1,\ x_2,\ x_3)$ and $u=(u_1,\ u_2,\ u_3)$ dual sets of variables (coordinates on $\mathbb{C}^3$ and $(\mathbb{C}^3)^\vee$). Then if symbolically

$$g=\alpha_x^2\otimes\tilde{\alpha}_u^2=\beta_x^2\otimes\tilde{\beta}_u^2\in V(2,\ 2)\subset\mathrm{Sym}^2(\mathbb{C}^3)^\vee\otimes\mathrm{Sym}^2\mathbb{C}^3$$

then

$$\tilde{\mathcal{P}}=(\alpha\,\beta\,(\tilde{\alpha}\times\tilde{\beta})$$

(where $\times$ is the vector product) and one easily checks that $\tilde{\mathcal{P}}$ is indeed nonzero. Moreover, if we write f in $V(0,4)$ symbolically as $f=\alpha_x^4=\beta_x^4=\gamma_x^4$, then the degree 3 covariant $\Psi\,:\,V(0,\ 4)\to V(2,\ 2)$ which we are interested in is given by

$$\begin{aligned} \Psi(f)=\pi_{2,\,2}\circ(\alpha\,\beta\,\gamma)^2\cdot\{&(\beta\,\gamma\,u)(\alpha\,\gamma\,u)\alpha_x\beta_x\\ &+(\gamma\,\alpha\,u)(\beta\,\alpha\,u)\gamma_x\beta_x\\ &+(\alpha\,\beta\,u)(\gamma\,\beta\,u)\alpha_x\gamma_x\} \end{aligned}$$

where $\pi_{2,\,2}\,:\,\mathrm{Sym}^2(\mathbb{C}^3)\otimes\mathrm{Sym}(\mathbb{C}^3)^\vee\to V(2,\ 2)$ is the equivariant projection; note that the previous equality includes a choice of normalization of Ψ which

we will adopt in this remark, since a priori it is only defined up to multiplication by a nonzero constant.
Now $\tilde{P} \circ \Psi$ is a degree 6 invariant of quartics! The degree 6 homogeneous piece of the invariant ring of ternary quartics is well-known ([Dix]): a vector space basis is given by the square of the degree 3 invariant of quartics

$$I_3 = (\alpha\,\beta\,\gamma)^4$$

and the catalecticant invariant I_6. We use the determinantal expression for I_6 given in [Dix]: if one writes a quartic in $V(0,\, 4)$ as

$$\begin{gathered} ax_1^4 + 4bx_1^3x_2 + 6cx_1^2x_2^2 + 4dx_1x_2^3 + ex_2^4 \\ +4fx_1^3x_3 + 12gx_1^2x_2x_3 + 12hx_1x_2^2x_3 + 4ix_2^3x_3 \\ +6jx_1^2x_3^2 + 12kx_1x_2x_3^2 + 6lx_2^2x_3^2 \\ +4mx_1x_3^3 + 4nx_2x_3^3 \\ +px_3^4 \end{gathered}$$

then

$$I_6 = \det \begin{pmatrix} a & c & j & g & f & b \\ c & e & l & i & h & d \\ j & l & p & n & m & k \\ g & i & n & l & k & h \\ f & h & m & k & j & g \\ b & d & k & h & g & c \end{pmatrix}.$$

Thus we have

$$\tilde{P} \circ \Psi = \lambda_1 I_6 + \lambda_2 I_3^2$$

for some constants λ_1 and λ_2. We computed them using `Macaulay2`, and found

$$\lambda_1 = 3,\ \lambda_2 = \frac{3}{16}$$

(where we normalize all invariants and covariants involved as described above). We do not see the geometric meaning of this particular linear combination of I_3^2 and I_6, but believe there should be one. Previously, we had hoped one might get a pure multiple of either I_3^2 or I_6 in the end.

3.2.4 From cubic to quadratic equations

We have to fix some further notation.

Definition 3.2.4.1. (1) Z is the affine cone in $V(2,2)$ over $\overline{\Psi(X)} \subset \mathbb{P}V(2,2)$.

(2) L is the linear subspace $L := \{g \in V(2,2) \,|\, \Phi(g, h_0) = 0\} \subset V(2,2)$.

(3) $\epsilon : V(0,4) \times V(0,2) \to V(2,2)$ is the unique (up to a nonzero factor) nontrivial $\mathrm{SL}_3\,\mathbb{C}$-equivariant bilinear map with the indicated source and target spaces (the explicit form is in Appendix A (66)).

(4) $\zeta : V(0,4) \times V(0,2) \to V(1,1)$ is the unique (up to factor) nontrivial $\mathrm{SL}_3\,\mathbb{C}$-equivariant map with the property that it is homogeneous of degree 2 in both factors of its domain (cf. Appendix A (67) for the explicit description). We put $\Gamma := \zeta(\cdot, h_0) : V(0,4) \to V(1,1)$.

Let us state explicitly what we are heading towards:

> The affine cone Z over the birational modification $\overline{\Psi(X)}$ of our $(\mathrm{SL}_3\,\mathbb{C}, \mathrm{SO}_3\,\mathbb{C})$-section $X \subset \mathbb{P}V_0 \subset \mathbb{P}V(0,4)$ (whose closure in $\mathbb{P}V(0,4)$ was seen to be an irreducible component of an algebraic set defined by 15 *cubic* equations) has the following wonderful properties: Z lies in L, the linear map $\epsilon(\cdot, h_0) : V(0,4) \to V(2,2)$ restricts to an $\mathrm{SO}_3\mathbb{C}$-equivariant isomorphism between $V(0,4)$ and L, and if, via this isomorphism, we transport Z into $V(0,4)$ and call this Y, then the equations for Y are given by Γ! More precisely, Y is the unique irreducible component of $\Gamma^{-1}(0)$ passing through the point h_0^2, and Γ maps $V(0,4)$ into a five-dimensional $\mathrm{SO}_3\,\mathbb{C}$-invariant subspace of $V(1,1)$!

Thus, if we have carried out this program, Y (or Z) will be proven to be an irreducible component of an algebraic set defined by 5 *quadratic equations*! This seems quite miraculous, but a satisfactory explanation why this happens probably requires an answer to the problem raised in remark 2.3.4.
We start with some preliminary observations: It is clear that $Z \subset L$ and $\mathbb{C}(\mathbb{P}V(0,4))^{\mathrm{SL}_3\,\mathbb{C}} \simeq \mathbb{C}(Z)^{\mathrm{SO}_3\,\mathbb{C} \times \mathbb{C}^*}$, $\mathbb{C}^*$ acting by homotheties. In the following, we need the decomposition into irreducibles of $\mathrm{SL}_3\,\mathbb{C}$-modules such as $V(2,2)$, $V(2,1)$, $V(1,1)$ and $V(0,4)$ as $\mathrm{SO}_3\,\mathbb{C}$-modules. The patterns according to which irreducible representations of a complex semi-simple algebraic group

decompose when restricted to a smaller semi-simple subgroup are generally known as *branching rules*. In our case the answer is

$$V(2,2) = V(2,2)_8 \oplus V(2,2)_6 \oplus V(2,2)_4 \oplus V(2,2)'_4 \oplus V(2,2)_0\,, \tag{3.12}$$

$$V(2,1) = V(2,1)_6 \oplus V(2,1)_4 \oplus V(2,1)_2\,, \tag{3.13}$$

$$V(1,1) = V(1,1)_4 \oplus V(1,1)_2\,, \tag{3.14}$$

$$V(0,4) = V(0,4)_8 \oplus V(0,4)_4 \oplus V(0,4)_0\,. \tag{3.15}$$

Here the subscripts indicate the numerical label of the highest weight of the respective $\mathrm{SO}_3\,\mathbb{C}$-submodule of the ambient $\mathrm{SL}_3\,\mathbb{C}$-module under consideration. Note also that $\mathrm{SO}_3\,\mathbb{C} \simeq \mathrm{PSL}_2\,\mathbb{C}$, so we are really back in the much classically studied theory of *binary forms*. It is not difficult (and fun) to check (12), (13), (14), (15) by hand; let us briefly digress on how this can be done (cf. [Fu-Ha]):
We fix the following notation. Let first $n = 2l+1$ be an odd integer, $\mathfrak{g} = \mathfrak{sl}_n\,\mathbb{C}$ the Lie algebra of $\mathrm{SL}_n\,\mathbb{C}$, and let $\mathfrak{t}_\mathfrak{g}$ its standard torus of diagonal matrices of trace 0, and define the standard weights $\epsilon_i \in \mathfrak{t}_\mathfrak{g}^\vee$, $i = 1,\dots,n$, by $\epsilon_i(\mathrm{diag}(x_1,\dots,x_n)) := x_i$. Inside $\mathfrak{g}$ we find $\mathfrak{h} := \mathfrak{so}_n\,\mathbb{C}$ defined by

$$\mathfrak{h} := \left\{ \begin{pmatrix} X & Y & U \\ Z & -X^t & V \\ -V^t & -U^t & 0 \end{pmatrix} \,\middle|\, X,\ Y,\ Z \in \mathfrak{gl}_l\,\mathbb{C},\ Y^t = -Y^t, \right.$$
$$\left. Z = -Z^t,\ U, V \in \mathbb{C}^l \right\}\,.$$

Then $\mathfrak{t}_\mathfrak{h} := \{\mathrm{diag}(x_1,\dots,x_l,-x_1,\dots,-x_l,0) \,|\, x_i \in \mathbb{C}\}$; by abuse of notation we denote the restrictions of the functions ϵ_i to $\mathfrak{t}_\mathfrak{h}$ by the same letters. The fundamental weights of $\mathfrak{g}$ are $\pi_i := \epsilon_1+\dots+\epsilon_i$, $i = 1,\dots,n-1$, the fundamental weights of $\mathfrak{h}$ are $\omega_i := \epsilon_1+\dots+\epsilon_i$, $(1 \le i \le l-1)$ and $\omega_l := (\epsilon_1+\dots+\epsilon_l)/2$. Let $\Lambda_\mathfrak{g}$ and $\Lambda_\mathfrak{h}$ be the corresponding weight lattices. $\Lambda_\mathfrak{g}^+$ and $\Lambda_\mathfrak{h}^+$ are the dominant weights. For $\mathfrak{g}$ (and similarly for $\mathfrak{h}$) an irreducible representation $V(\lambda)$ for $\lambda \in \Lambda_\mathfrak{g}^+$ comes with its *formal character*

$$\mathrm{ch}_\lambda := \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) e(\mu) \in \mathbb{Z}[\Lambda_\mathfrak{g}]\,,$$

an element of the group algebra $\mathbb{Z}[\Lambda_\mathfrak{g}]$ generated by the symbols $e(\lambda)$ for $\lambda \in \Lambda_\mathfrak{g}$, where $\Pi(\lambda)$ means the weights of $V(\lambda)$, and $m_\lambda(\mu)$ is the dimension of the weight space corresponding to μ in $V(\lambda)$. We have a formal character ch_V

for any finite-dimensional $\mathfrak{g}$-module $V = V(\lambda_1)\oplus\cdots\oplus V(\lambda_t)$, $\lambda_1,\dots,\lambda_t \in \Lambda_{\mathfrak{g}}^+$ defined by

$$\mathrm{ch}_V := \sum_{i=1}^{t} \mathrm{ch}_{\lambda_i}\,.$$

The important point is that V (i.e. its irreducible constituents) can be recovered from the formal character ch_V, meaning that in $\mathbb{Z}[\Lambda_{\mathfrak{g}}]$ we can write ch_V uniquely as a $\mathbb{Z}$-linear combination of characters corresponding to dominant weights $\lambda \in \Lambda_{\mathfrak{g}}^+$.

We go back to the case $l = 1$, $n = 3$. We have $\mathfrak{h} = \mathfrak{so}_3\,\mathbb{C} = \mathfrak{sl}_2\,\mathbb{C}$. The character $\mathrm{ch}_{V(a)}$ of the irreducible $\mathfrak{so}_3\,\mathbb{C}$-module $V(a) := V(a\omega_1)$ is not hard: The weights of $V(a)$ are

$$-a\omega_1,\ (-a+2)\omega_1,\ \dots,\ (a-2)\omega_1,\ a\omega_1$$

(all multiplicities are 1). It remains to understand the weights and their multiplicities in the irreducible $\mathfrak{g} = \mathfrak{sl}_3\,\mathbb{C}$-module $V(a,b) := V(a\pi_1 + b\pi_2)$. In fact noting that π_1 restricted to the diagonal torus of $\mathfrak{so}_3\,\mathbb{C}$ above is $2\omega_1$, and the restriction of π_2 is 0, we see that, once we know the formal character of $V(a,b)$ as $\mathfrak{sl}_3\,\mathbb{C}$-module, we simply substitute $2\omega_1$ for π_1 and 0 for π_2 in the result and obtain in this way the formal character of the $\mathfrak{so}_3\,\mathbb{C}$-*module* $V(a,b)$, and hence its decomposition into irreducible constituents as $\mathfrak{so}_3\,\mathbb{C}$-module.

Let us assume $a \geq b$ (otherwise pass to the dual representation); we describe the weights and their multiplicities of the $\mathfrak{sl}_3\,\mathbb{C}$-module $V(a,b)$ following [Fu-Ha], p. 175ff.: Imagine a plane with a chosen origin from which we draw two vectors of unit length, representing π_1 and π_2, such that the angle measured counterclockwise from π_1 to π_2 is 60°. Thus the points of the lattice spanned by π_1, π_2 are the vertices of a set of equilateral congruent triangles which gives a tiling of the plane.

The weights of $V(a,b)$ are the lattice points which lie on the edges of a sequence of b (not necessarily regular) hexagons H_i with vertices at lattice points, $i = 0,\dots,b-1$, and a sequence of $[(a-b)/3]+1$ triangles T_j, $j = 0,\dots,[(a-b)/3]$. The H_i and T_j are concentric around the origin, and H_i has one vertex at $(a-i)\pi_1 + (b-i)\pi_2$, T_j has one vertex at the point $(a-b-3j)\pi_1$, and H_i and T_j are otherwise determined by the condition that the lines through π_1, π_2, $\pi_2-\pi_1$ are axes of symmetry for them, i.e. they are

preserved by the reflections in these lines (one should make a picture now). The multiplicities of the weights obtained in this way are as follows: Weights lying on H_i have multiplicity $i+1$, and weights lying on one of the T_j have multiplicity b. This completely determines the formal character of $V(a,b)$.
Let us look at $V(2,2)$ for example. Here we get three concentric regular hexagons (one of them is degenerate and consists of the origin alone). The weights are thus:

$$2\pi_1 + 2\pi_2,\ 3\pi_2,\ -2\pi_1 + 4\pi_2,\ -3\pi_1 + 3\pi_2,\ -4\pi_1 + 2\pi_2,\ -3\pi_1,$$
$$-2\pi_1 - 2\pi_2,\ -3\pi_2,\ 2\pi_1 - 4\pi_2,\ 3\pi_1 - 3\pi_2,\ 4\pi_1 - 2\pi_2,\ 3\pi_1$$

(these are the ones on the outer hexagon, read counterclockwise, and have multiplicity one),

$$\pi_1 + \pi_2,\ -\pi_1 + 2\pi_2,\ -2\pi_1 + \pi_2,\ -\pi_1 - \pi_2,\ \pi_1 - 2\pi_2,\ 2\pi_1 - \pi_2$$

(these lie on the middle hexagon and have multiplicity two), and finally there is 0 with multiplicity 3 corresponding to the origin. Consequently, the formal character of $V(2,2)$ as a representation of $\mathfrak{so}_3\,\mathbb{C}$ is

$$e(-8\omega_1) + 2e(-6\omega_1) + 4e(-4\omega_1) + 4e(-2\omega_1) + 5e(0\omega_1)\,,$$
$$+4e(2\omega_1) + 4e(4\omega_1) + 2e(6\omega_1) + e(8\omega_1)$$

which is equal to $\mathrm{ch}_{V(8)} + \mathrm{ch}_{V(6)} + 2\mathrm{ch}_{V(4)} + \mathrm{ch}_{V(0)}$. This proves (12), and (13), (14) and (15) are similar.
We resume the discussion of the main content of subsection 2.4. Before stating the main theorem, we collect some preliminary facts in the following lemma.

Lemma 3.2.4.2. *(1) The following deccomposition of $L \subset V(2,2)$ as $\mathrm{SO}_3\,\mathbb{C}$-subspace of $V(2,2)$ holds (possibly after interchanging the roles of $V(2,2)_4$ and $V(2,2)'_4$):*

$$L = V(2,2)_8 \oplus V(2,2)_4 \oplus V(2,2)_0\,.$$

(2) The map $\epsilon(\cdot, h_0) : V(0,4) \to V(2,2)$ is an SO_3-equivariant isomorphism onto L.

(3) Putting $Y := \epsilon(\cdot, h_0)^{-1}(Z) \subset V(0,4)$, we have $h_0^2 \in Y$.

(4) One has $\Gamma(V(0,4)) \subset V(1,1)_4 \subset V(1,1)$*, and the inclusion* $Y \subset \Gamma^{-1}(0)$ *holds.*

Proof. (1): Using the explicit form of Φ one calculates that the dimension of the image of the $\mathrm{SO}_3\,\mathbb{C}$-equivariant map $\Phi(\cdot, h_0) : V(2,2) \to V(2,1)$ is 12. Thus, in view of the decomposition (13) of $V(2,1)$ as $\mathrm{SO}_3\,\mathbb{C}$-representation, we must have $\Phi(V(2,2), h_0) = V(2,1)_6 \oplus V(2,1)_4$. Since

$$\dim V(a,b) = \frac{1}{2}(a+1)(b+1)(a+b+2)\,, \tag{3.16}$$

the dimension of $V(2,2)$ is 27 and the kernel L of $\Phi(\cdot, h_0)$ has dimension 15; in fact, $V(2,2)_8$, $V(2,2)_0$ and (after possibly exchanging $V(2,2)_4$ and $V(2,2)'_4$) $V(2,2)_4$ must all be in the kernel, since these representations do not appear in the decomposition of the image.
(2): Using the explicit form of ϵ given in Appendix A (66), one calculates that the dimension of the image of $\epsilon(\cdot, h_0)$ is 15 whence this linear map is injective. Moreover, its image is contained in L, hence equals L, because the map $V(0,4) \times V(0,2) \to V(2,1)$ given by $(f,g) \mapsto \Phi(\epsilon(f,g), g)$ is identically zero since there is no $V(2,1)$ in the decomposition of $V(0,4) \otimes \mathrm{Sym}^2 V(0,2)$.
(3): As we saw in theorem 2.3.3 (1), $\mathbb{C}h_0^2 \in X$, and we have $0 \neq \Psi(h_0^2) \in Z$. From the decomposition (12), we get, $\Psi(h_0^2)$ being invariant, $\langle \Psi(h_0^2) \rangle_{\mathbb{C}} = L^{\mathrm{SO}_3\,\mathbb{C}}$. By the decomposition (15), we get that the preimage under $\epsilon(\cdot, h_0)$ of $\Psi(h_0^2)$ spans the $\mathrm{SO}_3\,\mathbb{C}$-invariants $V(0,4)_0$ which are thus in Y. So in particular, $h_0^2 \in Y$.
(4): The first part is straightforward: Just decompose $\mathrm{Sym}^2 V(0,4)$ as $\mathrm{SO}_3\,\mathbb{C}$-module by the methods explained above, and check that it does not contain any $\mathrm{SO}_3\,\mathbb{C}$-submodule the highest weight of which has numerical label 2 (this suffices by (14)). The second statement of (4) follows from the observation that the map $\zeta : V(0,4) \times V(0,2) \to V(1,1)$ (Def. 2.4.1 (4)) factors:

$$c \cdot \zeta = \tilde{\gamma} \circ \epsilon\,,\ c \in \mathbb{C}^*\,,$$

where $\tilde{\gamma} : V(2,2) \to V(1,1)$ is the unique (up to nonzero scalar) nontrivial $\mathrm{SL}_3\,\mathbb{C}$-equivariant map which is homogeneous of degree 2. This is because $V(1,1)$ occurs in the decomposition of $\mathrm{Sym}^2 V(0,4) \otimes \mathrm{Sym}^2 V(0,2)$ with multiplicity one, and $\tilde{\gamma} \circ \epsilon$ is not identically zero, as follows from the explicit form of these maps (cf. Appendix A, (66), (68)). Thus, defining $\tilde{\Gamma} : V(0,4) \to V(1,1)$ by $\tilde{\Gamma}(\cdot) := (\tilde{\gamma} \circ \epsilon)(\cdot, h_0)$ (which thus differs from Γ just by a nonzero scalar), we must show $\tilde{\Gamma}(Y) = 0$. But recalling the definitions of Y, $\tilde{\Gamma}$ and Z (Def. 2.4.1 (1)), it suffices to show that $\tilde{\gamma} \circ \Psi$ is identically zero; the latter is true since it is an $\mathrm{SL}_3\,\mathbb{C}$-equivariant map from $V(0,4)$ to $V(1,1)$, homogeneous of degree 6, but $\mathrm{Sym}^6 V(0,4)$ does not contain $V(1,1)$. This proves (4). □

Let us now pass from $\mathrm{SO}_3\,\mathbb{C}$ to the $\mathrm{PSL}_2\,\mathbb{C}$-picture and denote by $V(d)$ the space of binary forms of degree d in the variables z_1, z_2. This is of course consistent with our previous notation since, under the isomorphism $\mathfrak{so}_3\,\mathbb{C} \simeq \mathfrak{sl}_2\,\mathbb{C}$, $V(d)$ is just the irreducible $\mathfrak{so}_3\,\mathbb{C}$-module the highest weight of which has numerical label d; since we consider $\mathrm{PSL}_2\,\mathbb{C}$-representations, d is always even.
We will fix a covering $\mathrm{SL}_2\,\mathbb{C} \to \mathrm{SO}_3\,\mathbb{C}$ and thus an isomorphism $\mathrm{PSL}_2\,\mathbb{C} \simeq \mathrm{SO}_3\,\mathbb{C}$, and we will fix isomorphisms $\delta_1 : V(0) \oplus V(4) \oplus V(8) \to V(0,4)$ and $\delta_2 : V(4) \to V(1,1)_4$ such that $(1,0,0)$ maps to h_0^2 under δ_1 and both δ_1 and δ_2 are equivariant with respect to the isomorphism $\mathrm{PSL}_2\,\mathbb{C} \simeq \mathrm{SO}_3\,\mathbb{C}$; we will discuss in a moment how this is done, but for now this is not important. Look at the diagram

$$\begin{tikzcd}
h_0^2 & & (1,0,0) \arrow[ll, mapsto, "\delta_1"'] & U := \delta_1^{-1}(Y) \\
& & & \cap \\
Y \subset \Gamma^{-1}(0) \subset V(0,4) \arrow[d, "\Gamma|_{\Gamma^{-1}(0)}"', "\Gamma"] & & & V(0) \oplus V(4) \oplus V(8) \arrow[lll, "\simeq", "\delta_1"'] \arrow[d, "\delta := \delta_2^{-1} \circ \Gamma \circ \delta_1"] \\
0 \in V(1,1)_4 & & & V(4) \arrow[lll, "\simeq", "\delta_2"'] \\
\cap & & & \\
V(1,1) \simeq V(1,1)_4 \oplus V(1,1)_2 & & &
\end{tikzcd}$$

By part (4) of lemma 2.4.2, we have $\delta^{-1}(0) \supset U$, and by part (3) of the same lemma, $(1,0,0) \in U$. Moreover, recalling our construction of X in theorem 2.3.3, we see that $\dim X = \dim \mathbb{P}V(0,4) - \dim \mathbb{P}V(0,2) = 14 - 5 = 9$, whence, chasing through the definitions of Z, Y, U, we get $\dim U = 10$. But the explicit form of δ (we will see this in a moment) allows us to conclude, by explicit calculation of the rank of the differential of δ at the invariant point $(1,0,0)$, that $\dim T_{(1,0,0)} U = 10$, whence $T_{(1,0,0)} U = V(0) \oplus V(8)$. Therefore, as U is irreducible, it is the unique component of the (possibly reducible) variety $\delta^{-1}(0)$ passing through $(1,0,0)$. Moreover, it is clear the condition $\{\delta = 0\}$ amounts to 5 quadratic equations! We have proven

Theorem 3.2.4.3. *There is an isomorphism*

$$\mathbb{C}(\mathbb{P}V(0,4))^{\mathrm{SL}_3\mathbb{C}} \simeq \mathbb{C}(U)^{\mathrm{PSL}_2\mathbb{C}\times\mathbb{C}^*} \tag{3.17}$$

where

$$\delta : V(0) \oplus V(4) \oplus V(8) \to V(4)$$

is PSL_2*-equivariant and homogeneous of degree* 2*, and* U *is the unique irreducible component of* $\delta^{-1}(0)$ *passing through* $(1,0,0)$*. Moreover,* $\dim U = 10$ *and* $T_{(1,0,0)}U = V(0) \oplus V(8)$*.*

We close this section by describing the explicit form of the covering $\mathrm{SL}_2\mathbb{C} \to \mathrm{SO}_3\mathbb{C}$ and the maps δ_1, δ_2, and by making some remarks on transvectants and the final formula for the map δ.

Let e_1, e_2, e_3 be the standard basis in $\mathbb{C}^3$, and denote by x_1, x_2, x_3 the dual basis in $(\mathbb{C}^3)^\vee$. In this notation, $h_0^2 = x_1x_3 - x_2^2$. We may view the $x's$ as coordinates on $\mathbb{C}^3$ and identify $\mathbb{C}^3$ with the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ by assigning to (x_1, x_2, x_3) the matrix

$$X = \begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix} \in \mathfrak{sl}_2\mathbb{C}\,.$$

Consider the adjoint representation Ad of $\mathrm{SL}_2\mathbb{C}$ on $\mathfrak{sl}_2\mathbb{C}$. Clearly, for $X \in \mathfrak{sl}_2\mathbb{C}$, $A \in \mathrm{SL}_2\mathbb{C}$, the map $\mathrm{Ad}(A) : X \mapsto AXA^{-1}$ preserves the determinant of X, which is just our h_0; the kernel of Ad is the center $\{\pm 1\}$ of $\mathrm{SL}_2\mathbb{C}$, and since $\mathrm{SL}_2\mathbb{C}$ is connected, the image of Ad is $\mathrm{SO}_3\mathbb{C}$. This is how we fix the isomorphism $\mathrm{PSL}_2\mathbb{C} \simeq \mathrm{SO}_3\mathbb{C}$ explicitly, and how we view $\mathrm{SO}_3\mathbb{C}$ as a subgroup of $\mathrm{SL}_3\mathbb{C}$. Note that the induced isomorphism $\mathfrak{sl}_2\mathbb{C} \to \mathfrak{so}_3\mathbb{C}$ on the Lie algebra level can be described as follows:

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.18}$$

$$f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(where we view $\mathfrak{so}_3\,\mathbb{C}$ as a subalgebra of $\mathfrak{sl}_3\,\mathbb{C}$ in a way consistent with the inclusion on the group level described above). For example,

$$\begin{aligned}\mathrm{ad}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)(X) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix} \\ -\begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} x_3 & -2x_1 \\ 0 & -x_3 \end{pmatrix},\end{aligned}$$

so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

To give the isomorphism $\delta_1 : V(0) \oplus V(4) \oplus V(8) \to V(0,4)$ explicitly, we just have to find highest weight vectors inside $V(0)$, $V(4)$, $V(8)$ and corresponding highest weight vectors inside $V(0,4)$. For example, h acts on $z_2^4 \in V(4)$ by multiplication by 4, and z_2^4 is killed by e, so this is a highest weight vector inside $V(4)$. But if we compute

$$\begin{aligned}h \cdot (x_1x_3^3 - x_2^2x_3^2) &= (h \cdot x_1)x_3^3 + 3x_1(h \cdot x_3)x_3^2 - 2(h \cdot x_2)x_2x_3^2 \\ -2x_2^2(h \cdot x_3)x_3 &= (-2x_1)x_3^3 + 3x_1(2x_3)x_3^2 - 2 \cdot 0 \cdot x_2x_3^2 \\ -2x_2^2(2x_3)x_3 &= 4(x_1x_3^3 - x_2^2x_3^2) \quad \text{and} \\ e \cdot (x_1x_3^3 - x_2^2x_3^2) &= (e \cdot x_1)x_3^3 + 3x_1(e \cdot x_3)x_3^2 - 2(e \cdot x_2)x_2x_3^2 \\ -2x_2^2(e \cdot x_3)x_3 &= (-2x_2) \cdot x_3^3 + 3x_1 \cdot 0 \cdot x_3^2 - 2(-x_3)x_2x_3^2 \\ -2x_2^2 \cdot 0 \cdot x_3 &= 0\end{aligned}$$

(use (18) and remark that the x's are dual variables, so we have to use the dual action), then we find that a corresponding highest weight vector for the submodule of $V(0,4)$ isomorphic to $V(4)$ is $x_1x_3^3 - x_2^2x_3^2$. Proceeding in this way, we see that we can define δ_1 uniquely by the requirements:

$$\delta_1 \,:\, 1 \mapsto h_0^2\,,\; z_2^4 \mapsto x_1x_3^3 - x_2^2x_3^2\,,\; z_2^8 \mapsto x_3^4\,, \tag{3.19}$$

and using the Lie algebra action and linearity, we can compute the values of δ_1 on a set of basis vectors in $V(0) \oplus V(4) \oplus V(8)$.

To write down δ_2 explicitly, remark that $V(1,1)$ may be viewed as the $\mathrm{SL}_3\,\mathbb{C}$-submodule of $\mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee$ consisting of those tensors that are annihilated by

$$\Delta := \frac{\partial}{\partial e_1} \otimes \frac{\partial}{\partial x_1} + \frac{\partial}{\partial e_2} \otimes \frac{\partial}{\partial x_2} + \frac{\partial}{\partial e_3} \otimes \frac{\partial}{\partial x_3}\,.$$

We take again our highest weight vector $z_2^4 \in V(4)$, and all we have to do is to find a vector in $\mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee$ on which h acts by multiplication by 4 and which is annihilated by e and Δ. Indeed, $e_1 x_3$ is one such. Thus we define δ_2 by

$$\delta_2 \,:\, z_2^4 \mapsto e_1 x_3 \,.$$

Then it is easy to compute the values of δ_2 on basis elements of $V(4)$ in the same way as for δ_1.

Let us recall the classical notion of *transvectants* ("Überschiebung " in German). Let d_1, d_2, n be nonnegative integers such that $0 \leq n \leq \min(d_1, d_2)$. For $f \in V(d_1)$ and $g \in V(d_2)$ one puts

$$\psi_n(f,g) := \frac{(d_1-n)!}{d_1!} \frac{(d_2-n)!}{d_2!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\partial^n f}{\partial z_1^{n-i} \partial z_2^i} \frac{\partial^n g}{\partial z_1^i \partial z_2^{n-i}} \tag{3.20}$$

(cf. [B-S], p. 122). The map $(f,g) \mapsto \psi_n(f,g)$ is a bilinear and $\mathrm{SL}_2\,\mathbb{C}$-equivariant map from $V(d_1) \times V(d_2)$ onto $V(d_1+d_2-2n)$. The map

$$\begin{aligned} V(d_1) \otimes V(d_2) &\to \bigoplus_{n=0}^{\min(d_1,d_2)} V(d_1+d_2-2n) \\ (f,g) &\mapsto \sum_{n=0}^{\min(d_1,d_2)} \psi_n(f,g) \end{aligned}$$

is an isomorphism of $\mathrm{SL}_2\,\mathbb{C}$-modules ("Clebsch-Gordan decomposition"). Thus transvectants make the decomposition of $V(d_1) \otimes V(d_2)$ into irreducibles explicit.

The explicit form of δ that results from the computations is then

$$\begin{aligned} \delta(f_0, f_4, f_8) = &-\frac{6}{1225}\psi_6(f_8, f_8) + \frac{1}{840}\psi_4(f_8, f_4) \\ &+\frac{11}{54}\psi_2(f_4, f_4) - \frac{7}{36} f_4 f_0 \,, \end{aligned} \tag{3.21}$$

where $(f_0, f_4, f_8) \in V(0) \oplus V(4) \oplus V(8)$. Note that the fact that δ turns out to be such a linear combination of transvectants is no surprise in view of the Clebsch-Gordan decomposition: In fact, δ may be viewed as a map

$$\delta' : (V(0) \oplus V(4) \oplus V(8)) \otimes (V(0) \oplus V(4) \oplus V(8)) \to V(4)$$

and using the fact that δ is symmetric and collecting only those tensor products in the preceding formula for which $V(4)$ is a subrepresentation, we see that δ comes from a map

$$\delta'' : (V(0) \otimes V(4)) \oplus (V(4) \otimes V(4))$$
$$\oplus (V(8) \otimes V(4)) \oplus (V(8) \otimes V(8)) \to V(4)\,.$$

Thus it is clear from the beginning that δ will be a linear combination of ψ_6, ψ_4, ψ_2, ψ_0 as in formula (21), and the actual coefficients are easily calculated once we know δ explicitly!
In fact, the next lemma shows that the actual coefficients of the transvectants ψ_i's occurring in δ are not very important.

Lemma 3.2.4.4. *For $\lambda := (\lambda_0, \lambda_2, \lambda_4, \lambda_6) \in \mathbb{C}^4$ consider the homogeneous of degree 2 PSL_2-equivariant map*

$$\delta_\lambda : V(8) \oplus V(0) \oplus V(4) \to V(4)$$
$$f_8 + f_0 + f_4 \mapsto \lambda_6\psi_6(f_8, f_8) + 2\lambda_4\psi_4(f_8, f_4) + \lambda_2\psi_2(f_4, f_4) + 2\lambda_0 f_4 f_0\,.$$

Suppose that $\lambda_0 \neq 0$. Then:

(1) One has $1 \in \delta_\lambda^{-1}(0)$ and $T_1\,\delta_\lambda^{-1}(0) = V(8) \oplus V(0)$; thus there is a unique irreducible component U_λ of $\delta_\lambda^{-1}(0)$ passing through 1 on which 1 is a smooth point.

(2) If furthermore $\lambda \in (\mathbb{C}^*)^4$, then $\mathbb{P}U_\lambda$ is $\mathrm{PSL}_2\,\mathbb{C}$-equivariantly isomorphic to $\mathbb{P}U_{(1,6\epsilon,1,6)}$ for some $\epsilon \neq 0$ (depending on λ).

Proof. Part (1) is a straightforward calculation, and for part (2) we choose complex numbers μ_0, μ_4, μ_8 with the properties $6\mu_8^2 = \lambda_6$, $\mu_4\mu_8 = \lambda_4$, $\mu_0\mu_4 = \lambda_0$, and compute ϵ from $6\epsilon\mu_4^2 = \lambda_2$. Then the map from $\mathbb{P}U_\lambda$ to $\mathbb{P}U_{(1,6\epsilon,1,6)}$ given by sending $[f_0 + f_4 + f_8]$ to $[\mu_0 f_0 + \mu_4 f_4 + \mu_8 f_8]$ gives the desired isomorphism. $\square$

In the next section we will see that for any $\epsilon \neq 0$, the $\mathrm{PSL}_2\,\mathbb{C}$-quotient of $\mathbb{P}U_{(1,6\epsilon,1,6)}$ is rational, and so the same holds for $\mathbb{P}U_\lambda$ for any $\lambda \in (\mathbb{C}^*)^4$; note however that the reduction step in lemma 2.4.4 (2) just simplifies the subsequent calculations, but is otherwise not substantial.

3.3 Further sections and inner projections

3.3.1 Binary quartics again and a $(\mathrm{PSL}_2\,\mathbb{C}, \mathfrak{S}_4)$-section

All the subsequent constructions and calculations depend very much on the geometry of the $\mathrm{PSL}_2\,\mathbb{C}$-action on the module $V(4)$. In fact, the first main point in the proof that $\mathbb{P}U_\lambda/\mathrm{PSL}_2\,\mathbb{C}$ is rational will be the construction of a $(\mathrm{PSL}_2\,\mathbb{C}, \mathfrak{S}_4)$-section of this variety ($\mathfrak{S}_4$ being the group of permutations of 4 elements); this is done by using proposition 2.1.2 (2) for the projection of $V(8)\oplus V(0)\oplus V(4)$ to $V(4)$ and producing such a section for $V(4)$ via the concept of *stabilizer in general position* which we recall next.

Definition 3.3.1.1. Let G be a linear algebraic group G acting on an irreducible variety X. A stabilizer in general position (s.g.p.) for the action of G on X is a subgroup H of G such that the stabilizer of a general point in X is conjugate to H in G.

An s.g.p. (if it exists) is well-defined to within conjugacy, but it need not exist in general; however, for the action of a reductive group G on an irreducible smooth affine variety, an s.g.p. always exists by results of Richardson and Luna (cf. [Po-Vi], §7).

Proposition 3.3.1.2. *For the action of* $\mathrm{PSL}_2\,\mathbb{C}$ *on* $V(4)$, *an s.g.p. is given by the subgroup* H *generated by*

$$\omega := \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \quad \text{and} \quad \rho := \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right].$$

H is isomorphic to the Klein four-group $\mathfrak{V}_4 \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ and its normalizer $N(H)$ in $\mathrm{PSL}_2\,\mathbb{C}$ is isomorphic to $\mathfrak{S}_4$; one has $N(H)/H \simeq \mathfrak{S}_3$.
More explicitly, $N(H) = \langle \tau, \sigma\rangle$, where, putting $\theta := \exp(2\pi i/8)$, one has

$$\tau := \left[\begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}\right], \quad \sigma := \left[\frac{1}{\sqrt{2}}\begin{pmatrix} \theta^3 & \theta^7 \\ \theta^5 & \theta^5 \end{pmatrix}\right].$$

Proof. We will give a geometric proof due to Bogomolov ([Bogo2], p.18). A general homogeneous degree 4 binary form $f \in V(4)$ determines a set of 4 points $\Sigma \subset \mathbb{P}^1$; the double cover of $\mathbb{P}^1$ with branch points Σ is an elliptic curve; it is acted on by its subgroup of 2-torsion points $H_f \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, and this action commutes with the sheet exchange map, hence descends to an

action of H_f on $\mathbb{P}^1$ which preserves the point set Σ and thus the polynomial f; in general H_f will be the full automorphism group of the point set Σ since a general elliptic curve does not have complex multiplication.
Let us see that H_f is conjugate to H: H_f is generated by two commuting reflections γ_1, γ_2 acting on the Riemann sphere $\mathbb{P}^1$ (with two fixed points each). By applying a suitable projectivity, we see that H_f is conjugate to $\langle \omega, \gamma_2' \rangle$ where γ_2' is another reflection *commuting* with ω; thus ω interchanges the fixed points of γ_2' and also the fixed points of ρ: Thus if we change coordinates via a suitable dilation (a projectivity preserving the fixed points of ω), γ_2' goes over to ρ, and thus H_f is conjugate to H.
One computes that σ and τ normalize H; in fact, $\sigma^{-1}\omega\sigma = \rho$, $\sigma^{-1}\rho\sigma = \omega\rho$, and $\tau^{-1}\omega\tau = \omega\rho$, $\tau^{-1}\rho\tau = \rho$. Moreover, τ has order 4 and σ order 3, $(\tau\sigma)^2 = 1$, thus one has the relations

$$\tau^4 = \sigma^3 = (\tau\sigma)^2 = 1\,.$$

It is known that $\mathfrak{S}_4$ is the group on generators R, S with relations $R^4 = S^2 = (RS)^3 = 1$; mapping $R \mapsto \tau^{-1}$, $S \mapsto \tau\sigma$, we see that the group $\langle \tau, \sigma \rangle < N(H)$ is a quotient of $\mathfrak{S}_4$; since $\langle \tau, \sigma \rangle$ contains elements of order 4 and order 3, its order is at least 12, but since there are no normal subgroups of order 2 in $\mathfrak{S}_4$, $\mathfrak{S}_4 = \langle \tau, \sigma \rangle$. To finish the proof, it therefore suffices to note that the order of $N(H)$ is at most 24: For this one just has to show that the centralizer of H in $\mathrm{PSL}_2\,\mathbb{C}$ is just H, for then $N(H)/H$ is a subgroup of the group of permutations of the three nontrivial elements $H - \{1\}$ in H (in fact equal to it). Elements in $\mathrm{PGL}_2\,\mathbb{C}$ commuting with ω must be of the form

$$\left[\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right] \quad \text{or} \quad \left[\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right],$$

and if these commute also with ρ, the elements 1, ω, ρ, $\omega\rho$ are the only possibilities. ☐

Corollary 3.3.1.3. *The variety $(V(4)^H)^0 \subset V(4)$ consisting of those points whose stabilizer in $\mathrm{PSL}_2\,\mathbb{C}$ is exactly H is a $(\mathrm{PSL}_2\,\mathbb{C}, N(H))$-section of $V(4)$.*

Proof. The fact that the orbit $\mathrm{PSL}_2\,\mathbb{C} \cdot (V(4)^H)^0$ is dense in $V(4)$ follows since a general point in $V(4)$ has stabilizer conjugate to H; the assertion $\forall g \in \mathrm{PGL}_2\,\mathbb{C}$, $\forall x \in (V(4)^H)^0 : gx \in (V(4)^H)^0 \implies g \in N(H)$ is clear by definition. ☐

Let us recall the representation theory of $N(H) = \mathfrak{S}_4$ viewed as the group of permutations of four letters $\{a,\ b,\ c,\ d\}$; the character table is as follows (cf. [Se]).

	1	(ab)	$(ab)(cd)$	(abc)	$(abcd)$
χ_0	1	1	1	1	1
ϵ	1	-1	1	1	-1
θ	2	0	2	-1	0
ψ	3	1	-1	0	-1
$\epsilon\psi$	3	-1	-1	0	1

V_{χ_0} is the trivial 1-dimensional representation, V_ϵ is the 1-dimensional representation where $\epsilon(g)$ is the sign of the permutation g; $\mathfrak{S}_4 = N(H)$ being the semidirect product of $N(H)/H = \mathfrak{S}_3$ by the normal subgroup H, V_θ is the irreducible two-dimensional representation induced from the representation of $\mathfrak{S}_3$ acting on the elements of $\mathbb{C}^3$ which satisfy $x+y+z=0$ by permutation of coordinates. V_ψ is the extension to $\mathbb{C}^3$ of the natural representation of $\mathfrak{S}_4$ on $\mathbb{R}^3$ as the group of rigid motions stabilizing a regular tetrahedron; finally, $V_{\epsilon\psi} = V_\epsilon \otimes V_\psi$.
We want to decompose $V(8) \oplus V(0) \oplus V(4)$ as $N(H)$-module; we fix the notation:

$$
\begin{gathered}
a_0 := 1; \quad a_1 := z_1^4 + z_2^4,\ a_2 := 6z_1^2z_2^2,\ a_3 := z_1^4 - z_2^4, \\
a_4 := 4(z_1^3z_2 - z_1z_2^3),\ a_5 := 4(z_1^3z_2 + z_1z_2^3); \\
e_1 := 28(z_1^6z_2^2 - z_1^2z_2^6),\ e_2 := 56(z_1^7z_2 + z_1^5z_2^3 - z_1^3z_2^5 - z_1z_2^7), \\
e_3 := 56(z_1^7z_2 - z_1^5z_2^3 - z_1^3z_2^5 + z_1z_2^7),\ e_4 := z_1^8 - z_2^8 \\
e_5 := 8(z_1^7z_2 - 7z_1^5z_2^3 + 7z_1^3z_2^5 - z_1z_2^7), \\
e_6 := 8(z_1^7z_2 + 7z_1^5z_2^3 + 7z_1^3z_2^5 + z_1z_2^7), \\
e_7 := z_1^8 + z_2^8,\ e_8 := 28(z_1^6z_2^2 + z_1^2z_2^6),\ e_9 := 70z_1^4z_2^4\,.
\end{gathered} \tag{3.22}
$$

Lemma 3.3.1.4. *One has the following decompositions as $N(H)$-modules:*

$$V(0) = V_{\chi_0},\ V(4) = V_\psi \oplus V_\theta,\ V(8) = V_{\epsilon\psi} \oplus V_\psi \oplus V_\theta \oplus V_{\chi_0}\,. \tag{3.23}$$

More explicitly,

$$
\begin{gathered}
V(0) = \langle a_0\rangle,\ V(4) = \langle a_3, a_4, a_5\rangle \oplus \langle a_1, a_2\rangle, \\
V(8) = \langle e_4, e_5, e_6\rangle \oplus \langle e_1, e_2, e_3\rangle \oplus \langle e_8, 7e_7 - e_9\rangle \oplus \langle 5e_7 + e_9\rangle\,.
\end{gathered} \tag{3.24}
$$

Here $\langle e_4, e_5, e_6\rangle$ corresponds to $V_{\epsilon\psi}$ and $\langle e_1, e_2, e_3\rangle$ corresponds to V_ψ. Moreover,

$$V(0)^H = \langle a_0\rangle,\ V(4)^H = \langle a_1, a_2\rangle,\ V(8)^H = \langle e_7, e_8, e_9\rangle\,. \tag{3.25}$$

Proof. We will prove (25) first; one observes that quite generally for $k \geq 0$, $V(2k)^H = (V(2k)^\rho)^\omega$ (ρ and ω commute) and that the monomials $z_1^j z_2^{2k-j}$, $j = 0, \ldots, 2k$, are invariant under ρ if $j + k$ is even, and otherwise anti-invariant, so if $k = 2s$, $\dim V(2k)^\rho = 2s+1$, and if $k = 2s+1$, $\dim V(2k)^\rho = 2s+1$. Since ω is also a reflection, we have $2\dim(V(2k)^\rho)^\omega - \dim V(2k)^\rho = \mathrm{tr}(\omega|_{V(2k)^\rho})$, and the trace is 1 for $k = 2s$, and -1 for $k = 2s+1$, thus

$$\dim V(2k)^H = s+1,\ k = 2s, \quad \dim V(2k)^H = s,\ k = 2s+1\,.$$

In particular, the H-invariants in $V(0)$, $V(4)$, $V(8)$ have the dimensions as claimed in (25), and one checks that the elements given there are indeed invariant.

To prove (23), we use the Clebsch-Gordan formula $V(2k) \otimes V(2) = V(2k+2) \oplus V(2k) \oplus V(2k-2)$ (cf. (20)) iteratively together with the fact that the character of the tensor product of two representations of a finite group is the product of the characters of each of the factors; since $V(2)$ has dimension 3 and $\dim V(2)^H = 0$, $V(2)$ is irreducible; the value of the character of the $N(H)$-module $V(2)$ on τ is 1, so $V(2) = V_{\epsilon\psi}$. Now $V(2) \otimes V(2) = V(4) \oplus V(2) \oplus V(0)$, and looking at the character table, one checks that

$$(\epsilon\psi)^2 = \chi_0 + (\epsilon\psi) + (\psi) + (\theta)\,.$$

This proves the decomposition in (23) for $V(4)$. The decomposition for $V(8)$ is proven similarly (one proves $V(6) = V_\psi \oplus V_{\epsilon\psi} \oplus V_\epsilon$ first).

The proof of (24) now amounts to checking that the given spaces are invariant under σ and τ; finally note that $V_{\epsilon\psi}$ corresponds to $\langle e_4, e_5, e_6\rangle$ since the value of the character on τ is 1. $\square$

Recall from Lemma 2.4.4 that we want to prove the rationality of $(\mathbb{P}\, U_\lambda)/\mathrm{PSL}_2\,\mathbb{C}$ and we can and will always assume in the sequel that $\lambda = (1, 6\epsilon, 1, 6)$ for $\epsilon \neq 0$. In view of Lemma 3.1.4 it will be convenient for subsequent calculations to write the map $\delta_\lambda : V(8) \oplus V(0) \oplus V(4) \to V(4)$ in terms of the basis $(e_1, \ldots, e_9, a_0, a_1, \ldots, a_5)$ in the source and the basis $(a_1, \ldots, a_5)$ in the target. Denote coordinates in $V(8) \oplus V(0) \oplus V(4)$ with respect to the chosen basis by $(x_1, \ldots, x_9, s_0, s_1, \ldots, s_5) =: (x, s)$. Then one may write

$$\delta_\lambda(x, s) = \begin{pmatrix} Q_1(x, s) \\ \vdots \\ Q_5(x, s) \end{pmatrix} \tag{3.26}$$

with $Q_1(x, s), \ldots, Q_5(x, s)$ quadratic in (x, s); their values may be computed using formulas (20), (22), and the definition of δ_λ in Lemma 2.4.4, and they can be found in Appendix B.

Theorem 3.3.1.5. *Let $\tilde{\mathcal{Q}}_\lambda \subset V(8) \oplus V(0) \oplus V(4)$ be the subvariety defined by the equations $Q_1 = \cdots = Q_5 = 0$, $s_3 = s_4 = s_5 = 0$. There is exactly one 7-dimensional irreducible component $\mathcal{Q}_\lambda$ of $\tilde{\mathcal{Q}}_\lambda$ passing through the $N(H)$-invariant point $5e_7 + e_9$ in $V(8)$; $\mathcal{Q}_\lambda$ is $N(H)$-invariant and*

$$\mathbb{C}(\mathbb{P}\, U_\lambda)^{\mathrm{PSL}_2 \mathbb{C}} = \mathbb{C}(\mathbb{P}\, \mathcal{Q}_\lambda)^{N(H)}. \tag{3.27}$$

Proof. We want to use Proposition 2.1.2, (2).
Note that $5e_7 + e_9 \in U_\lambda$: In fact, δ_λ maps the $N(H)$-invariants in $V(8) \oplus V(0) \oplus V(4)$ to the $N(H)$-invariants in $V(4)$ which are 0. Since U_λ is the unique irreducible component of $\delta_\lambda^{-1}(0)$ passing through $a_0 = 1$, U_λ contains the whole plane of invariants $\langle a_0, 5e_7 + e_9 \rangle$.
If we denote by $p : V(8) \oplus V(0) \oplus V(4) \to V(4)$ the projection, then $\tilde{\mathcal{Q}}_\lambda = p^{-1}(V(4)^H) \cap \delta_\lambda^{-1}(0)$. Clearly, $\tilde{\mathcal{Q}}_\lambda$ is $N(H)$-invariant, and one only has to check that $5e_7+e_9$ is a nonsingular point on it with tangent space of dimension 7 by direct calculation: Then there is a unique 7-dimensional irreducible component $\mathcal{Q}_\lambda$ of $\tilde{\mathcal{Q}}_\lambda$ passing through $5e_7+e_9$ which is $N(H)$-invariant (since $5e_7 + e_9$ is an invariant point on it and this point is nonsingular on $\tilde{\mathcal{Q}}_\lambda$).
It remains to prove (27): $\mathcal{Q}_\lambda$ is an irreducible component of $p^{-1}(V(4)^H) \cap U_\lambda$ and $\mathcal{Q}_\lambda^0 = \mathcal{Q}_\lambda \cap p^{-1}((V(4)^H)^0)$ is a dense $N(H)$-invariant open subset of $\mathcal{Q}_\lambda$ dominating $(V(4)^H)^0$. Thus by Proposition 2.1.1 (2),

$$\mathbb{C}(\mathbb{P}\, U_\lambda)^{\mathrm{PSL}_2 \mathbb{C}} \simeq \mathbb{C}(\mathbb{P}\, \mathcal{Q}_\lambda^0)^{N(H)} \simeq \mathbb{C}(\mathbb{P}\, \mathcal{Q}_\lambda)^{N(H)}.$$

□

3.3.2 Dividing by the action of H

Next we would like to "divide out" the action by H, so that we are left with an invariant theory problem for the group $N(H)/H = \mathfrak{S}_3$. Look back at the action of $N(H)$ on $M := \{s_3 = s_4 = s_5 = 0\} \subset V(8) \oplus V(0) \oplus V(4)$ which is explained in formulas (23), (24); we will adopt the notational convention to denote the irreducible $N(H)$-submodule of $V(8)$ isomorphic to V_ψ by $V(8)_{(\psi)}$ and so forth; thus

$$M = V(0)_{(\chi_0)} \oplus V(4)_{(\theta)} \oplus V(8)_{(\chi_0)} \oplus V(8)_{(\theta)} \oplus V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)}\,, \quad (3.28)$$

and looking at the character table of $\mathfrak{S}_4$, we see that the action of H is nontrivial only on $V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)} = \langle e_1, e_2, e_3\rangle \oplus \langle e_4, e_5, e_6\rangle$ where x_1, x_2, x_3 and x_4, x_5, x_6 are coordinates; in terms of these, we have

$$\begin{aligned}
(\omega)(x_1, \ldots, x_6) &= (-x_1,\ x_2,\ -x_3,\ -x_4,\ x_5,\ -x_6)\,, \qquad (3.29)\\
(\rho)(x_1, \ldots, x_6) &= (x_1,\ -x_2,\ -x_3,\ x_4,\ -x_5,\ -x_6)\,,\\
(\omega\rho)(x_1, \ldots, x_6) &= (-x_1,\ -x_2,\ x_3,\ -x_4,\ -x_5,\ x_6)\,,
\end{aligned}$$

and

$$\begin{aligned}
\tau(x_1, \ldots, x_6) &= (-x_1,\ -ix_3,\ -ix_2,\ x_4,\ -ix_6,\ -ix_5)\,, \qquad (3.30)\\
\sigma(x_1, \ldots, x_6) &= \left(4x_3,\ -\frac{i}{4}x_1,\ ix_2,\ -8x_6,\ -\frac{i}{8}x_4,\ -ix_5\right)\,.
\end{aligned}$$

Thus we see that the map

$$\begin{aligned}
&\mathbb{P}(V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)}) - \{x_1x_2x_3 = 0\} \to R \times \mathbb{P}^2\,,\\
&(x_1, \ldots, x_6) \mapsto \left(\left(\frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3}\right)\,,\ \left(\frac{1}{x_1^2} : \frac{1}{x_2^2} : \frac{1}{x_3^2}\right)\right)\,,
\end{aligned}$$

where $R = \mathbb{C}^3$, is dominant with fibres H-orbits, and furthermore $N(H)$-equivariant for a suitable action of $N(H)$ on $R \times \mathbb{P}^2$: In fact, we will agree to write

$$\left(\frac{1}{x_1^2} : \frac{1}{x_2^2} : \frac{1}{x_3^2}\right) = \left(\frac{x_2x_3}{x_1} : \frac{x_3x_1}{x_2} : \frac{x_1x_2}{x_3}\right)$$

and remark that the subspaces

$$R = \left\langle \frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3} \right\rangle\,, \quad T := \left\langle \frac{x_2x_3}{x_1}, \frac{x_3x_1}{x_2}, \frac{x_1x_2}{x_3} \right\rangle$$

of the field of fractions of $\mathbb{C}\left[V(8)_{(\psi)} \oplus V(8)_{(\epsilon\psi)}\right]$ are invariant under σ and τ (thus $\mathbb{P}^2 = \mathbb{P}(T)$). If we denote the coordinates with respect to the basis vectors in R resp. T given above by r_1, r_2, r_3 resp. y_1, y_2, y_3, then the actions of τ and σ are described by

$$\tau(r_1, r_2, r_3) = (-r_1, r_3, r_2)\,,\ \sigma(r_1, r_2, r_3) = (-2r_3, r_1/2, -r_2)$$
$$\tau(y_1, y_2, y_3) = (y_1, -y_3, -y_2)\,,\ \sigma(y_1, y_2, y_3) = ((1/16)y_3, -16y_1, -y_2)\,.$$

Thus the only $N(H)$-invariant lines in R resp. T are the ones spanned by $(2, 1, -1)$ resp. $(-1, 16, -16)$ on which τ acts by multiplication by -1 resp. by $+1$ and hence

$$R = R_{(\epsilon)} \oplus R_{(\theta)}\,,\ T = T_{(\chi_0)} \oplus T_{(\theta)}\,. \tag{3.31}$$

We see that the morphism

$$\begin{aligned} \pi\ :\ \mathbb{P}(M) - \{x_1x_2x_3 = 0\} \\ \to R \times \mathbb{P}(T \oplus V(8)_{(\chi_0)} \oplus V(8)_{(\theta)} \oplus V(0)_{(\chi_0)} \oplus V(4)_{(\theta)}) \simeq R \times \mathbb{P}^8\,, \\ \pi(x, s) := \left(\left(\frac{x_4}{x_1}, \frac{x_5}{x_2}, \frac{x_6}{x_3} \right)\,,\ \left(\frac{x_2x_3}{x_1} : \frac{x_3x_1}{x_2} : \frac{x_1x_2}{x_3} \right) \right. \\ \left. : x_7 : x_8 : x_9 : s_0 : s_1 : s_2 \right) \end{aligned} \tag{3.32}$$

is $N(H)$-equivariant, dominant, and all fibres are H-orbits. If we consider $(x_7, x_8, x_9, s_0, s_1, s_2)$ as coordinates in $V(8)_{(\chi_0)} \oplus V(8)_{(\theta)} \oplus V(0)_{(\chi_0)} \oplus V(4)_{(\theta)}$ in the *target* of the map π (as we do in formula (32)) we denote them by $(y_7, y_8, y_9, y_{10}, y_{11}, y_{12})$ to achieve consistency with [Kat96].
How do we get equations which define the image

$$\pi(\mathbb{P}\,\tilde{\mathcal{Q}}_\lambda \cap \{x_1x_2x_3 \neq 0\}) \subset R \times (\mathbb{P}^8 - \{y_1y_2y_3 = 0\})$$

in $\mathbb{P}^8 - \{y_1y_2y_3 = 0\}$ from the quadrics $Q_1(x, s), \dots, Q_5(x, s)$ in formula (26)? We can set $s_3 = s_4 = s_5 = 0$ in $Q_1, \dots, Q_5$ to obtain equations $\bar{Q}_1, \dots, \bar{Q}_5$ for $\mathbb{P}\,\tilde{\mathcal{Q}}_\lambda$ in $\mathbb{P}(M)$; the point is now that the quantities

$$\bar{Q}_1,\ \bar{Q}_2,\ \frac{\bar{Q}_3}{x_1},\ \frac{\bar{Q}_4}{x_2},\ \frac{\bar{Q}_4}{x_3}$$

are H-invariant (as one sees from the equations in Appendix B). Moreover, the map

$$\pi : \mathbb{P}(M) - \{x_1x_2x_3 = 0\} \to R \times (\mathbb{P}^8 - \{y_1y_2y_3 = 0\})$$

is a geometric quotient for the action of H on the source (by [Po-Vi], Thm. 4.2), so we can write

$$\bar{Q}_1 = q_1(r_1, \dots, y_{12}),\ \bar{Q}_2 = q_2(r_1, \dots, y_{12}),\ \frac{\bar{Q}_3}{x_1} = q_3(r_1, \dots, y_{12}),$$
$$\frac{\bar{Q}_4}{x_2} = q_4(r_1, \dots, y_{12}),\ \frac{\bar{Q}_4}{x_3} = q_5(r_1, \dots, y_{12})$$

where $q_1, \dots, q_5$ are polynomials in (r_1, r_2, r_3), $(y_1, y_2, y_3, y_7, \dots, y_{12})$ which one may find written out in Appendix B. Here we just want to emphasize their structural properties which will be most important for the subsequent arguments:

(1) The polynomials q_1, q_2 are homogeneous of degree 2 in the set of variables $(y_1, \dots, y_{12})$; the coefficients of the monomials in the y's are (inhomogeneous) polynomials of degrees ≤ 2 in r_1, r_2, r_3. For $r_1 = r_2 = r_3 = 0$, q_1, q_2 do not vanish identically.

(2) The polynomials q_3, q_4, q_5 are homogeneous linear in $(y_1, \dots, y_{12})$; the coefficients of the monomials in the y's are (inhomogeneous) polynomials of degrees ≤ 2 in r_1, r_2, r_3. For $r_1 = r_2 = r_3 = 0$, q_3, q_4, q_5 do not vanish identically.

Theorem 3.3.2.1. *Let $\tilde{Y}_\lambda$ be the subvariety of $R \times \mathbb{P}^8$ defined by the equations $q_1 = q_2 = q_3 = q_4 = q_5 = 0$. There is an irreducible $N(H)$-invariant component Y_λ of $\tilde{Y}_\lambda$ with $\pi([x^0]) \in Y_\lambda$, where $x^0 := 13i(5e_7 + e_9) + 5(4e_1 - ie_2 + e_3)$, such that*

$$\mathbb{C}(\mathbb{P}\,\mathcal{Q}_\lambda)^{N(H)} \simeq \mathbb{C}(Y_\lambda)^{N(H)}\,. \tag{3.33}$$

Proof. The variety Y_λ will be the closure of the image $\pi(\mathbb{P}\,\mathcal{Q}_\lambda \cap \{x_1x_2x_3 \neq 0\})$ in $R \times \mathbb{P}^8$.
It remains to see that $x^0 \in \mathcal{Q}_\lambda$. Recall from Theorem 3.1.5 that $\mathcal{Q}_\lambda$ is the unique irreducible component of $\tilde{\mathcal{Q}}_\lambda$ passing through the $N(H)$-invariant point $5e_7 + e_9$, and that this point is a nonsingular point on $\tilde{\mathcal{Q}}_\lambda$; thus, if we can find an irreducible subvariety of $\tilde{\mathcal{Q}}_\lambda$ which contains both $5e_7 + e_9$ and x^0, we are done. The sought-for subvariety is $\tilde{\mathcal{Q}}_\lambda \cap V(8)^\sigma$, where $V(8)^\sigma$ are the

elements in $V(8)$ invariant under $\sigma \in N(H)$. One sees that x^0 and $5e_7 + e_9$ lie on it, and computing

$$V(8)^\sigma = \langle 5e_7 + e_9, 8e_4 - ie_5 - e_6, 4e_1 - ie_2 + e_3 \rangle \,,$$
$$V(4)^\sigma = \langle 2(z_1^4 - z_2^4) + 4(z_1^3 z_2 + z_1 z_2^3) + 4i(z_1^3 z_2 - z_1 z_2^3) \rangle \,,$$

and using $\delta_\lambda(V(8)^\sigma) \subset V(4)^\sigma$, we find that $\tilde{\mathcal{Q}}_\lambda \cap V(8)^\sigma$ is a quadric in $V(8)^\sigma$ which is easily checked to be irreducible. $\square$

Thus it remains to prove the rationality of $Y_\lambda / N(H) = Y_\lambda / \mathfrak{S}_3$.

3.3.3 Inner projections and the "no-name" method

The variety $\tilde{Y}_\lambda$ comes with the two projections

$$\begin{array}{ccc} \tilde{Y}_\lambda & \xrightarrow{p_{\mathbb{P}^8}} & \mathbb{P}^8 \\ {\scriptstyle p_R}\downarrow & & \\ R & & \end{array}$$

Recall from (32) that $N := \mathbb{P}(V(8)_\theta \oplus V(4)_\theta) \subset \mathbb{P}^8$ is an $N(H)$-invariant 3-dimensional projective subspace of $\mathbb{P}^8$. We will show $\mathbb{C}(Y_\lambda)^{N(H)} \simeq \mathbb{C}(R \times N)^{N(H)}$ via the following theorem.

Theorem 3.3.3.1. *There is an open $N(H)$-invariant subset $R_0 \subset R$ containing $0 \in R$ with the following properties:*

(1) For all $r \in R_0$ the fibre $p_R^{-1}(r) \subset \tilde{Y}_\lambda$ is irreducible of dimension 3, and $p_R^{-1}(R_0)$ is an open $N(H)$-invariant subset of Y_λ.

(2) There exist $N(H)$-sections σ_1, σ_2 of the $N(H)$-equivariant projection $R_0 \times \mathbb{P}^8 \to R_0$ such that $N(r) := \langle \sigma_1(r), \sigma_2(r), (1:0:0:\cdots:0), (0:1:0:\cdots:0), (0:0:1:0:\cdots:0) \rangle \subset \mathbb{P}^8$, $r \in R_0$, is an $N(H)$-invariant family of 4-dimensional projective subspaces in $\mathbb{P}^8$ with the properties:

(i) $N(r)$ is disjoint from N for all $r \in R_0$.

(ii) The fibre $p_{\mathbb{P}^8}(p_R^{-1}(r)) \subset \mathbb{P}^8$ contains the line $\langle \sigma_1(r), \sigma_2(r) \rangle \subset N(r)$ for all $r \in R_0$.

(iii) The projection $\pi_r : \mathbb{P}^8 \dashrightarrow N$ from $N(r)$ to N maps the fibre $p_{\mathbb{P}^8}(p_R^{-1}(r)) \subset \mathbb{P}^8$ dominantly onto N for all $r \in R_0$.

Before turning to the proof, let us note the following corollary.

Corollary 3.3.3.2. *One has the field isomorphism*

$$\mathbb{C}(Y_\lambda)^{N(H)} \simeq \mathbb{C}(R \times N)^{N(H)}\,,$$

and the latter field is rational. Hence $\mathfrak{M}_3$ is rational.

Proof. (of corollary) The $N(H)$-invariant set $p_R^{-1}(R_0)$ is an open subset of Y_λ. Let us see that the projection $\pi_r : F_r := p_{\mathbb{P}^8}(p_R^{-1}(r)) \dashrightarrow N$ is birational. In fact, F_r is of dimension 3 and irreducible and the intersection of a 3-codimensional linear subspace and two quadrics in $\mathbb{P}^8$. Moreover, $F_r \cap N(r)$ contains a line L_r by Theorem 3.3.1 (2), (ii). Thus for a general point P in N, $F_r \cap \langle L_r, P\rangle$ consists of L_r and a single point (namely the point of intersection of the two lines which are the residual intersections of each of the two quadrics defining F_r with $\langle L_r, P\rangle$, the other component being L_r itself). Thus π_r is generically one-to-one whence birational.
Thus one has a birational $N(H)$-isomorphism $p_R^{-1}(R_0) \dashrightarrow R_0 \times N$, given by sending $(r, [y])$ to $(r, \pi_r([y]))$. Thus one gets the field isomorphism in Corollary 3.3.2.
By the no-name lemma (cf. e.g. [Dolg1], section 4), $\mathbb{C}(R \times N)^{N(H)} \simeq \mathbb{C}(N)^{N(H)}(T_1, T_2, T_3)$, where T_1, T_2, T_3 are indeterminates, thus it suffices to show that the quotient of N by $N(H)$ is stably rational of level ≤ 3. This in turn follows from the same lemma, since clearly, if we take the representation of $\mathfrak{S}_3$ in $\mathbb{C}^3$ by permutation of coordinates, the quotient of $\mathbb{P}(\mathbb{C}^3)$ by $\mathfrak{S}_3$, a unirational surface, is rational. □

Proof. (of theorem) The proof will be given in several steps.
Step 1. (Irreducibility of the fibre over 0*)* We have to show that the variety $p_{\mathbb{P}^8}(p_R^{-1}(0)) \subset \mathbb{P}^8$ is irreducible and 3-dimensional. We have explicit equations for it (namely the ones that arise if we substitute $r_1 = r_2 = r_3 = 0$ in $q_1, \dots, q_5$, which are thus 3 linear and 2 quadratic equations); the assertions can then be checked with a computer algebra system such as `Macaulay 2`. Recall from Theorem 3.2.1 that Y_λ contains $\pi([x^0])$. In fact,

$$\pi([x^0]) = \left((0,0,0), \left(-\frac{5}{4} : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0\right)\right)\,, \tag{3.34}$$

as follows from the definition of x^0 in Theorem 3.2.1 and the definition of π in (32). Thus $\pi([x^0])$ lies in the fibre over 0 of p_R^{-1} and thus, since there

is an open subset around 0 in R over which the fibres are irreducible and 3-dimensional, assertion (1) of Theorem 3.3.1 is established.
Step 2. (Construction of σ_1) To obtain σ_1, we just assign to $r \in R$ the point $(r, \sigma_1(r))$ with $\sigma_1(r) = (0:0:0:0:0:0:1:0:0)$, i.e. $y_{10} = 1$, the other y's being 0. This always is in the fibre $p_{\mathbb{P}^8}(p_R^{-1}(r))$ as one sees on substituting in the equations $q_1, \dots, q_5$. Moreover, this is an $N(H)$-section, since y_{10} is a coordinate in the space $V(0)_{\chi_0}$ in formula (32).
Step 3. (Construction of σ_2; decomposition of $V := \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$) The construction of a section σ_2, $\sigma_2(r) = (\sigma_2^{(1)}(r) : \dots : \sigma_2^{(9)}(r))$, involves a little more work. Let us look back at the construction of Y_λ in subsection 3.2 for this, especially the definition of the projection π in formula (32), and the decomposition of the linear subspace $M \subset V(8) \oplus V(0) \oplus V(4)$. By definition of R, the family of codimension 3 linear subspaces

$$L(r) := \{[(x,s)] \,|\, x_4 = r_1 x_1,\ x_5 = r_2 x_2,\ x_6 = r_3 x_3\} \subset \mathbb{P}(M)\,, \tag{3.35}$$

$r = (r_1, r_2, r_3) \in R$, is $N(H)$-invariant, i.e. $gL(r) = L(gr)$, for $g \in N(H)$. It is natural to intersect this family with $\mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ which, as we will see, has dimension 3 and look for an H-orbit $\mathfrak{O}_r$ in the intersection of $\mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ with the open set of $L(r)$ where $x_1x_2x_3 \neq 0$. Moreover, we will see that for $r = 0$, the point $[x^0]$ is in this intersection. Thus passing to the quotient we may put

$$(r, \sigma_2(r)) := \pi(\mathfrak{O}_r) \tag{3.36}$$

to obtain a σ_2 with the required properties. Indeed, note that we will have $\sigma_2^{(7)}(r) = \sigma_2^{(8)} = \sigma_2^{(9)} = 0$ which ensures that σ_2 and σ_1 span a line. Moreover,

$$\sigma_2(0) = \left(-\frac{5}{4} : 20 : -20 : 65 : 0 : 13 : 0 : 0 : 0\right)\,, \tag{3.37}$$

by formula (34), which allows us to check assertions (2), (i) and (iii) of Theorem 3.3.1, which are open properties on the base R, by explicit computation for the fibre over 0. Property (2), (ii) stated in the theorem is clear by construction. Let us now carry out this program. We will start by explicitly decomposing $V := \mathbb{P}(\delta_\lambda^{-1}(0) \cap V(8))$ into irreducible components.
To guess what V might be, note that according to the definition of δ_λ in Lemma 2.4.4, δ_λ vanishes on $f_8 \in V(8)$ if for the transvectant ψ_6 one has $\psi_6(f_8, f_8) = 0$; but looking back at the definition of transvectants in formula (20), we see that $\psi_6 : V(8) \times V(8) \to V(4)$ vanishes if f_8 is a linear

combination of z_1^8, $z_1^7 z_2$ and $z_1^6 z_2^2$ (since we differentiate at least 3 times with respect to z_2 in one factor in the summands in formula (20)). Thus $X_1 := \overline{\mathrm{PSL}_2\,\mathbb{C}\cdot\langle z_1^8,\ z_1^7 z_2,\ z_1^6 z_2^2\rangle}$, the variety of forms of degree 8 with a six-fold zero, is contained in V, and one computes that the differential of $\delta_\lambda|_{V(8)}$ in $z_1^6 z_2^2$ is surjective, so that X_1 is an irreducible component of V.
The dimension of X_1 is clearly three. Weyman, in [Wey], Cor. 4, computed the Hilbert function of $X_{p,g}$, the variety of binary forms of degree g having a root of multiplicity $\geq p$ which is

$$H(X_{p,g}, d) = (dp+1)\binom{g-p+d}{g-p} - (d(p+1)-1)\binom{g-p+d-1}{g-p-1}.$$

For $d=6$, $g=8$, the leading term in d in this expression is $3d^3$, which shows

$$\deg X_1 = 18\,. \tag{3.38}$$

Moreover, we know already that $5e_7+e_9$ is in V from the proof of Theorem 3.1.5; thus set $X_2 := \overline{\mathrm{PSL}_2\,\mathbb{C}\cdot\langle 5e_7+e_9\rangle}$. We know that the stabilizer of $5e_7+e_9$ in $\mathrm{PSL}_2\,\mathbb{C}$ contains $N(H)$ because $5e_7+e_9 = 5z_1^8+5z_2^8+70z_1^4z_2^4$ spans the $N(H)$-invariants in $V(8)$ by Lemma 3.1.4. The claim is that the stabilizer is not larger. An easy way to check this is to use the beautiful theory developed in [Ol], p. 188 ff., using differential invariants and signature curves, which allows the explicit determination of the order of the symmetry group of a complex binary form. More precisely we have (cf. [Ol], Cor. 8.68):

Theorem 3.3.3.3. *Let $Q(p)$ be a binary form of degree n (written in terms of the inhomogeneous coordinate $p = z_1/z_2$) which is not equivalent to a monomial. Then the cardinality k of the symmetry group of $Q(p)$ satisfies*

$$k \leq 4n-8\,,$$

provided that U is not a constant multiple of H^2, where U and H are the following polynomials in p: $H := (1/2)(Q,Q)^{(2)}$, $T := (Q,H)^{(1)}$, $U := (Q,T)^{(1)}$ where, if Q_1 is a binary form of degree n_1, and Q_2 is a binary form of degree n_2, we put

$$\begin{aligned}(Q_1,Q_2)^{(1)} &:= n_2Q_1'Q_2 - n_1Q_1Q_2'\,,\\ (Q_1,Q_2)^{(2)} &:= n_2(n_2-1)Q_1''Q_2 - 2(n_2-1)(n_1-1)Q_1'Q_2'\\ &\quad +n_1(n_1-1)Q_1Q_2''\,.\end{aligned}$$

(these are certain transvectants).

Applying this result in our case, we find the upper bound 24 for the symmetry group of $5e_7 + e_9$, which is indeed the order of $N(H) = \mathfrak{S}_4$. X_2 is irreducible of dimension 3, and computing that the differential of $\delta_\lambda|_{V(8)}$ is surjective in $5e_7 + e_9$, we get that X_2 is another irreducible component of V. But let us intersect X_2 with the codimension 3 linear subspace in $V(8)$ consisting of forms with zeroes $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{P}^1$; there is a unique projectivity carrying these to three roots of $5e_7 + e_9$, which are all distinct, thus there are $8 \cdot 7 \cdot 6$ such projectivities, and $\deg X_2 \geq (8 \cdot 7 \cdot 6)/|N(H)|$. But one checks easily that V itself has dimension 3 and is the intersection of 5 quadrics in $\mathbb{P}(V(8))$, thus has degree ≤ 32. Thus we must have

$$\deg X_2 = 14,\ V = X_1 \cup X_2,\ \deg V = 32. \tag{3.39}$$

Note also that

$$[x^0] \in X_2 \cap L(0)\,. \tag{3.40}$$

In fact, from the proof of Theorem 3.2.1, we know $[x^0] \in V$, and $[x^0] \in L(0)$ being clear, we just check that x^0 has no root of multiplicity ≥ 6.

Step 4. (Construction of σ_2; intersecting V with a family of linear spaces in $\mathbb{P}(M)$) Let $L^0(r)$ be the open subset of $L(r) \subset \mathbb{P}(M)$ where $x_1x_2x_3 \neq 0$. According to the strategy outlined at the beginning of Step 3, we would like to compute the cardinalities

$$|L^0(r) \cap X_1|, \quad |L^0(r) \cap X_2|,$$

for r varying in a small neighbourhood of 0 in R. It is, however, easier from a computational point of view to determine the number of intersection points of X_1 resp. X_2 with certain boundary components of $L^0(r)$ in $L(r)$ first; the preceding cardinalities will afterwards fall out as the residual quantities needed to have $\deg X_1 = 18$, $\deg X_2 = 14$. Thus let us introduce the following additional strata of $L(r)\backslash L^0(r)$:

$$\begin{aligned}
L_0 &:= \{[(x,s)]\,|\,x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0\}, \qquad (3.41)\\
L_1(r) &:= \{[(x,s)]\,|\,x_1 \neq 0,\ x_4 = r_1x_1,\ x_2 = x_3 = x_5 = x_6 = 0\},\\
L_2(r) &:= \{[(x,s)]\,|\,x_2 \neq 0,\ x_5 = r_2x_2,\ x_1 = x_3 = x_4 = x_6 = 0\},\\
L_3(r) &:= \{[(x,s)]\,|\,x_3 \neq 0,\ x_6 = r_3x_3,\ x_1 = x_2 = x_4 = x_5 = 0\},\\
\tilde{L}_1(r) &:= \{[(x,s)]\,|\,x_2x_3 \neq 0,\ x_5 = r_2x_2,\ x_6 = r_3x_3,\ x_1 = x_4 = 0\}\\
\tilde{L}_2(r) &:= \{[(x,s)]\,|\,x_1x_3 \neq 0,\ x_4 = r_1x_1,\ x_6 = r_3x_3,\ x_2 = x_5 = 0\}\\
\tilde{L}_3(r) &:= \{[(x,s)]\,|\,x_1x_2 \neq 0,\ x_4 = r_1x_1,\ x_5 = r_2x_2,\ x_3 = x_6 = 0\}.
\end{aligned}$$

$L(r)$ is the disjoint union of these and $L^0(r)$. From the equations describing δ_λ one sees that V is defined in $\mathbb{P}(V(8))$ with coordinates $x_1,\dots,x_9$ by

$$\begin{gathered}-192x_6^2-192x_3x_6+384x_3^2-192x_5^2-192x_2x_5+384x_2^2 \\ -12x_1x_4+12x_7x_8+180x_8x_9=0,\end{gathered} \tag{3.42}$$
$$\begin{gathered}64x_6^2-192x_3x_6-128x_3^2-64x_5^2+192x_2x_5+128x_2^2 \\ -2x_4^2+16x_1^2+2x_7^2-16x_8^2-50x_9^2=0,\end{gathered} \tag{3.43}$$
$$\begin{gathered}96x_5x_6-672x_3x_5-672x_2x_6+1248x_2x_3 \\ -12x_1x_7+12x_4x_8+180x_1x_9=0,\end{gathered} \tag{3.44}$$
$$\begin{gathered}6x_4x_6+42x_3x_4+84x_1x_6+156x_1x_3 \\ -6x_5x_7-42x_2x_7+24x_5x_8-264x_2x_8+30x_5x_9-30x_2x_9=0,\end{gathered} \tag{3.45}$$
$$\begin{gathered}-6x_4x_5-42x_2x_4+84x_1x_5+156x_1x_2 \\ +6x_6x_7+42x_3x_7+24x_6x_8-264x_3x_8-20x_6x_9+30x_3x_9=0,\end{gathered} \tag{3.46}$$

and thus

$$\tilde{L}_i(r)\cap V=\emptyset \quad \forall i=1,2,3 \tag{3.47}$$

for r in a Zariski open neighbourhood of $0\in R$ (for $\tilde{L}_1(r)$ consider equation (44) and assume $96r_2r_3-672r_2-672r_3+1248\neq 0$, for $\tilde{L}_2(r)$ we see that (45) cannot hold if $6r_1r_3+42r_1+84r_3+156\neq 0$, and for $\tilde{L}_3(r)$ equation (46) is impossible provided that $-6r_1r_2-42r_1+84r_2+156\neq 0$).
Let us consider the intersection $V\cap L_0$. We have to solve the equations

$$12x_7x_8+180x_8x_9=0, \quad 2x_7^2-16x_8^2-50x_9^2=0,$$

which have the four distinct solutions $(x_7,x_8,x_9)=(5,0,\pm 1)$, $(x_7,x_8,x_9)=(15,\pm 5,-1)$, whence

$$L_0\cap V=\{[5e_7\pm e_9],[15e_7\pm 5e_8-e_9]\}\,. \tag{3.48}$$

We will also have to determine the intersection $V\cap L_1(r)$ explicitly. We have to solve the equations

$$\begin{gathered}-12r_1x_1^2+12x_7x_8+180x_8x_9=0,\\ -2r_1^2x_1^2+16x_1^2+2x_7^2-16x_8^2-50x_9^2=0,\\ -12x_1x_7+12r_1x_1x_8+180x_1x_9=0,\end{gathered}$$

in the variables x_1, x_7, x_8, x_9. We can check (e.g. with `Macaulay 2`) that the subscheme they define has dimension 0 (and degree 8) for $r_1 = 0$. We already know four solutions with $x_1 = 0$, namely the ones given in formula (48). Then it suffices to check that

$$(x_1, x_7, x_8, x_9) = (\pm 1, r_1, 1, 0),\ (x_1, x_7, x_8, x_9) = (\pm a, (90 - 5r_1^2), -5r_1, 6),$$

where a is a square-root of $25(r_1^2 - 36)$, are also solutions (with $x_1 \neq 0$ in a neighbourhood of 0 in R, and obviously all distinct there). Thus

$$\begin{gathered} L_1(r) \cap V = \{[\pm(e_1 + r_1 e_4) + r_1 e_7 + e_8], \\ [\pm(a e_1 + r_1 a e_4) + (90 - 5r_1^2)e_7 - 5r_1 e_8 + 6e_9]\}\,. \end{gathered} \tag{3.49}$$

We still have to see how the intersection points $L_0 \cap V$ and $L_1(r) \cap V$ are distributed among X_1 and X_2: Suppose $f \in V(8)$ is a binary octic such that $[f] \in L_0 \cap \mathbb{P}(V(8))$ or $[f] \in L_1(r) \cap \mathbb{P}(V(8))$; then f is a linear combination of the binary octics e_1, e_4, e_7, e_8, e_9 defined in (22), which involve only even powers of z_1 and z_2; thus if $(a : b) \in \mathbb{P}^1$ is a root of one of them, so is its negative $(a : -b)$ whence

> $[f]$ lies in X_1 if and only if $(1 : 0)$ or $(0 : 1)$ is a root of multiplicity ≥ 6.

Applying this criterion, we get, using (48) and (49)

$$\begin{gathered} L_0 \cap X_1 = \emptyset,\ L_0 \cap X_2 = \{[5e_7 \pm e_9], [15e_7 \pm 5e_8 - e_9]\}, \\ L_1(r) \cap X_1 = \{[\pm(e_1 + r_1 e_4) + r_1 e_7 + e_8]\}, \\ L_1(r) \cap X_2 = \{[\pm(a e_1 + r_1 a e_4) + (90 - 5r_1^2)e_7 - 5r_1 e_8 + 6e_9]\}\,. \end{gathered} \tag{3.50}$$

The reader may be glad to hear now that we do not have to repeat this entire procedure for $L_2(r)$ and $L_3(r)$; in fact, $L_1(r)$, $L_2(r)$, $L_3(r)$ are permuted by $N(H)$ in the following way: For the element $\sigma \in N(H)$ we have

$$\sigma \cdot L_1(r) = L_2(\sigma \cdot r), \quad \sigma \cdot L_2(r) = L_3(\sigma \cdot r), \quad \sigma \cdot L_3(r) = L_1(\sigma \cdot r)\,,$$

which follows from (30) (and (28)) and the definition of R. Thus we get that generally for $i = 1, 2, 3$

$$L_i(r) \cap X_1 = \{P_1(r),\ P_2(r)\},\ L_i(r) \cap X_2 = \{Q_1(r),\ Q_2(r)\} \tag{3.51}$$

where $P_1(r)$, $P_2(r)$, $Q_1(r)$, $Q_2(r)$ are mutually distinct points, and this is valid in a Zariski open $N(H)$-invariant neighbourhood of $0 \in R$. It remains to check that

$L(0) \cap V$ consists of 32 reduced points.

We check (with `Macaulay 2`) that if we substitute $x_4 = x_5 = x_6 = 0$ in equations (42)-(46), they define a zero-dimensional reduced subscheme of degree 32 in the projective space with coordinates $x_1, x_2, x_3, x_7, x_8, x_9$. Taking into account (47), (50), (51), we see that all the intersections in equations (50), (51) are free of multiplicities in an open $N(H)$-invariant neighbourhood of $0 \in R$ and moreover, since $\deg X_1 = 18$, $\deg X_2 = 14$, we must have there

$L^0(r) \cap X_1$ consists of 12 reduced points, and $L^0(r) \cap X_2$ consists of 4 reduced points.

Now these 4 points make up the H-orbit $\mathfrak{O}_r$ we wanted to find in Step 3: Clearly $L^0(r) \cap X_2$ is H-invariant, and H acts with trivial stabilizers in $L^0(r)$ (as is clear from (29)). Thus we have completed the program outlined at the beginning of Step 3. It just remains to notice that $[x^0] \in X_2 \cap L^0(0)$. This is clear since $[x^0] \in V$, but x^0 does not have a root of multiplicity ≥ 6.

Step 5. (Verification of the properties of $N(r)$) For the completion of the proof of Theorem 3.3.1, it remains to verify the properties of the subspace $N(r)$ in parts (2), (i) and (iii) of that theorem. First of all, it is clear that

$$N(r) = \langle \sigma_1(r), \sigma_2(r), (1:0:0:\cdots:0), \\ (0:1:0:\cdots:0), (0:0:1:\cdots:0) \rangle$$

is $N(H)$-invariant in the sense that $g \cdot N(r) = N(g \cdot r)$ for $g \in N(H)$ by the construction of σ_1, σ_2 and because the last three vectors in the preceding formula are a basis in the invariant subspace $\mathbb{P}(T) \subset \mathbb{P}^8$ (where by (31) $T = T_{(\chi_0)} \oplus T_{(\theta)}$). Moreover, by the definition of σ_1 in Step 2, and the formula (37) for $\sigma_2(0)$, one has $\dim N(0) = 4$, which thus holds also for $r \in R$ sufficiently close to 0.

Recall that N was defined to be $N := \mathbb{P}(V(8)_{(\theta)} \oplus V(4)_{(\theta)}) \subset \mathbb{P}^8$, and as such can be described in terms of the coordinates $(y_1 : y_2 : y_3 : y_7 : y_8 : \cdots : y_{12})$ in $\mathbb{P}^8$ as

$$N = \{y_1 = y_2 = y_3 = y_7 + 7y_9 = y_{10} = 0\}$$

(cf. (24)). Thus we get that $N(0) \cap N = \emptyset$, and the same holds in an open $N(H)$-invariant neighbourhood of 0 in R.

For Theorem 3.3.1, (2), (iii), it suffices to check that π_0 maps the fibre $p_{\mathbb{P}^8}(p_R^{-1}(0))$ dominantly onto N, which can be done by direct calculation. This concludes the proof. $\square$

3.3.4 Appendix A: Collection of formulas for section 2

We start with some remarks on how to calculate equivariant projections, and then we give explicit formulas for the equivariant maps in section 2.
Let a, b be nonnegative integers, $m := \min(a, b)$, and let $G := \mathrm{SL}_3\,\mathbb{C}$. We denote the irreducible G-module whose highest weight has numerical labels a, b by $V(a, b)$. For $k = 0, \dots, m$ we define $V^k := \mathrm{Sym}^{a-k}\,\mathbb{C}^3 \otimes \mathrm{Sym}^{b-k}(\mathbb{C}^3)^\vee$. Let e_1, e_2, e_3 be the standard basis in $\mathbb{C}^3$ and x_1, x_2, x_3 the dual basis in $(\mathbb{C}^3)^\vee$.
There are G-equivariant linear maps $\Delta^k : V^k \to V^{k+1}$ for $k = 0, \dots, m-1$ and $\delta^k : V^k \to V^{k-1}$ for $k = 1, \dots, m$ given by

$$\Delta^k := \sum_{i=1}^{3} \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i}, \quad \delta^k := \sum_{i=1}^{3} e_i \otimes x_i\,. \tag{3.52}$$

(The superscript k thus only serves as a means to remember the sources and targets of the respective maps). If for some positive integers α, β the G-module V^k contains a G-submodule isomorphic to $V(\alpha, \beta)$ we will denote it by $V^k(\alpha, \beta)$ to indicate the ambient module (this is unambiguous because it is known that all such modules occur with multiplicity one).
It is clear that Δ^k is surjective and δ^k injective; one knows that $\ker(\Delta^k) = V^k(a-k, b-k)$ whence

$$V^k = \bigoplus_{i=k}^{m} V^k(a-i, b-i)\,. \tag{3.53}$$

We want to find a formula for the G-equivariant projection of $V^0 = \mathrm{Sym}^a\,\mathbb{C}^3 \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee$ onto the subspace $V^0(a-i, b-i)$ for $i = 0, \dots, m$. We call this linear map $\pi^i_{a,b}$.
We remark that, by (53), one can decompose each vector $v \in V^0$ as $v = v_0 + \dots + v_m$ where $v_i \in V^0(a-i, b-i)$, and this decomposition is unique. Note that

$$\delta^1 \dots \delta^i(\ker \Delta^i) = V^0(a-i, b-i) \tag{3.54}$$

so that

$$V^0 = \ker\Delta^0 \oplus \delta^1(\ker\Delta^1) \oplus \delta^1\delta^2(\ker\Delta^2) \oplus \cdots \oplus \delta^1\ldots\delta^i(\ker\Delta^i) \\ \oplus \cdots \oplus \delta^1\ldots\delta^m(V^m)\,.$$

Of course, $\pi^i_{a,b}(v) = v_i$. It will be convenient to put

$$L^i := \delta^1 \circ \delta^2 \circ \cdots \circ \delta^i \circ \Delta^{i-1} \circ \cdots \circ \Delta^1 \circ \Delta^0\,, \quad i = 0, \ldots, m \tag{3.55}$$

(whence L^0 is the identity) and

$$U^i := \Delta^{i-1} \circ \Delta^{i-2} \circ \cdots \circ \Delta^0 \circ \delta^1 \circ \cdots \circ \delta^{i-1} \circ \delta^i\,, \quad i = 0, \ldots, m \tag{3.56}$$

(U^0 being again the identity). By Schur's lemma, we have

$$U^i|_{V^i(a-i,b-i)} = c_i \cdot \mathrm{id}_{V^i(a-i,b-i)}$$

for some nonzero rational number $c_i \in \mathbb{Q}^*$. This is easy to calculate: For example, since $e_1^{a-i} \otimes x_2^{b-i} \in \ker\Delta^i = V^i(a-i, b-i)$, we have that c_i is the unique number such that

$$U^i(e_1^{a-i} \otimes x_2^{b-i}) = c_i \cdot e_1^{a-i} \otimes x_2^{b-i}\,. \tag{3.57}$$

We will now calculate $\pi^{m-l}_{a,b}$ for $l = 0, \ldots, m$ by induction on l; the case $l = 0$ can be dealt with as follows:
Write $v = v_1 + \cdots + v_m \in V^0$ as before. Then $v_m = \delta^1\delta^2\ldots\delta^m(u_m)$ for some $u_m \in V^m$. Now

$$L^m(v) = L^m(v_m) = L^m(\delta^1\delta^2\ldots\delta^m(u_m)) \\ = \delta^1\delta^2\ldots\delta^m \circ U^m(u_m) = c_m v_m$$

so we set

$$\pi^m_{a,b} := \frac{1}{c_m} L^m\,. \tag{3.58}$$

Now assume, by induction, that $\pi^{m-l}_{a,b}$, $\pi^{m-l+1}_{a,b}, \dots, \pi^{m}_{a,b}$ have already been determined. We show how to calculate $\pi^{m-l-1}_{a,b}$.
Now, by (54), $v_{m-l-1} \in \delta^1 \dots \delta^{m-l-1}(\ker \Delta^{m-l-1})$. We write $v_{m-l-1} = \delta^1 \dots \delta^{m-l-1}(u_{m-l-1})$, for some $u_{m-l-1} \in \ker \Delta^{m-l-1} = V^{m-l-1}(a-(m-l-1), b-(m-l-1))$, and using (57) we get

$$\begin{aligned} L^{m-l-1}\left(v - \sum_{i=0}^{l} \pi^{m-i}_{a,b}(v)\right) &= L^{m-l-1}(v_0 + v_1 + \dots + v_{m-l-1}) \\ &= L^{m-l-1}(v_{m-l-1}) = L^{m-l-1}(\delta^1 \dots \delta^{m-l-1}(u_{m-l-1})) \\ &= \delta^1 \dots \delta^{m-l-1} \circ \Delta^{m-l-2} \dots \Delta^0 \circ \delta^1 \dots \delta^{m-l-1}(u_{m-l-1}) \\ &= \delta^1 \dots \delta^{m-l-1} \circ U^{m-l-1}(u_{m-l-1}) = c_{m-l-1} v_{m-l-1} \,. \end{aligned}$$

So we put

$$\pi^{m-l-1}_{a,b} := \frac{1}{c_{m-l-1}} \left(L^{m-l-1} \left(\mathrm{id}_{V^0} - \sum_{i=0}^{l} \pi^{m-i}_{a,b} \right) \right) . \tag{3.59}$$

Formulas (52), (55), (56), (57), (58), (59) contain the algorithm to compute the G-equivariant linear projection

$$\pi^i_{a,b} : V^0 \to V^0(a-i, b-i) \subset V^0$$

and thus to compute the associated G-equivariant bilinear map

$$\beta^i_{a,b} : V(a,0) \times V(0,b) \to V(a-i, b-i)$$

in suitable bases in source and target (remark that $V(a,0) = \mathrm{Sym}^a\, \mathbb{C}^3$ and $V(0,b) = \mathrm{Sym}^b\, (\mathbb{C}^3)^\vee$).

In particular, we obtain for $a=2$, $b=1$ the map

$$\begin{gathered} \pi^0_{2,1} \,:\, V^0 = \mathrm{Sym}^2\mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee \to V(2,1) \subset V^0 \\ \pi^0_{2,1} = \mathrm{id} - \frac{1}{4}\delta^1\Delta^0 \,, \end{gathered} \tag{3.60}$$

for $a = b = 2$ the map

$$\begin{aligned} \pi^0_{2,2} \,:\, V^0 = \mathrm{Sym}^2\mathbb{C}^3 \otimes \mathrm{Sym}^2(\mathbb{C}^3)^\vee \to V(2,2) \subset V^0 \\ \pi^0_{2,2} = \mathrm{id} - \frac{1}{5}\delta^1\Delta^0 + \frac{1}{40}\delta^1\delta^2\Delta^1\Delta^0 \,, \end{aligned} \tag{3.61}$$

and for $a = b = 1$ the map

$$\begin{aligned} \pi^0_{1,1} \,:\, V^0 = \mathbb{C}^3 \otimes (\mathbb{C}^3)^\vee \to V(1,1) \subset V^0 \\ \pi^0_{1,1} = \mathrm{id} - \frac{1}{3}\delta^1\Delta^0 \,. \end{aligned} \tag{3.62}$$

In the following, we will often view elements $x \in V(a,b)$ as tensors $x = (x^{i_1,\dots,i_b}_{j_1,\dots,j_a}) \in (\mathbb{C}^3)^{\otimes a} \otimes (\mathbb{C}^{3\vee})^{\otimes b} =: T^b_a\mathbb{C}^3$ (the indices ranging from 1 to 3) which are covariant of order b and contravariant of order a via the natural inclusions

$$V(a,b) \subset \mathrm{Sym}^a\mathbb{C}^3 \otimes \mathrm{Sym}^b(\mathbb{C}^3)^\vee \subset T^b_a\mathbb{C}^3$$

(the first inclusion arises since $V(a,b)$ is the kernel of Δ^0, the second is a tensor product of symmetrization maps). In particular, we have the determinant tensor $\det \in T^3_0\mathbb{C}^3$ and its inverse $\det^{-1} \in T^0_3\mathbb{C}^3$. In formulas involving several tensors, we will also adopt the summation convention throughout. Finally, we define

$$\begin{aligned} \mathrm{can} \,:\, T^b_a\mathbb{C}^3 \to \mathrm{Sym}^a\,\mathbb{C}^3 \otimes \mathrm{Sym}^b\,(\mathbb{C}^3)^\vee \,, \\ e_{j_1} \otimes \cdots \otimes e_{j_a} \otimes x_{i_1} \otimes \cdots \otimes x_{i_b} \mapsto e_{j_1} \cdot \cdots \cdot e_{j_a} \otimes x_{i_1} \cdot \cdots \cdot x_{i_b} \end{aligned} \tag{3.63}$$

as the canonical projection.
We write down the explicit formulas for the equivariant maps in section 2. The map $\Psi : V(0,4) \to V(2,2)$ (degree 3) is given by

$$\begin{aligned} \Psi(f) := \pi^0_{2,2}(\mathrm{can}(g))\,, \\ g^{i_1i_2}_{j_1j_2} := f^{i_1i_2i_3i_4} f^{i_5i_6i_7i_8} f^{i_9i_{10}i_{11}i_{12}} \det\nolimits^{-1}_{i_3i_5i_9} \det\nolimits^{-1}_{i_4i_6i_{10}} \det\nolimits^{-1}_{j_1i_7i_{11}} \det\nolimits^{-1}_{j_2i_8i_{12}} \,. \end{aligned} \tag{3.64}$$

The map $\Phi : V(2,2) \times V(0,2) \to V(2,1)$ (bilinear) is given by

$$\begin{aligned} \Phi(g,h) := \pi^0_{2,1}(\mathrm{can}(r))\,, \\ r^{i_1}_{j_1j_2} := g^{i_1i_2}_{j_1i_3} h^{i_3i_4} \det\nolimits^{-1}_{i_2i_4j_2} \,. \end{aligned} \tag{3.65}$$

The map $\epsilon : V(0,4) \times V(0,2) \to V(2,2)$ (bilinear) is

$$\epsilon(f,h) := \mathrm{can}(g),\ g^{i_1 i_2}_{j_1 j_2} := f^{i_3 i_4 i_1 i_2} h^{i_5 i_6} \mathrm{det}^{-1}_{i_3 j_1 i_5} \mathrm{det}^{-1}_{i_4 j_2 i_6}\,. \tag{3.66}$$

The map $\zeta : V(0,4) \times V(0,2) \to V(1,1)$ (homogeneous of degree 2 in both factors) is given by

$$\zeta(f,h) := \pi^0_{1,1}(a)\,, \tag{3.67}$$
$$a^{i_1}_{j_1} := h^{i_1 i_2} h^{i_3 i_4} f^{i_5 i_6 i_7 i_8} f^{i_9 i_{10} i_{11} i_{12}} \mathrm{det}^{-1}_{i_5 i_9 j_1} \mathrm{det}^{-1}_{i_6 i_{10} i_2} \mathrm{det}^{-1}_{i_7 i_{11} i_3} \mathrm{det}^{-1}_{i_8 i_{12} i_4}\,.$$

The map $\tilde{\gamma} : V(2,2) \to V(1,1)$ (homogeneous of degree 2) is given by

$$\tilde{\gamma} := \pi^0_{1,1}(u)\,,\ u^{i_1}_{j_1} := g^{i_1 i_2}_{i_3 i_4} g^{i_3 i_4}_{j_1 i_2}\,. \tag{3.68}$$

3.3.5 Appendix B: Collection of formulas for section 3

In section 3.1, we saw (formula (26)) that

$$\delta_\lambda = Q_1(x,s)a_1 + Q_2(x,s)a_2 + Q_3(x,s)a_3 + Q_4(x,s)a_4 + Q_5(x,s)a_5\,. \tag{3.69}$$

We collect here the explicit values of the $Q_i(x,s)$ (recall $\lambda = (1, 6\epsilon, 1, 6)$, $\epsilon \neq 0$):

$$Q_1(x,s) = \hat{Q}_1(x) + 2x_7 s_1 + 12 x_8 s_2 + 2 x_9 s_1 + \epsilon(12 s_1 s_2) + 2 s_0 s_1 \tag{3.70}$$
$$+48 x_2 s_4 - 48 x_3 s_5 - 2 x_4 s_3 + 16 x_5 s_4 - 16 x_6 s_5 + \epsilon(-12 s_4^2 - 12 s_5^2)\,,$$
$$Q_2(x,s) = \hat{Q}_2(x) + 4 x_8 s_1 + 12 x_9 s_2 + \epsilon(2 s_1^2 - 6 s_2^2) + 2 s_0 s_2 \tag{3.71}$$
$$-4 x_1 s_3 + 16 x_2 s_4 + 16 x_3 s_5 - 16 x_5 s_4 - 16 x_6 s_5 + \epsilon(-2 s_3^2 - 4 s_4^2 + 4 s_5^2)\,,$$
$$Q_3(x,s) = \hat{Q}_3(x) + 2 x_4 s_1 + 12 x_1 s_2 + 64 x_2 s_5 + 64 x_3 s_4 \tag{3.72}$$
$$-2 x_7 s_3 + 2 x_9 s_3 + \epsilon(12 s_2 s_3 - 24 s_4 s_5) + 2 s_0 s_3\,,$$
$$Q_4(x,s) = \hat{Q}_4(x) + 4 x_5 s_1 + 12 x_2 s_1 - 12 x_5 s_2 + 12 x_2 s_2 \tag{3.73}$$
$$-8 x_1 s_5 - 16 x_3 s_3 + 8 x_8 s_4 - 8 x_9 s_4 + \epsilon(-6 s_1 s_4 - 6 s_2 s_4 + 6 s_3 s_5) + 2 s_0 s_4\,,$$
$$Q_5(x,s) = \hat{Q}_5(x) + 4 x_6 s_1 + 12 x_3 s_1 + 12 x_6 s_2 - 12 x_3 s_2 \tag{3.74}$$
$$+8 x_1 s_4 - 16 x_2 s_3 - 8 x_8 s_5 - 8 x_9 s_5 + \epsilon(6 s_1 s_5 - 6 s_2 s_5 - 6 s_3 s_4) + 2 s_0 s_5\,,$$

where

$$\hat{Q}_1(x) = -192x_6^2 - 192x_3x_6 + 384x_3^2 - 192x_5^2 - 192x_2x_5 + 384x_2^2 \\ -12x_1x_4 + 12x_7x_8 + 180x_8x_9, \tag{3.75}$$

$$\hat{Q}_2(x) = 64x_6^2 - 192x_3x_6 - 128x_3^2 - 64x_5^2 + 192x_2x_5 + 128x_2^2 \\ -2x_4^2 + 16x_1^2 + 2x_7^2 - 16x_8^2 - 50x_9^2, \tag{3.76}$$

$$\hat{Q}_3(x) = 96x_5x_6 - 672x_3x_5 - 672x_2x_6 + 1248x_2x_3 \\ -12x_1x_7 + 12x_4x_8 + 180x_1x_9, \tag{3.77}$$

$$\hat{Q}_4(x) = 6x_4x_6 + 42x_3x_4 + 84x_1x_6 + 156x_1x_3 \\ -6x_5x_7 - 42x_2x_7 + 24x_5x_8 - 264x_2x_8 + 30x_5x_9 - 30x_2x_9, \tag{3.78}$$

$$\hat{Q}_5(x) = -6x_4x_5 - 42x_2x_4 + 84x_1x_5 + 156x_1x_2 \\ +6x_6x_7 + 42x_3x_7 + 24x_6x_8 - 264x_3x_8 - 20x_6x_9 + 30x_3x_9\,. \tag{3.79}$$

The polynomials $q_1, \ldots, q_5$ defining $\tilde{Y}_\lambda \subset R \times \mathbb{P}^8$ (cf. Theorem 3.2.1) are:

$$q_1 = (-192r_3^2 - 192r_3 + 384)y_1y_2 + (-192r_2^2 - 192r_2 + 384)y_1y_3 \\ +(-12r_1)y_2y_3 + 12y_7y_8 + 180y_8y_9 + 2y_7y_{11} + 12y_8y_{12} \\ +2y_9y_11 + \epsilon(12y_{11}y_{12}) + 2y_{10}y_{11}\,, \tag{3.80}$$

$$q_2 = (64r_3^2 - 192r_3 - 128)y_1y_2 + (-64r_2^2 + 192r_2 + 128)y_1y_3 \\ +(-2r_1^2 + 16)y_2y_3 + 2y_7^2 - 16y_8^2 - 50y_9^2 + 4y_8y_{11} + 12y_9y_{12} \\ +\epsilon(2y_{11}^2 - 6y_{12}^2) + 2y_{10}y_{12}\,, \tag{3.81}$$

$$q_3 = (96r_2r_3 - 672r_2 - 672r_3 + 1248)y_1 \\ -12y_7 + 12r_1y_8 + 180y_9 + 2r_1y_{11} + 12y_{12}\,, \tag{3.82}$$

$$q_4 = (6r_1r_3 + 42r_1 + 84r_3 + 156)y_2 + (-6r_2 - 42)y_7 + (24r_2 - 264)y_8 \\ +(30r_2 - 30)y_9 + (4r_2 + 12)y_{11} + (-12r_2 + 12)y_{12}\,, \tag{3.83}$$

$$q_5 = (-6r_1r_2 - 42r_1 + 84r_2 + 156)y_3 + (6r_1 + 42)y_7 + (24r_3 - 264)y_8 \\ +(-30r_3 + 30)y_9 + (4r_3 + 12)y_{11} + (12r_3 - 12)y_{12}\,. \tag{3.84}$$

Chapter 4

The rationality of the moduli spaces of plane curves of sufficiently large degree

4.1 Introduction

In this chapter we deal with the question whether the orbit spaces $\mathbb{P}/G$ are rational in an "asymptotic way" (where as usual $\mathbb{P}$ is a projective space and G is a reductive algebraic group acting linearly in $\mathbb{P}$). We first recall some general structural results. If G is not assumed connected, in fact for G a finite solvable group, D. Saltman has shown in [Sa] that the answer to this question is negative in general (Emmy Noether had apparently conjectured that the quotient should be rational in this case). No counterexamples are known for connected complex reductive groups G.

For simply connected classical groups except $\mathrm{Spin}_n(\mathbb{C})$ for $n > 12$, the quotients $\mathbb{P}/G$ are known to be stably rational, cf. [Bogo1], [CT-S]. [Bogo1] claims the result for all $\mathrm{Spin}_n(\mathbb{C})$ but the proof contains a mistake pointed out by P. Katsylo. The question whether stably rational varieties are always rational is the well-known Zariski problem which Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer [B-CT-S-SwD] answered in the negative as well: There are three-dimensional conic bundles X over rational surfaces which are irrational, but $X \times \mathbb{P}^3$ is rational. This uses the method of intermediate Jacobians by Clemens-Griffiths [C-G] which, however, seems to work only for threefolds. In general, it is rather hard to distinguish stably rational and rational varieties. The method connected with the birational invariance of the Brauer-Grothendieck group used previously by Artin and Mumford [A-M] to obtain more elementary examples of unirational non-rational threefolds, is insensitive to this distinction (unirational varieties constitute a strictly bigger class than stably rational ones; e.g. Saltman's counterexamples mentioned above are not even stably rational). The reader may find these and other methods to prove irrationality, including the use of Noether-Fano inequalities via untwisting of birational maps and Kollár's method of differential forms in characteristic p to prove non-rationality of some general hypersurfaces, in the survey by V. A. Iskovskikh and Yu. G. Prokhorov [Is-Pr].

The geometrically most relevant case of the general question discussed above seems to be the case of the moduli space of projective hypersurfaces of degree d in $\mathbb{P}^n$, which we denote by $\mathrm{Hyp}(d, n)$. Here rationality is known in the following cases:

- $n = 1$ (the classical case of binary forms resp. sets of points on the projective line), d odd [Kat83], d even [Bo-Ka], [Bogo2]
- $n = 2$, $d \leq 3$ (well known), $d = 4$ ([Kat92/2], [Kat96]), $d \equiv 1 \pmod 4$ ([Shep]), $d \equiv 1 \pmod 9$ and $d \geq 19$ ([Shep]); $d \equiv 0 \pmod 3$ and $d \geq 1821$ [Kat84] (this article contains the remark that the author obtained the result also for $d \geq 210$, unpublished); the same paper also gives some results for congruences to the modulus 39; furthermore, there are some unpublished additional cases in the case of plane curves which we do not try to enumerate.
- $n = 3$, $d \leq 2$ (obvious), $d = 3$ (Clebsch and Salmon; but see [Be]).
- $n > 3$, $d \leq 2$ (obvious).

This represents what we could extract from the literature. It is hard to say if it is exhaustive. The reader may consult the very good (though not recent) survey article [Dolg1] for much more information on the rationality problem for fields of invariants.
The main theorem of the present chapter is

Theorem 4.1.0.1. *The moduli space of plane curves of sufficiently large degree $d >> 0$ under projective equivalence is rational.*

More precisely, for $d = 3n$, $d \geq 1821$, this was proven by Katsylo [Kat84] as a glance back at the preceding summary shows. We use this result and don't improve the bound for d. For $d \equiv 1 \pmod 3$, we obtain rationality for $d \geq 37$. For $d \equiv 2$ (mod3), we need $d \geq 65$.

Let us turn to some open problems. First of all, the method used in this paper seems to generalize and -provided the required genericity properties hold and can be verified computationally- could yield a proof of the rationality of $\mathrm{Hyp}(d,\, n)$ for *fixed* n if the degree d is large enough and $n+1$ does not divide d. The latter case might be amenable to the techniques of [Kat84] in general. Thus the case of the moduli spaces of surfaces of degree d in $\mathbb{P}^3$ seems now tractable with some diligence and effort. But we do not see how one could obtain results on $\mathrm{Hyp}(d,\, n)$ *for all* n (and d sufficiently large compared to n).

More importantly, whereas we think that it is highly plausible that $\mathrm{Hyp}(d, n)$ is always rational if d is sufficiently large compared to n, we do not want to hazard any guess in the case where d is small. In fact, we do not know any truely convincing philosophical reason why $\mathrm{Hyp}(d,\, n)$ should be rational in general; the present techniques of proving rationality always seem to force one into assuming that d is sufficiently large if one wants to obtain an infinite series of rational examples by a uniform method. Moreover, it can be quite painstaking and tricky to get a hold of the situation if d is small as Katsylo's tour de force proof for $\mathfrak{M}_3$ (i.e. Hyp(4, 2)) in [Kat92/2], [Kat96] amply illustrates. The maybe easiest unsolved cases are Hyp(6, 2) (plane sextics) and Hyp(4, 3) (quartic surfaces). Note that the former space is birational to the moduli space of polarized K3 surfaces $(S,\, h)$ of degree 2 (thus S is a nonsingular projective K3 surface and $h \in \mathrm{Pic}(S)$ is the class of an ample divisor with $h^2 = 2$), and the latter space is birational to the moduli space of polarized K3 surfaces of degree 4.

We would like to thank Fedor Bogomolov and Yuri Tschinkel for suggesting this problem and helpful discussions.

4.2 Proof of rationality

4.2.1 Outline of proof

The structural pattern of the proof is similar to [Shep]; there the so called *method of covariants* is introduced, and we learnt a lot from studying that source.
We fiber the space $\mathbb{P}\left(\mathrm{Sym}^d(\mathbb{C}^3)^\vee\right)$ of degree d plane curves over the space of plane quartics, if $d \equiv 1 \pmod 3$, and over the space of plane octics, if $d \equiv 2 \pmod 3$, i.e. we construct $\mathrm{SL}_3(\mathbb{C})$-equivariant maps

$$S_d \colon \mathrm{Sym}^d(\mathbb{C}^3)^\vee \to \mathrm{Sym}^4(\mathbb{C}^3)^\vee$$

and

$$T_d \colon \mathrm{Sym}^d(\mathbb{C}^3)^\vee \to \mathrm{Sym}^8(\mathbb{C}^3)^\vee$$

(S_d coincides with the covariant used in [Shep] for the case $d = 9n + 1$). These maps are of degree 4 as polynomials in the coordinates on the source spaces, i.e. of degree 4 in the curve coefficients. They are constructed via the symbolic method recalled in section 4.2.2. Furthermore they induce *dominant* rational maps on the associated projective spaces. We remark here that the properties of S_d and T_d essential for the proof are that they are of fixed low degree in the curve coefficients, take values in spaces of curves of fixed low degree, and are sufficiently generic.

We now focus on the case $d \equiv 1 \pmod 3$. The proof has three main steps:

(1) $\mathrm{Hyp}(4,\,2)$ is stably rational, more precisely its product with $\mathbb{P}^8$ is rational; cf. [Bo-Ka], Theorem 1.1 for this.

(2) We find a linear subspace $L_S \subset \mathrm{Sym}^d(\mathbb{C}^3)^\vee$ such that $\mathbb{P}(L_S)$ is contained in the base locus B_{S_d} of S_d with a full triple structure, i.e. $I^3_{\mathbb{P}(L_S)} \supset I_{B_{S_d}}$, and consider the projection π_{L_S} away from $\mathbb{P}(L_S)$ onto $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee/L_S)$. We show that a general fibre of S_d is birationally a vector bundle over a rational base.

(3) The quotient map

$$\mathbb{P}\mathrm{Sym}^4(\mathbb{C}^3)^\vee \dashrightarrow \left(\mathbb{P}\mathrm{Sym}^4(\mathbb{C}^3)^\vee\right)/\mathrm{PGL}_3(\mathbb{C})$$

has a section σ_4. Pulling back the linear fibrations constructed in (2) via σ_4 we show that the moduli space of plane curves of degree d is birational to $\mathrm{Hyp}(4,\,2) \times \mathbb{P}^N$, where N is large, whence we conclude by (1).

The main computational difficulty occurs in (2) where we have to establish that L_S is sufficiently generic. Projecting from $\mathbb{P}(L_S)$ we obtain a diagram

$$\begin{tikzcd} \mathbb{P}\mathrm{Sym}^d(\mathbb{C}^3)^\vee \arrow[r, dashed, "S_d"] \arrow[d, dashed, "\pi_{L_S}"] & \mathbb{P}\mathrm{Sym}^4(\mathbb{C}^3)^\vee \\ \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee/L_S) & \end{tikzcd}$$

We show that

(∗) for a particular (hence a general) $\bar{g} \in \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee/L_S)$ the map

$$S_d|_{\mathbb{P}(L_S+\mathbb{C}g)} \colon \mathbb{P}(L_S + \mathbb{C}g) \dashrightarrow \mathbb{P}\mathrm{Sym}^4(\mathbb{C}^3)^\vee$$

is surjective.

Note that $S_d|_{\mathbb{P}(L_S+\mathbb{C}g)}$ is linear since L_S is contained in the base locus with full triple structure and S_d is of degree 4 in the curve coefficients. It is therefore enough to explicitly construct points in the image that span $\mathbb{P}\mathrm{Sym}^4(\mathbb{C}^3)^\vee$. From this it follows at once that a general fibre of S_d is mapped dominantly by π_{L_S} whence we may view such a fibre birationally as a vector bundle over a rational base. To understand better why the dominance of S_d is not sufficient here, it is instructive to keep the following example in mind:

Example 4.2.1.1. Consider the rational map

$$S \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^1, \quad x \mapsto (Q_1(x) : Q_2(x))$$

where Q_1, Q_2 are quadric cones with vertex L. S is dominant. Projection from the vertex L to $\mathbb{P}^2$ is also dominant, but the quadric cones (i.e. the fibers of S) do not map dominantly to $\mathbb{P}^2$ (see Figure **??**). The projection fibers are the lines through L and indeed each such line is contained completely in one cone in the pencil $\lambda Q_1 + \mu Q_2$.

The base locus B of S consists of 4 lines that meet in L. If on the other hand we project from a smooth point $L' \subset B$ then a general fiber of S maps dominantly to $\mathbb{P}^2$. Indeed a general line through L' intersects all cones.

The complications in proving (∗) arise due to the fact that the natural description of L_S is in terms of the *monomials* which span it, whereas S_d can be most easily evaluated on forms which are written *as sums of powers of linear forms*. These two points of view do not match, and we cannot repose on methods in [Shep]. Instead we introduce new techniques in section 4.2.4 to solve this difficulty:

- We use *interpolation polynomials* to write down elements in L_S as *sums of powers of linear forms*.
- Next we employ considerations of *leading terms* (or, geometrically, jets at infinity) to eliminate the interpolation polynomials, from our formulae.
- For large enough $d = 3n+1$, we finally reduce $(*)$ to the property that a certain matrix $M(n)$ has full rank. The size of $M(n)$ is independent of n while (*and this is the main point*) its entries are of the form

$$\sum_{\nu} \rho_\nu^n P_\nu(n)$$

  where P_ν are polynomials of fixed degree (i.e. independent of n), ρ_ν are constants, and the number of summands in the expression is independent of n. This is possible only because we eliminated the interpolation polynomials in the previous step.
- By choosing a point g with integer coefficients we can arrange that ρ_ν and $P_\nu(n)$ are defined over $\mathbb{Q}$ with denominators that are not divisible by a small prime $\wp$ which we call the precision of our calculation. Thus if we work over *the finite field* $\mathbb{F}_\wp$, the matrix $M(n)$ is periodic in n with period $\wp(\wp-1)$. A computer algebra program is then used to check that these matrices all have full rank. By semicontinuity, this proves that $M(n)$ has full rank for all n in characteristic 0.

In a rather round-about sense, we have also been guided by the principle that *evaluation of a polynomial* at a special point can be much cheaper than *computing the polynomial*.

4.2.2 Notation and definition of the covariants

For definiteness, the base field will be $\mathbb{C}$, the field of complex numbers, though one might replace it by any algebraically closed field of characteristic 0 throughout.
Let $G := \mathrm{SL}_3(\mathbb{C})$, and let $\bar{G} := \mathrm{PGL}_3(\mathbb{C})$ be the adjoint form of G. We denote by $V(k)$ the irreducible G-representation $\mathrm{Sym}^k(\mathbb{C}^3)^\vee$. We fix a positive integer d not divisible by 3, $d = 3n+1$ or $d = 3n+2$, $n \in \mathbb{N}$.
The symbol $[k]$, $k \in \mathbb{N}$, denotes the set of integers from 0 (incl.) to k (incl.).

Let $x_1,\ x_2,\ x_3 \in (\mathbb{C}^3)^\vee$ denote the basis dual to the standard basis in $\mathbb{C}^3$ and put $\mathbf{x} := (x_1,\ x_2,\ x_3)$. We will use Schwartz's multi-index notation and denote multi-indices by lower case boldface letters. Thus we write a general homogeneous form $f \in V(d)$ of degree d as

$$f = \sum_{\substack{\mathbf{i}\in[d]^3\\ |\mathbf{i}|=d}} \frac{d!}{\mathbf{i}!} A_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\,, \tag{4.1}$$

where $\mathbf{i}! = i_1!i_2!i_3!$, $|\mathbf{i}| := i_1 + i_2 + i_3$, $A_{\mathbf{i}} = A_{i_1 i_2 i_3}$, $\mathbf{x}^{\mathbf{i}} = x_1^{i_1}x_2^{i_2}x_3^{i_3}$. We will use the symbolical method introduced by Aronhold and Clebsch to write down G-equivariant maps (covariants) from $V(d)$ to $V(4)$ (if $d = 3n + 1$) or to $V(8)$ (if $d = 3n + 2$). It is explained in [G-Y] and, from a modern point of view, in [Dolg2], chapter 1. We denote by $\alpha = (\alpha_1,\ \alpha_2,\ \alpha_3)$ a vector of symbolic variables, and also introduce vectors $\beta,\ \gamma,\ \delta$, similarly. We write $\alpha_{\mathbf{x}} = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, and similarly $\beta_{\mathbf{x}},\ \gamma_{\mathbf{x}},\ \delta_{\mathbf{x}}$. Moreover we define the bracket factor $(\alpha\,\beta\,\gamma)$ by

$$(\alpha\,\beta\,\gamma) := \det\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

and write $(\alpha\,\beta\,\delta)$ etc. similarly. The idea in this calculus is to write $f \in V(d)$ symbolically as a power of a linear form in several ways:

$$f = \alpha_{\mathbf{x}}^d = \beta_{\mathbf{x}}^d = \gamma_{\mathbf{x}}^d = \delta_{\mathbf{x}}^d\,, \tag{4.2}$$

whence the identities

$$A_{\mathbf{i}} = \alpha^{\mathbf{i}} = \beta^{\mathbf{i}} = \gamma^{\mathbf{i}} = \delta^{\mathbf{i}}\,. \tag{4.3}$$

If $d = 3n + 1$, define a covariant $S_d : V(d) \to V(4)$ of order 4 and degree 4 by the following prescription:

$$I(\alpha,\ \beta,\ \gamma,\ \delta) := (\alpha\,\beta\,\gamma)(\alpha\,\beta\,\delta)(\alpha\,\gamma\,\delta)(\beta\,\gamma\,\delta) \tag{4.4}$$

$$S_d(\alpha,\ \beta,\ \gamma,\ \delta) := I^n\alpha_{\mathbf{x}}\beta_{\mathbf{x}}\gamma_{\mathbf{x}}\delta_{\mathbf{x}}\,. \tag{4.5}$$

The formula for S_d should be read in the following way: The right hand side of (5), when we multiply it out formally, is a sum of monomials $\alpha^{\mathbf{i}}\beta^{\mathbf{j}}\gamma^{\mathbf{k}}\delta^{\mathbf{l}}\mathbf{x}^{\mathbf{e}}$, $\mathbf{i},\ \mathbf{j},\ \mathbf{k},\ \mathbf{l} \in [d]^3$, $\mathbf{e} \in [4]^3$, and $|\mathbf{i}| = \cdots = |\mathbf{l}| = d$, $|\mathbf{e}| = 4$. Thus one can use equations (1) and (3) to rewrite the right-hand-side unambiguously in terms of the coefficients $A_{\mathbf{i}}$ of $f \in V(d)$. Hence S_d may be viewed as a map from $V(d)$ to $V(4)$, homogeneous of degree 4 in the coefficients $A_{\mathbf{i}}$, which is clearly G-equivariant. By abuse of notation, we denote the induced rational map

$$S_d : \mathbb{P}V(d) \dashrightarrow \mathbb{P}V(4) \tag{4.6}$$

by the same letter. Note that I defined by equation (4) may be viewed as an invariant of plane cubics $I : V(3) \to \mathbb{C}$ of degree 4 in the coefficients of the cubic. In fact, this is the famous Clebsch invariant, vanishing on the locus of Fermat cubics, or vanishing on the equi-anharmonic cubics, i.e. nonsingular plane cubics which can be written as a double cover of $\mathbb{P}^1$ branched in four points with equi-anharmonic cross-ratio. Equi-anharmonic cross-ratio means cross-ratio equal to minus a cube root of 1. Equi-anharmonic quadruples of points in $\mathbb{P}^1$ are one of the two possible $\mathrm{PGL}_2\,\mathbb{C}$-orbits of 4 points in $\mathbb{P}^1$ with non-trivial isotropy group (the other orbit being quadruples with harmonic cross-ratio, i.e. equal to -1, $1/2$ or 2). See [D-K], (5.13), for details.
The letter S in S_d was chosen in honor of the 19th century Italian geometer Gaetano Scorza, who studied in detail the map S_4, called the *Scorza map* (cf. [D-K], §6 and §7, and [Dolg3], section 6.4.1).

Similarly, for $d = 3n + 2$, we define a covariant $T_d : V(d) \to V(8)$ of order 8 and degree 4 by

$$T_d(\alpha,\, \beta,\, \gamma,\, \delta) := I^n \alpha_{\mathbf{x}}^2 \beta_{\mathbf{x}}^2 \gamma_{\mathbf{x}}^2 \delta_{\mathbf{x}}^2 \,. \tag{4.7}$$

and denote the induced rational map $T_d : \mathbb{P}V(d) \to \mathbb{P}V(8)$ by the same letter.

We remark that it is hard to calculate the values of S_d (or T_d) on a general homogeneous form f of degree d without knowing the entire expression of S_d (resp. T_d) as a polynomial in the coefficients $A_{\mathbf{i}}$, which is awkward. One can, however, work directly with the symbolic expressions given in (5) and (7) if one writes f as a linear combination of d-th powers of linear forms:

$$f = \lambda_1 l_1^d + \dots + \lambda_N l_N^d, \quad \text{some} \quad N \in \mathbb{N}\,. \tag{4.8}$$

For linear forms $l_i,\, l_j,\, l_k,\, l_p \in \mathbb{C}[x_1,\, x_2,\, x_3]_1$ we use the notation

$$I(l_i,\, l_j,\, l_k,\, l_p), \quad S_d(l_i,\, l_j,\, l_k,\, l_p), \quad T_d(l_i,\, l_j,\, l_k,\, l_p) \tag{4.9}$$

which is defined via formulas (4), (5), (7), but where for the vectors α, β, γ, δ of symbolic variables we substitute the *vectors of coordinates* w.r.t. x_1, x_2, x_3 of l_i, l_j, l_k, and l_p. One then has the following easy, but fundamental multi-linearity properties of S_d and T_d whose proof is a straight-forward computation and therefore omitted.

Lemma 4.2.2.1. *We have*

$$S_d(f) = 24 \sum \lambda_i \lambda_j \lambda_k \lambda_p S_d(l_i,\, l_j,\, l_k,\, l_p) \tag{4.10}$$

$$T_d(f) = 24 \sum \lambda_i \lambda_j \lambda_k \lambda_p T_d(l_i,\, l_j,\, l_k,\, l_p) \tag{4.11}$$

The right-hand sums run over all $(i,\, j,\, k,\, p)$ *with* $1 \le i < j < k < p \le N$.

4.2.3 Special linear subspaces of the base loci

The group $G = \mathrm{SL}_3 \mathbb{C}$ is a rank 2 complex semisimple algebraic group, and choosing the standard torus T of diagonal matrices as maximal torus, and the group of upper-triangular matrices as Borel subgroup, one has the notions of roots, positive and simple roots, and simple coroots H_1, H_2 available. Corresponding to H_1, H_2 one has one-parameter subgroups $\lambda_{H_1}, \lambda_{H_2} : \mathbb{C}^* \to T$ given by

$$\lambda_{H_1}(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \lambda_{H_2}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}. \tag{4.12}$$

For $d = 3n+1$, we may view the covariant S_d as an element in

$$\left(\mathrm{Sym}^4 V(d)^\vee \otimes V(4)\right)^G, \tag{4.13}$$

a G-invariant polynomial of degree 4 in the curve coefficients $A_{\mathbf{i}}$, $\mathbf{i} \in [d]^3$, $|\mathbf{i}| = d$, with values in $V(4)$. As such it is a linear combination of monomials

$$A_{\mathbf{i}} A_{\mathbf{j}} A_{\mathbf{k}} A_{\mathbf{l}} \mathbf{x}^{\mathbf{e}}, \tag{4.14}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l} \in [d]^3$, $|\mathbf{i}| = \cdots = |\mathbf{k}| = d$, $\mathbf{e} \in [4]^3$, $|\mathbf{e}| = 4$.
Similarly, for $d = 3n+2$, T_d can be viewed as an element of

$$\left(\mathrm{Sym}^4 V(d)^\vee \otimes V(8)\right)^G, \tag{4.15}$$

i.e. a G-invariant polynomial of degree 4 in the curve coefficients $A_{\mathbf{i}}$, $\mathbf{i} \in [d]^3$, $|\mathbf{i}| = d$, with values in $V(8)$. It is a linear combination of monomials

$$A_{\mathbf{i}} A_{\mathbf{j}} A_{\mathbf{k}} A_{\mathbf{l}} \mathbf{x}^{\mathbf{e}}, \tag{4.16}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l} \in [d]^3$, $|\mathbf{i}| = \cdots = |\mathbf{k}| = d$, $\mathbf{e} \in [8]^3$, $|\mathbf{e}| = 8$.
The following proposition is an important ingredient in the proof of rationality.

Proposition 4.2.3.1. *For $d = 3n+1$, the projectivization of the linear space*

$$L_S = x_1^{2n+3} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-2} \subset V(d) \tag{4.17}$$

is contained in the base scheme B_S of the rational map

$$S_d : \mathbb{P}V(d) \dashrightarrow \mathbb{P}V(4)$$

with a full triple structure, i.e.

$$\mathcal{I}^3_{\mathbb{P}(L_S)} \supset \mathcal{I}_{B_S}\,.$$

Similarly, for $d = 3n+2$, the projectivization of the linear space

$$L_T = x_1^{2n+5} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-3} \subset V(d) \tag{4.18}$$

is contained in the base scheme B_T of

$$T_d : \mathbb{P}V(d) \dashrightarrow \mathbb{P}V(8)$$

with a full triple structure.

Proof. Regardless of whether $d = 3n+1$ or $d = 3n+2$, the conditions that the monomials in (14) or (16) are invariant under the actions of the one-parameter subgroups λ_{H_1} resp. λ_{H_2} read

$$i_1 + j_1 + k_1 + l_1 - i_2 - j_2 - k_2 - l_2 - e_1 + e_2 = 0 \quad \text{resp.} \tag{4.19}$$
$$i_2 + j_2 + k_2 + l_2 - i_3 - j_3 - k_3 - l_3 - e_2 + e_3 = 0\,. \tag{4.20}$$

Now for $d = 3n+1$ we get

$$\begin{aligned} 4(3n+1) &= |\mathbf{i}| + |\mathbf{j}| + |\mathbf{k}| + |\mathbf{l}| \\ &= 3(i_2 + j_2 + k_2 + l_2) + (e_1 - e_2) + (e_3 - e_2) \\ &= 3(i_2 + j_2 + k_2 + l_2) + 4 - 3e_2. \end{aligned} \tag{4.21}$$

and for $d = 3n + 2$ one has

$$\begin{aligned} 4(3n+2) &= |\mathbf{i}| + |\mathbf{j}| + |\mathbf{k}| + |\mathbf{l}| \\ &= 3(i_2 + j_2 + k_2 + l_2) + (e_1 - e_2) + (e_3 - e_2) \\ &= 3(i_2 + j_2 + k_2 + l_2) + 8 - 3e_2. \end{aligned} \tag{4.22}$$

In both cases then it follows that

$$\begin{aligned} i_1 + j_1 + k_1 + l_1 &= 4n + e_1, \\ i_2 + j_2 + k_2 + l_2 &= 4n + e_2, \\ i_3 + j_3 + k_3 + l_3 &= 4n + e_3. \end{aligned} \tag{4.23}$$

In particular, for $d = 3n + 1$, $i_1 + j_1 + k_1 + l_1 \leq 4n + 4$, which means that at most 1 out of the 4 indices i_1, j_1, k_1, l_1 can be $\geq (4n + 4)/2 + 1 = 2n + 3$. Since $\mathcal{I}_{L_S}$ is generated by those $A_{\mathbf{i}}$ with $i_1 < 2n + 3$, this proves the first assertion.

For $d = 3n + 2$, $i_1 + j_1 + k_1 + l_1 \leq 4n + 8$, whence at most 1 out of i_1, j_1, k_1, l_1 can be $\geq (4n + 8)/2 + 1 = 2n + 5$, which proves the proposition. □

Remark 4.2.3.2. By construction, L_S (resp. L_T) have the following basic property: For $g \in V(d) \backslash L_S$ (resp. $g \in V(d) \backslash L_T$), the restriction $S_d \mid_{\mathbb{P}(L_S + \mathbb{C}g)}$ (resp. $T_d \mid_{\mathbb{P}(L_T + \mathbb{C}g)}$) is linear.

4.2.4 Fiberwise surjectivity of the covariants

To begin with, we will show how some elements of L_S (resp. L_T) can be written as sums of powers. For this let K be a positive integer.

Definition 4.2.4.1. Let $\mathbf{b} = (b_1, \ldots, b_K) \in \mathbb{C}^K$ be given. Then we denote by

$$p_i^{\mathbf{b}}(c) := \prod_{\substack{j \neq i \\ 1 \leq j \leq K}} \frac{c - b_j}{b_i - b_j} \tag{4.24}$$

for $i = 1, \ldots, K$ the interpolation polynomials of degree $K - 1$ w.r.t. $\mathbf{b}$ in the one variable c.

Lemma 4.2.4.2. *Let* $\mathbf{b} = (b_1, \ldots, b_K) \in \mathbb{C}^K$*,* $b_i \neq b_j$ *for* $i \neq j$*, and set* $x = x_1$*,* $y = \lambda x_2 + \mu x_3$*,* $(\lambda, \mu) \neq (0, 0)$*. Suppose* $d > K$ *and put* $l_i := b_i x + y$*. Then for each* $c \in \mathbb{C}$ *with* $c \neq b_i$*,* $\forall i$*,*

$$f(c) = p_1^{\mathbf{b}}(c) l_1^d + \cdots + p_K^{\mathbf{b}}(c) l_K^d - (cx + y)^d \tag{4.25}$$

is nonzero and divisible by x^K*.*

Proof. The coefficient of the monomial $x^A y^B$ in $f(c)$ is equal to

$$\binom{d}{A} (p_1^{\mathbf{b}}(c) b_1^A + \cdots + p_K^{\mathbf{b}}(c) b_K^A - c^A).$$

For $A \leq K - 1$ one has

$$c^A = p_1^{\mathbf{b}}(c) b_1^A + \cdots + p_K^{\mathbf{b}}(c) b_K^A\,.$$

for all c by interpolation. □

Choosing $K = 2n + 3$, we obtain elements $f(c) \in L_S$, and for $K = 2n + 5$ elements $f(c) \in L_T$. Now for $d = 3n + 1$ consider the diagram

$$\begin{tikzcd} \mathbb{P}(L_S + \mathbb{C}g) \arrow[d] & \subset & \mathbb{P}V(d) \arrow[r, dashed, "S_d"] \arrow[d, dashed, "\pi_{L_S}"] & \mathbb{P}V(4) \\ \bar{g} & \in & \mathbb{P}(V(d)/L_S) & \end{tikzcd}$$

or for $d = 3n + 2$ the diagram

$$\begin{tikzcd} \mathbb{P}(L_T + \mathbb{C}g) \arrow[d] & \subset & \mathbb{P}V(d) \arrow[r, dashed, "T_d"] \arrow[d, dashed, "\pi_{L_T}"] & \mathbb{P}V(8) \\ \bar{g} & \in & \mathbb{P}(V(d)/L_T) & \end{tikzcd}$$

The aim of this section is to prove

Proposition 4.2.4.3. *Let* $d = 3n + 1 \geq 37$*. Then there exists a* $g \in V(d)$ *such that*

$$S_d|_{\mathbb{P}(L_S + \mathbb{C}g)} \colon \mathbb{P}(L_S + \mathbb{C}g) \dashrightarrow \mathbb{P}V(4)$$

is surjective. For $d = 3n + 2 \geq 65$ *there exists a* $g \in V(d)$ *such that*

$$T_d|_{\mathbb{P}(L_T + \mathbb{C}g)} \colon \mathbb{P}(L_T + \mathbb{C}g) \dashrightarrow \mathbb{P}V(8)$$

is surjective.

We will prove the case $d = 3n + 1$ first. The case $d = 3n + 2$ is very similar, and we will deal with it afterwards.

We start by constructing points in the image of S_d:

Lemma 4.2.4.4. *Consider $S_d(f(c) + g)$ as an element of $\mathbb{C}[x_1,\ x_2,\ x_3,\ c]$ and write*

$$S_d(f(c) + g) = Q_d c^d + \cdots + Q_0 \tag{4.26}$$

with $Q_i \in \mathbb{C}[x_1,\ x_2,\ x_3]_4$. Then $[Q_i] \in S_d(\mathbb{P}(L_S + \mathbb{C}g))$ for all i.

Proof. The map

$$\begin{aligned} \varphi\ : \mathbb{A}^1 &\to \mathbb{P}V(4) \\ c &\mapsto S_d(f(c) + g) \end{aligned} \tag{4.27}$$

gives a rational curve X in $S_d(\mathbb{P}(L_S + \mathbb{C}g))$. Since by Remark 4.2.3.2, $S_d\ |_{\mathbb{P}(L_S+\mathbb{C}g)}$ is linear, the linear span of X is contained in $S_d(\mathbb{P}(L_S + \mathbb{C}g))$. Now $\langle X \rangle = \langle Q_0, \ldots, Q_d \rangle$ which proves the claim. □

Surprisingly, for i large enough, the Q_i do not depend on the vector $\mathbf{b} = (b_1, \ldots, b_K)$ chosen to construct $f(c)$:

Proposition 4.2.4.5. *If*

$$S_d(f(c) + g) = Q_d c^d + \cdots + Q_0, \quad Q_i \in \mathbb{C}[x_1,\ x_2,\ x_3]_4\,,$$

and

$$S_d(-(cx + y)^d + g) = Q'_d c^d + \cdots + Q'_0, \quad Q'_i \in \mathbb{C}[x_1,\ x_2,\ x_3]_4\,,$$

then $Q_i = Q'_i$ for $i \geq K$.

Proof. Write g as a sum of dth powers of linear forms

$$g = m_1^d + \cdots + m_{\text{const}}^d\,, \tag{4.28}$$

where const is a positive integer that will be fixed (independently of n) in the later discussion. Then (using $\mathcal{I}_{B_S} \subset \mathcal{I}^3_{\mathbb{P}(L_S)}$ and Lemma 4.2.2.1)

$$\begin{aligned} S_d(f(c)+\epsilon g) &= S_d\Big(p_1^{\mathbf{b}}(c)l_1^d + \cdots + p_K^{\mathbf{b}}(c)l_K^d - (cx+y)^d \\ &\qquad + \epsilon m_1^d + \cdots + \epsilon m_{\mathrm{const}}^d\Big) \\ &= 24\Big(\epsilon^3 \sum_{\substack{i\\ j<k<p}} p_i^{\mathbf{b}}(c) I(l_i,\, m_j,\, m_k,\, m_p)^n l_i m_j m_k m_p \\ &\qquad - \epsilon^3 \sum_{j<k<p} I(cx+y,\, m_j,\, m_k,\, m_p)^n (cx+y) m_j m_k m_p \\ &\qquad + \epsilon^4 \sum_{i<j<k<p} I(m_i,\, m_j,\, m_k,\, m_p)^n m_i m_j m_k m_p\Big) \end{aligned} \tag{4.29}$$

For $\epsilon = 1$ we find

$$\begin{aligned} S_d(f(c)+g) = &\sum_{i,\,j,\,k,\,p} p_i^{\mathbf{b}}(c) I(l_i,\, m_j,\, m_k,\, m_p)^n l_i m_j m_k m_p \\ &+ S_d(-(cx+y)^d + g)\,. \end{aligned} \tag{4.30}$$

Since $\deg p_i^{\mathbf{b}} = K-1$, the assertion follows. $\square$

Next we will investigate the dependence of Q_t on n for $t \geq K$.
We choose a fixed constant $\wp \in \mathbb{N}$ (the "precision") with $d - \wp + 1 \geq K$ and $\wp \leq n$ (later $\wp$ will be a prime number).

Lemma 4.2.4.6. *For $3n+1-\wp \leq s \leq 3n$, the coefficient of c^s in $I(cx+y,\, m_i,\, m_j,\, m_k)^n$ is of the form*

$$\varrho^n P(n) \tag{4.31}$$

where $\varrho \in \mathbb{C}$ (independent of n and s) and $P(n)$ is a polynomial of degree $3n - s < \wp$. $P(n)$ is, as a polynomial in $\mathbb{C}[n]$ divisible by

$$\binom{n}{n - \lceil \frac{s}{3} \rceil}.$$

If the coefficients of the m_i are integers and $\wp$ is a prime number, then the reduction of the coefficient of c^s in $I(cx+y,\, m_i,\, m_j,\, m_k)^n$ modulo $\wp$ is still of the form

$$\varrho^n P(n) \tag{4.32}$$

with $\varrho \in \mathbb{F}_\wp$ and $P(n) \in \mathbb{F}_\wp[n]$ satisfying the same independence and divisibility conditions as above.

Proof. Calculating either over $\mathbb{C}$ or over $\mathbb{F}_\wp$, we have

$$\begin{aligned} & I(cx+y,\, m_i,\, m_j,\, m_k)^n \qquad (4.33)\\ &= (cx+y,\, m_i,\, m_k)^n (cx+y,\, m_i,\, m_j)^n (cx+y,\, m_j,\, m_k)^n (m_i,\, m_j,\, m_k)^n \\ &= (\xi_{ik}c+\eta_{ik})^n(\xi_{ij}c+\eta_{ij})^n(\xi_{jk}c+\eta_{jk})^n(m_i,\, m_j,\, m_k)^n \end{aligned}$$

where the ξ's and η's are constants (fixed once the m's are fixed). If any of the ξ's vanishes the polynomial $I(cx+y,\, m_i,\, m_j,\, m_k)^n$ is of degree $\leq 2n$ in c. Since $s > 3n - \wp \geq 2n$, in this situation the coefficient of c^s is 0 and we are finished. Assume therefore that the ξ's are invertible.

The above expression expands to

$$\left(\sum_{p=1}^{n} \binom{n}{p} \xi_{ik}^p c^p \eta_{ik}^{n-p}\right) \cdot \left(\sum_{q=1}^{n} \binom{n}{q} \xi_{ij}^q c^q \eta_{ij}^{n-q}\right) \cdot \left(\sum_{r=1}^{n} \binom{n}{r} \xi_{jk}^r c^r \eta_{jk}^{n-r}\right) \cdot (m_i,\, m_j,\, m_k)^n$$

and the coefficient of c^s is

$$(m_i,\, m_j,\, m_k)^n \sum_{p+q+r=s} \binom{n}{p}\binom{n}{q}\binom{n}{r} \xi_{ik}^p \xi_{ij}^q \xi_{jk}^r \eta_{ik}^{n-p} \eta_{ij}^{n-q} \eta_{jk}^{n-r}\,.$$

Put $p' = n-p$, $q' = n-q$, $r' = n-r$ and rewrite this as

$$((m_i,\, m_j,\, m_k)\xi_{ik}\xi_{ij}\xi_{jk})^n \sum_{p'+q'+r'=3n-s} \binom{n}{p'}\binom{n}{q'}\binom{n}{r'} \xi_{ik}^{-p'} \xi_{ij}^{-q'} \xi_{jk}^{-r'} \eta_{ik}^{p'} \eta_{ij}^{q'} \eta_{jk}^{r'}.$$

The first claim of the lemma over $\mathbb{C}$ is obvious now. The reductions of the binomial coefficients modulo $\wp$ are polynomials in n over $\mathbb{F}_\wp$ if $p', q', r' < \wp$. Our conditions on s imply this, since

$$p', q', r' \leq p' + q' + r' = 3n - s < \wp.$$

As for the stated divisibility property in $\mathbb{C}[n]$ and $\mathbb{F}_\wp[n]$, remark that in

$$\binom{n}{p'}\binom{n}{q'}\binom{n}{r'}$$

with $p'+q'+r' = 3n-s$, at least one of p', q', q' is $\geq n - \lceil \frac{s}{3} \rceil$. $\square$

Proposition 4.2.4.7. *For $d-\wp+1 \le t \le d$, the coefficient of each monomial $\mathbf{x}^{\mathbf{i}}$ in Q_t is of the form*

$$\sum_{\nu=1}^{\binom{\mathrm{const}}{3}} \varrho_\nu^n P_\nu(n) \tag{4.34}$$

where $\varrho_\nu \in \mathbb{C}$ are constants (independent of n), and $P_\nu(n)$ are polynomials of degree $\le d-t < \wp$, which are divisible by

$$\binom{n}{n-\lceil \frac{t}{3} \rceil}.$$

If g can be written as sum of powers with integer coefficients and $\wp$ is a prime number, the same is true for the reduction of Q_t mod $\wp$.

Proof. Q_t is the coefficient of c^t in

$$(-24) \sum_{1\le i< j< k\le \mathrm{const}} I(cx+y,\ m_i,\ m_j,\ m_k)^n (cx+y)m_i m_j m_k \tag{4.35}$$

(cf. (29)), so we may apply Lemma 4.2.4.6 with $s=t$ and $s=t-1$. □

Definition 4.2.4.8. For $d-\wp+1 \le t \le d$, we put

$$R_t := \frac{Q_t}{\binom{n}{n-\lceil \frac{t}{3} \rceil}}. \tag{4.36}$$

Proof of Proposition 4.2.4.3 (for $d=3n+1$). Let const $=9$ and consider

$$g = m_1^d + \cdots + m_{\mathrm{const}}^d$$

with

$$\begin{array}{lll} m_1 = x_1 + 3x_2 + 9x_3 & m_4 = x_1 + 6x_2 - 10x_3 & m_7 = -3x_2 + 2x_3 \\ m_2 = -10x_1 + x_2 + 4x_3 & m_5 = 4x_1 - 8x_2 - 10x_3 & m_8 = 8x_1 - 4x_2 - 4x_3 \\ m_3 = 8x_1 + 4x_2 + 6x_3 & m_6 = -3x_1 + 7x_2 - 4x_3 & m_9 = -10x_1 + 4x_2 + 6x_3. \end{array}$$

For $\wp = 11$ we perform our construction with $x = x_1$ and two different values for y, namely $y_1 = x_2$ and $y_2 = x_3$. We obtain 22 quartics

$R_d^{y_1}, \dots, R_{d-10}^{y_1}, R_d^{y_2}, \dots, R_{d-10}^{y_2}$. By Lemma 4.2.4.4 and Proposition 4.2.4.5 these quartics are in the image of $S_d|_{\mathbb{P}(L_S+\mathbb{C}g)}$ if

$$d - 10 \geq K \iff 3n + 1 - 10 \geq 2n + 3 \iff n \geq 12.$$

The coefficients of the $R_j^{y_i}$ form a 15×22 matrix $M(n)$ with entries of the form $\sum_{\nu=1}^{84} \varrho_\nu^n P_\nu(n)$ by Proposition 4.2.4.7. Modulo 11 this matrix becomes periodic in n with period $11 \cdot 10 = 110$. With a computer algebra program it is straightforward to check that all these matrices have full rank 15. A `Macaulay2` script doing this can be found at [BvB08-1a]. This proves the claim for $d = 3n + 1$. □

Let us turn to the case $d = 3n + 2$. The whole procedure is similar in this case. If we take $K = 2n + 5$ Lemma 4.2.4.2, Proposition 4.2.4.5, Lemma 4.2.4.6 and Proposition 4.2.4.7 remain true as stated and Definition 4.2.4.8 still makes sense.

Proof of Proposition 4.2.4.3 (for $d = 3n + 2$). Let const $= 9$ and consider

$$g = m_1^d + \cdots + m_{\text{const}}^d$$

with m_i as above.

For $\wp = 19$ we perform our construction with $x = x_1$ and three different values for y, namely $y_1 = x_2$, $y_2 = x_3$ and $y_3 = x_2 + x_3$. We obtain 57 octics $R_d^{y_1}, \dots, R_{d-18}^{y_1}, R_d^{y_2}, \dots, R_{d-18}^{y_2}, R_d^{y_3}, \dots, R_{d-18}^{y_3}$. By Lemma 4.2.4.4 and Proposition 4.2.4.5 these octics are in the image of $S_d|_{\mathbb{P}(L_T+\mathbb{C}g)}$ if

$$d - 18 \geq K \iff 3n + 2 - 18 \geq 2n + 5 \iff n \geq 21.$$

The coefficients of the $R_j^{y_i}$ from a 45×57 matrix $M(n)$ with entries of the form $\sum_{\nu=1}^{84} \varrho_\nu^n P_\nu(n)$ by Proposition 4.2.4.7. Modulo 19 this matrix becomes periodic in n with period $19 \cdot 18 = 342$. With a computer algebra program it is straightforward to check that all these matrices have full rank 45. A `Macaulay2` script doing this can be found at [BvB08-1a]. This proves the claim for $d = 3n + 2$. □

4.2.5 Sections of principal bundles and proof of rationality

We will now show how to conclude the proof in the case $d = 3n+1$. We make some comments on the case $d = 3n+2$ when they are in order, but otherwise leave the obvious modifications to the reader. Let $\big(\mathbb{P}\,V(4)\big)_{\mathrm{vs}} \subset \mathbb{P}\,V(4)$ be the open subset of very stable points with respect to the action of $\bar{G}$ and the $\bar{G}$-linearized line bundle $\mathcal{O}(3)$ (very stable means stable with trivial stabilizer). Now the essential point is

Proposition 4.2.5.1. *The quotient morphism*

$$\big(\mathbb{P}\,V(4)\big)_{\mathrm{vs}} \to \big(\mathbb{P}\,V(4)\big)_{\mathrm{vs}}/\bar{G}$$

is a principal $\bar{G}$-bundle in the Zariski topology.

Proof. See [Shep], Prop. 2. This holds also true with $V(4)$ replaced with $V(8)$. □

It follows that this $\bar{G}$-bundle has a section defined generically which we will denote by σ_4.

Proof of Theorem 4.1.0.1. Consider the graph

$$X = \{(g, \bar{g}, f) \mid \pi_{L_S}(g) = \bar{g}, S_d(g) = f\} \subset \mathbb{P}V(d) \times \mathbb{P}\big(V(d)/L_S\big) \times \mathbb{P}V(4)$$

and the diagram

$$\begin{tikzcd}
X \arrow[r, dashed, leftrightarrow, "1:1", "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbb{P}V(d) \arrow[r, dashed] & \mathbb{P}V(d)/\bar{G} \arrow[dd, dashed, "\bar{S}_d"] \\
\mathbb{P}\big(V(d)/L_S\big) \times \mathbb{P}V(4) \arrow[d] & & \\
\mathbb{P}V(4) \arrow[rr, dashed] & & \mathbb{P}V(4)/\bar{G}. \arrow[ll, dashed, bend left, "\sigma_4"]
\end{tikzcd}$$

By Proposition 4.2.4.3 the projection pr_{23} is dominant. It follows then from Remark 4.2.3.2 that X is birational to a vector bundle over $\mathbb{P}\big(V(d)/L_S\big) \times \mathbb{P}V(4)$ and hence also over $\mathbb{P}V(4)$. After replacing σ_4 by a translate, we can assume that σ_4 meets an open set $U \subset \mathbb{P}V(4)$ over which this vector bundle is trivial. Since $\bar{G}$ acts generically freely on $\mathbb{P}V(4)$, we can pull back the above vector bundle structure via σ_4 and obtain that $\mathbb{P}V(d)/\bar{G}$ is birational to $\mathbb{P}V(4)/\bar{G} \times \mathbb{P}^N$ with $N = \dim V(d) - \dim V(4)$. If $d \geq 37$ as in Proposition 4.2.4.3, then certainly $N \geq 8$ and since $\big(\mathbb{P}V(4)/\bar{G}\big) \times \mathbb{P}^8$ is rational, $\mathbb{P}V(d)/\bar{G}$ is rational. In the case $d = 3n + 2$ the same argument works since the space of octics is also stably rational of level 8. This proves Theorem 4.1.0.1. □

4.3 Another proof for the dominance of S_d

In this section we give another proof of the dominance of S_d which illustrates the usefulness of writing forms as sums of powers of linear forms nicely.

4.3.1 Configurations of lines and Scorza map

Let us recall the main properties of the Scorza map

$$\begin{aligned} S_4 \,:\, \mathbb{P}V(4) &\dashrightarrow \mathbb{P}V(4) \\ S_4(\alpha,\, \beta,\, \gamma,\, \delta) &:= (\alpha\,\beta\,\gamma)(\alpha\,\beta\,\delta)(\alpha\,\gamma\,\delta)(\beta\,\gamma\,\delta)\alpha_{\mathbf{x}}\beta_{\mathbf{x}}\gamma_{\mathbf{x}}\delta_{\mathbf{x}} \end{aligned}$$

which we will use. If $[\mathbf{a}] = [(a_1,\, a_2,\, ,a_3)] \in \mathbb{P}(\mathbb{C}^3)$ is a point in $\mathbb{P}^2$ with homogeneous coordinates $a_1,\, a_2,\, a_3$, and $f \in \mathbb{P}V(4)$ is a quartic, then

$$P_{\mathbf{a}}(f) := \sum_{i=1}^{3} a_i \frac{\partial f}{\partial x_i}$$

is the *first polar* of f with respect to the point $[\mathbf{a}]$. It is a cubic curve passing through the points of tangency with the quartic f of lines passing through the point $[\mathbf{a}]$. The geometric interpretation of the Scorza map is as follows: It assigns to a general plane quartic f the quartic $S_4(f)$ consisting of those points $[\mathbf{a}]$ of $\mathbb{P}^2$ such that the first polar of f with respect to $[\mathbf{a}]$ is an equianharmonic cubic.
One has the following fundamental theorem (cf. [D-K], section 7, for a proof).

Theorem 4.3.1.1. *The Scorza map S_4 is a dominant rational map of degree 36 from the space of plane quartics to itself. More precisely, for a general quartic f the image $S_4(f)$ carries naturally an even theta-characteristic $\vartheta_{\mathrm{Scorza}}$, and the rational map from the moduli space $\mathfrak{M}_3$ of curves of genus 3 to $\mathfrak{M}_3^{\mathrm{ev}}$, the moduli space of genus 3 curves with an even theta-characteristic*

$$\mathfrak{M}_3 \dashrightarrow \mathfrak{M}_3^{\mathrm{ev}}\,,$$

given by sending f to $(S_4(f),\, \vartheta_{\mathrm{Scorza}})$, is a birational isomorphism.

Let us go back to the general degree 4 covariant $S_d : V(d) \to V(4)$. As we saw, it is hard to calculate the values of S_d on a general homogeneous form f of degree d without knowing the entire expression of S_d as a polynomial in the coefficients $A_{\mathbf{i}}$, which is awkward. Recall, however, that one can work directly with the symbolic expression if one writes f as a linear combination of d-th powers of linear forms:

$$f = \lambda_1 l_1^d + \cdots + \lambda_N l_N^d, \quad \text{some} \quad N \in \mathbb{N}\,.$$

We recall that if

$$\mathbf{l}_i = l_i^{(1)} x_1 + l_i^{(2)} x_2 + l_i^{(3)} x_3, \quad i = 1, \dots, N$$

and if one writes

$$\mathbf{l}_i = (l_i^{(1)},\, l_i^{(2)},\, l_i^{(3)})\,,$$

then, instead of the vectors of symbolic variables α, β, γ, δ, one may substitute vectors

$$\mathbf{l}_i,\, \mathbf{l}_j,\, \mathbf{l}_k,\, \mathbf{l}_p, \quad i,\, j,\, k,\, p \in \{1, \dots, N\}$$

of complex numbers into the expression for S_d, to obtain a homogeneous degree 4 form in x_1, x_2, x_3, which we write as

$$S_d(\mathbf{l}_i,\, \mathbf{l}_j,\, \mathbf{l}_k,\, \mathbf{l}_p)\,.$$

Then we saw

Lemma 4.3.1.2. *We have*

$$S_d(f) = \sum \lambda_i \lambda_j \lambda_k \lambda_p S_d(\mathbf{l}_i,\, \mathbf{l}_j,\, \mathbf{l}_k,\, \mathbf{l}_p)\,.$$

The right-hand sum runs over all $(i,\, j,\, k,\, p) \in \{1, \dots, N\}^4$.

To make use of this, we require

Lemma 4.3.1.3. *A general quartic form* $f \in V(4)$ *can, after projective change of coordinates, be written as*

$$f = x_1^4 + x_2^4 + x_3^4 + (ax_1 + bx_2)^4 + (cx_1 + dx_3)^4 + (ex_2 + fx_3)^4\,,$$

where a, b, c, d, e, f *are complex scalars.*

Proof. If we denote by $Y \subset \mathbb{P}(V(4))$ the subvariety which is the closure of the image of $\mathbb{C}^6 \to \mathbb{P}(V(4))$, $(a, b, c, d, e, f) \mapsto [f]$, then one has to show that $\overline{\mathrm{SL}_3(\mathbb{C}) \cdot Y}$ has dimension 14, so is all of $\mathbb{P}(V(4))$. For this it is sufficient to exhibit a point $y \in Y$ such that the set of $g \in \mathrm{SL}_3(\mathbb{C})$ with $g \cdot y \in Y$ is finite (so the fibre of $\mathrm{SL}_3(\mathbb{C}) \times Y \to \mathbb{P}(V(4))$ over y is finite), which one checks by explicit computation with e.g. `Macaulay 2`. $\square$

Proposition 4.3.1.4. *The rational map S_d*

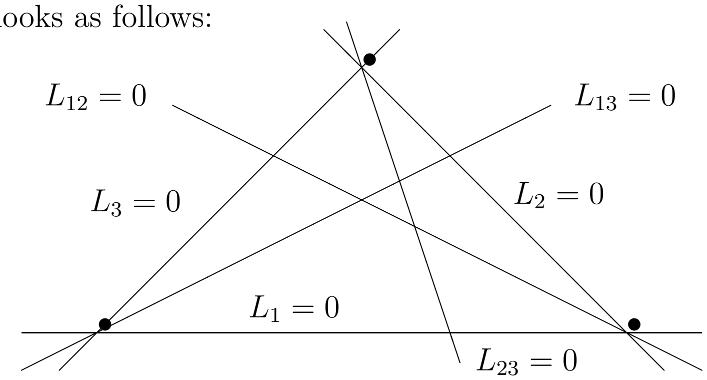
$$S_d \, : \, \mathbb{P}V(d) \dashrightarrow \mathbb{P}V(4)$$

is dominant for all $d = 3m + 1$ with $m \geq 1$.

Proof. By Lemma 4.3.1.3 we may write a general $f \in V(4)$ as

$$f = L_1^4 + L_2^4 + L_3^4 + L_{12}^4 + L_{13}^4 + L_{23}^4 \, ,$$

where the L's are linear forms whose zero sets in $\mathbb{P}^2$ form a hexagon which looks as follows:



We will prove that, for general f, $S_4(f)$ is also in the image of S_d by exhibiting an explicit preimage.

Associated with L's we have their vectors of components with respect to x_1, x_2, x_3 which we will write

$$\mathcal{S} := \{\mathbf{L}_1, \, \mathbf{L}_2, \, \mathbf{L}_3, \, \mathbf{L}_{12}, \, \mathbf{L}_{13}, \, \mathbf{L}_{23}\},$$

and we will regard $\mathcal{S}$ as an ordered set with the indicated total order ($\mathbf{L}_1$ being the smallest and $\mathbf{L}_{23}$ being the greatest element). Looking back at the definition of the Clebsch invariant I, we can define complex scalars

$$I(\mathbf{A}, \, \mathbf{B}, \, \mathbf{C}, \, \mathbf{D}), \quad \text{for} \quad \mathbf{A}, \, \mathbf{B}, \, \mathbf{C}, \, \mathbf{D} \in \mathcal{S} \, .$$

For $\mathbf{A}$, $\mathbf{B} \in \mathcal{S}$ two unequal elements, we write $[\mathbf{A},\, \mathbf{B}]$ for the complex scalar which is obtained from plugging into I the elements from the complement $\mathcal{S}\backslash\{\mathbf{A},\, \mathbf{B}\}$ in the order in which they occur in $\mathcal{S}$.
Then only

$$[\mathbf{L}_3,\, \mathbf{L}_{12}],\ [\mathbf{L}_2,\, \mathbf{L}_{13}],\ [\mathbf{L}_1,\, \mathbf{L}_{23}],\ [\mathbf{L}_2,\, \mathbf{L}_3],\ [\mathbf{L}_1,\, \mathbf{L}_3],\ [\mathbf{L}_1,\, \mathbf{L}_2]$$

are different from zero. Thus for the value $S_4(f)$ of the Scorza map on f we obtain

$$\begin{aligned} S_4(f) = 24\cdot\, ([\mathbf{L}_3,\, \mathbf{L}_{12}]&L_1L_2L_{13}L_{23} + [\mathbf{L}_2,\, \mathbf{L}_{13}]L_1L_3L_{12}L_{23} \\ &+ [\mathbf{L}_1,\, \mathbf{L}_{23}]L_2L_3L_{12}L_{13} + [\mathbf{L}_2,\, \mathbf{L}_3]L_1L_{12}L_{13}L_{23} \\ &+[\mathbf{L}_1,\, \mathbf{L}_3]L_2L_{12}L_{13}L_{23} + [\mathbf{L}_1,\, \mathbf{L}_2]L_3L_{12}L_{13}L_{23})\ . \end{aligned}$$

We define

$$\begin{aligned} g_1 &:= [\mathbf{L}_1,\, \mathbf{L}_3][\mathbf{L}_1,\, \mathbf{L}_2], \\ g_2 &:= [\mathbf{L}_1,\, \mathbf{L}_2][\mathbf{L}_2,\, \mathbf{L}_3], \\ g_3 &:= [\mathbf{L}_2,\, \mathbf{L}_3][\mathbf{L}_1,\, \mathbf{L}_3], \\ g_{12} &:= [\mathbf{L}_1,\, \mathbf{L}_2][\mathbf{L}_3,\, \mathbf{L}_{12}], \\ g_{13} &:= [\mathbf{L}_1,\, \mathbf{L}_3][\mathbf{L}_2,\, \mathbf{L}_{13}], \\ g_{23} &:= [\mathbf{L}_2,\, \mathbf{L}_3][\mathbf{L}_1,\, \mathbf{L}_{23}],\ . \end{aligned}$$

Look at

$$\begin{aligned} g :=&(g_1)^{m-1}L_1^d + (g_2)^{m-1}L_2^d + (g_3)^{m-1}L_3^d \\ &+(g_{12})^{m-1}L_{12}^d + (g_{13})^{m-1}L_{13}^d + (g_{23})^{m-1}L_{23}^d\,. \end{aligned}$$

Using Lemma 4.3.1.2 we obtain

$$\begin{aligned} S_d(g) = 24\cdot\, \big(&(g_1g_2g_{13}g_{23})^{m-1}[\mathbf{L}_3,\, \mathbf{L}_{12}]^m L_1L_2L_{13}L_{23} \\ &+ (g_1g_3g_{12}g_{23})^{m-1}[\mathbf{L}_2,\, \mathbf{L}_{13}]^m L_1L_3L_{12}L_{23} \\ &+ (g_2g_3g_{12}g_{13})^{m-1}[\mathbf{L}_1,\, \mathbf{L}_{23}]^m L_2L_3L_{12}L_{13} \\ &+ (g_1g_{12}g_{13}g_{23})^{m-1}[\mathbf{L}_2,\, \mathbf{L}_3]^m L_1L_{12}L_{13}L_{23} \\ &+ (g_2g_{12}g_{13}g_{23})^{m-1}[\mathbf{L}_1,\, \mathbf{L}_3]^m L_2L_{12}L_{13}L_{23} \\ &+ (g_3g_{12}g_{13}g_{23})^{m-1}[\mathbf{L}_1,\, \mathbf{L}_2]^m L_3L_{12}L_{13}L_{23}\big) \end{aligned}$$

and thus

$$\begin{aligned}S_d(g) = \big(&[\mathbf{L}_3,\ \mathbf{L}_{12}][\mathbf{L}_2,\ \mathbf{L}_{13}][\mathbf{L}_1,\ \mathbf{L}_{23}]\\ &\cdot[\mathbf{L}_1,\ \mathbf{L}_2]^2[\mathbf{L}_1,\ \mathbf{L}_3]^2[\mathbf{L}_2,\ \mathbf{L}_3]^2\big)^{m-1} \cdot S_4(f)\,.\end{aligned}$$

Therefore, for generic f, $S_4(f)$ is in the image of S_d, which proves the dominance of S_d in view of Theorem 4.3.1.1. $\square$

Chapter 5

The rationality of some moduli spaces of plane curves of small degree

5.1 Introduction

In this chapter we prove an analogue of the classical Clebsch-Gordan formula (which deals with the group $\mathrm{SL}_2(\mathbb{C})$) for $\mathrm{SL}_3(\mathbb{C})$ (Theorem 5.2.1.1). This allows one to obtain explicit matrix representatives for all $\mathrm{SL}_3(\mathbb{C})$-equivariant bilinear maps $U \otimes V \to W$ (U, V, W finite dimensional $\mathrm{SL}_3(\mathbb{C})$-representations) and to treat them in an algorithmically efficient way. In subsection 5.3 we introduce further computational techniques, based on writing a homogeneous polynomial as a sum of powers of linear forms and interpolation, which apply to a special class of maps $U \otimes V \to W$ that are ubiquitous in applications. Applications are Theorem 5.4.2.2, and in particular Theorem 5.4.1.3 where, combining the methods of this Chapter with those of Chapter 4, we prove the rationality of the moduli spaces of plane curves of degree d for all d with the possible exception of 15 values of d for which rationality remains unsettled.

5.2 A Clebsch-Gordan formula for $\mathrm{SL}_3(\mathbb{C})$

5.2.1 The elementary maps

Let $G := \mathrm{SL}_3(\mathbb{C})$ and denote as usual by $V(a,\, b)$ the irreducible G-module whose highest weight has numerical labels $a,\, b$ where $a,\, b$ are non-negative integers. We put

$$S^a := \mathrm{Sym}^a(\mathbb{C}^3), \quad D^b := \mathrm{Sym}^b(\mathbb{C}^3)^\vee$$

and denote by $e_1,\, e_2,\, e_3$ and $x_1,\, x_2,\, x_3$ dual bases in $\mathbb{C}^3$ resp. $(\mathbb{C}^3)^\vee$ so that $V(a,\, b)$ can be realized concretely as the kernel of the map

$$\Delta := \sum_{i=1}^{3} \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} \;:\; S^a \otimes D^b \to S^{a-1} \otimes D^{b-1}\,; \tag{5.1}$$

we will always view $V(a,\, b)$ in this way in the following. Our purpose is to determine an explicit basis of the G-equivariant maps

$$\mathrm{Hom}_G(V(a,\, b) \otimes\, V(c,\, d),\, V(e,\, f))$$

(if $V(e,\, f)$ is a subrepresentation of $V(a,\, b) \otimes V(c,\, d)$). To this end we define the following elementary maps:

$$\alpha \,:\, (S^a \otimes D^b) \otimes (S^c \otimes D^d) \to (S^{a-1} \otimes D^b) \otimes (S^c \otimes D^{d-1}) \tag{5.2}$$
$$\alpha := \sum_{i=1}^{3} \frac{\partial}{\partial e_i} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \frac{\partial}{\partial x_i}$$

$$\beta \,:\, (S^a \otimes D^b) \otimes (S^c \otimes D^d) \to (S^a \otimes D^{b-1}) \otimes (S^{c-1} \otimes D^d) \tag{5.3}$$
$$\beta := \sum_{i=1}^{3} \mathrm{id} \otimes \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial e_i} \otimes \mathrm{id}$$

$$\vartheta \,:\, (S^a \otimes S^c) \otimes (D^{b+d}) \to (S^{a-1} \otimes S^{c-1}) \otimes D^{b+d+1} \tag{5.4}$$
$$\vartheta := \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) \frac{\partial}{\partial e_{\sigma(1)}} \otimes \frac{\partial}{\partial e_{\sigma(2)}} \otimes x_{\sigma(3)}$$

$$\begin{aligned} \omega \,:\, S^{a+c} \otimes (D^b \otimes D^d) &\to S^{a+c+1} \otimes (D^{b-1} \otimes D^{d-1}) \\ \omega := \sum_{\sigma \in \mathfrak{S}_3} \operatorname{sgn}(\sigma) e_{\sigma(1)} &\otimes \frac{\partial}{\partial x_{\sigma(2)}} \otimes \frac{\partial}{\partial x_{\sigma(3)}} \end{aligned} \tag{5.5}$$

Note that an easier way of defining ϑ and ω is by saying that ϑ is multiplication by the determinant $x_1 \wedge x_2 \wedge x_3$ and ω multiplication by its inverse $e_1 \wedge e_2 \wedge e_3$.
The theorem we want to prove is the following.

Theorem 5.2.1.1. *Suppose that $V(e, f)$ occurs in the decomposition of $V(a, b) \otimes V(c, d)$ into irreducible constituents. Define integers*

$$s := \frac{(a+c-e) + 2(b+d-f)}{3}, \quad t := \frac{(a+c-e) - (b+d-f)}{3}.$$

Let $\pi_{e,\,f}$ be the equivariant projection from $S^e \otimes D^f$ onto $V(e,\, f)$. Then a basis of $\operatorname{Hom}_G(V(a,\, b) \otimes V(c,\, d),\, V(e,\, f))$ is given by the following maps:

(A) In case $t \geq 0$ the (restrictions to $V(a,\, b) \otimes V(c,\, d)$ of the) maps of the form

$$\pi_{e,\,f} \circ \vartheta^t \circ \beta^{s-i} \circ \alpha^i$$

where i is an arbitrary non-negative integer which is subject to the following set of inequalities

(A.1) $i \leq a,\ i \leq d$.

(A.2) $i \leq s,\ s - i \leq b,\ s - i \leq c$.

(A.3) $a - i \geq t,\ c - (s - i) \geq t$.

(B) In case $t < 0$ the (restrictions to $V(a,\, b) \otimes V(c,\, d)$ of the) maps of the form

$$\pi_{e,\,f} \circ \omega^{-t} \circ \beta^{s+t-i} \circ \alpha^i$$

where i is again an otherwise arbitrary non-negative integer constrained only by the inequalities

(B.1) $i \leq a,\ i \leq d$.

(B.2) $i \leq s+t$, $s+t-i \leq b$, $s+t-i \leq c$.

(B.3) $b-(s+t-i) \geq -t$, $d-i \geq -t$.

A few explanatory remarks are in order.

Remark 5.2.1.2. When writing a composition like $\pi_{e,\,f} \circ \vartheta^t \circ \beta^{s-i} \circ \alpha^i$, we suppress the obvious multiplication maps from the notation: after the map

$$\beta^{s-i} \circ \alpha^i \,:\, (S^a \otimes D^b) \otimes (S^c \otimes D^d) \to (S^{a-i} \otimes D^{b-(s-i)}) \otimes (S^{c-(s-i)} \otimes D^{d-i})$$

we perform the multiplication map

$$(S^{a-i} \otimes D^{b-(s-i)}) \otimes (S^{c-(s-i)} \otimes D^{d-i}) \to S^{a-i} \otimes S^{c-(s-i)} \otimes D^{b+d-s}$$

and apply ϑ^t to land in $S^{a-i-t} \otimes S^{c-(s-i)-t} \otimes D^{b+d-s+t}$; before applying the equivariant projection $\pi_{e,\,f}$ we multiply again to map to

$$S^{a+c-s-2t} \otimes D^{b+d-s+t}$$

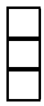
which one, looking back at the definition of t and s, identifies as $S^e \otimes D^f$. The composition $\pi_{e,\,f} \circ \omega^{-t} \circ \beta^{s+t-i} \circ \alpha^i$ has to be interpreted in a similar fashion. This simplification of notation should cause no confusion.

Remark 5.2.1.3. The definition of the integers s and t and the inequalities (A.1)-(A.3) and (B.1)-(B.3) may seem rather unmotivated at first glance, but, in fact, they have a very simple-minded meaning: (A.1) means that one can apply i-times the map α without obtaining the zero-map for positivity reasons (because the target is $(S^{a-i} \otimes D^b) \otimes (S^c \otimes D^{d-i})$); in a similar fashion, (A.2) and (A.3) mean that application of β^{s-i} resp. ϑ^t does not land us in spaces which are zero. (B.1), (B.2) resp. (B.3) have the same meaning in case $t < 0$ for the maps α^i, β^{s+t-i} resp. ω^{-t}. The numbers s and t are then chosen to eventually map to the space $S^e \otimes D^f$; their meaning (including the fact that they are indeed integers) will become clear in the next section when we interpret them in terms of Young diagrams.

The proof of Theorem 5.2.1.1 will occupy the next two sections. In the next subsection we prove that in both cases $t \geq 0$ and $t < 0$, the number of maps given in the Theorem equals the multiplicity of $V(e,\, f)$ in $V(a,\, b) \otimes V(c,\, d)$; we do it by directly relating the maps to the combinatorial data of the Littlewood Richardson rule. In subsection 5.1.3 we prove the linear independence of the maps in the Theorem, in particular that they are all non-zero.

5.2.2 Relation to the Littlewood Richardson rule

It is well known that isomorphism classes of irreducible $\mathrm{GL}_n(\mathbb{C})$-modules correspond bijectively to n-tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ via associating to such a representation its highest weight $\lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$ where ϵ_i is the i-th coordinate function of the standard diagonal torus in $\mathrm{GL}_n(\mathbb{C})$. The space of the corresponding irreducible representation will be denoted $\Sigma^\lambda(\mathbb{C}^n)$. Here Σ^λ is called the *Schur functor* (cf. [Fu-Ha]). If all λ_j are non-negative, one associates to λ the corresponding *Young diagram* whose number of boxes in its i-th row is λ_i; λ will often be identified with this Young diagram. For example,

$$\Sigma^{1,1,1}(\mathbb{C}^3) \quad \longleftrightarrow \quad \Lambda^3(\mathbb{C}^3) \quad \longleftrightarrow$$


We list some properties of the Schur functors for future use:

- One has $\Sigma^\lambda(\mathbb{C}^n) \simeq \Sigma^\mu(\mathbb{C}^n)$ as $\mathrm{SL}_n(\mathbb{C})$-representations if and only if $\lambda_i - \mu_i =: h$ is constant for all i. In fact, in this case
$$\Sigma^\lambda(\mathbb{C}^n) \simeq \Sigma^\mu(\mathbb{C}^n) \otimes (\Lambda^n(\mathbb{C}^n))^{\otimes h} .$$
- $\Sigma^{(\lambda_1,\, \lambda_2, \ldots,\, \lambda_n)}(\mathbb{C}^n)^\vee \simeq \Sigma^{(-\lambda_n,\, -\lambda_{n-1}, \ldots,\, -\lambda_1)}(\mathbb{C}^n)$.
- The representation $V(a, b)$ of $G = \mathrm{SL}_3(\mathbb{C})$ is isomorphic to $\Sigma^{(a+b,\, b,\, 0)}(\mathbb{C}^3)$.
- For a Young diagram λ with more than n rows one has $\Sigma^\lambda(\mathbb{C}^n) = 0$ by definition.

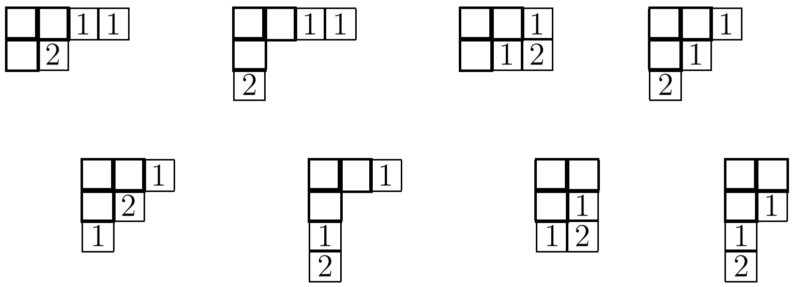
The Littlewood-Richardson rule to decompose $\Sigma^\lambda \otimes \Sigma^\mu$ into irreducible factors where λ, μ are Young diagrams (cf. [Fu-Ha], §A.1) says the following (in this notation we suppress the space which the Schur functors are applied to, since it plays no role): label each box of μ with the number of the row it belongs to. Then expand the Young diagram λ by adding the boxes of μ to the rows of λ subject to the following rules:

(a) The boxes with labels $\leq i$ of μ together with the boxes of λ form again a Young diagram;

(b) No column contains boxes of μ with equal labels.

(c) When the integers in the boxes added are listed from right to left and from top down, then, for any $0 \leq s \leq$ (number of boxes of μ), the first s entries of the list satisfy: each label l ($1 \leq l \leq$ (number of rows of μ)-1) occurs at least as many times as the label $l+1$.

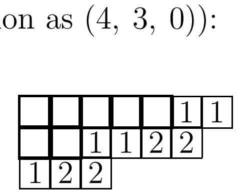
We will call this configuration of boxes (together with the labels) a *μ-expansion of λ*. Then the multiplicity of Σ^{ν} in $\Sigma^{\lambda} \otimes \Sigma^{\mu}$ is the number of times the Young diagram ν can be obtained by expanding λ by μ according to the above rules, forgetting the labels.
For $\Sigma^{(2,\,1,\,0)} \otimes \Sigma^{(2,\,1,\,0)}$ the following expansions are possible:



Hence

$$V(1,\ 1) \otimes V(1,\ 1) = V(2,\ 2) \oplus V(3,\ 0) \oplus V(0,\ 3) \oplus 2\,V(1,\ 1) \oplus V(0,\ 0)\,.$$

A typical μ-expansion of λ we are interested in will look like this (here $\lambda = (5,\ 2,\ 0)$, $\mu = (5,\ 4,\ 0)$; $\nu = (7,\ 6,\ 3)$ is depicted which yields the same $\mathrm{SL}_3\,(\mathbb{C})$-representation as $(4,\ 3,\ 0)$):



Reverting to labelling by highest weights, we consider the case $V(1,\ 3) \subset V(3,\ 2) \otimes V(1,\ 4)$ here.
We turn to the proof of Theorem 5.2.1.1 now. We put $\lambda := (a+b,\ b,\ 0)$ and $\mu := (c+d,\ d,\ 0)$, and let $\nu := (\nu_1,\ \nu_2,\ \nu_3)$ be the unique Young diagram corresponding to $V(e,\ f)$ in the μ-expansions of λ. We want to prove:

Proposition 5.2.2.1. *In the cases $t \geq 0$ resp. $t < 0$ there are natural bijections between the sets of non-negative integers i satisfying the inequalities (A.1)-(A.3) resp. (B.1)-(B.3) and the set of μ-expansions of λ of type ν.*

To begin with, we remark that

> the number s in Theorem 5.2.1.1 is the number of boxes in the third row of ν.

In fact, $3\nu_3 + f + (e+f)$ is equal to the total number of boxes in ν which is $2b + a + 2d + c$ which proves the preceding claim. It follows in particular that s is a non-negative integer, and $t = s - (b + d - f)$ is then an integer, too. There is a more useful interpretation of t:

> The number t is the difference of the number of 1's in the second row of each μ-expansion of λ of type ν and the number of 2's in the third row of the μ-expansion of λ of type ν.

This is so because $s - b + f$ is the number of boxes of μ added to the second row of λ in order to obtain ν; and d is the total number of 2's in the μ-expansion of λ of type ν.
We need an auxiliary concept before proceeding:

Definition 5.2.2.2. • Suppose $t \geq 0$. Let i be a non-negative integer satisfying $\boxed{i \leq s}$ and $\boxed{i \leq d}$. By the *pseudo-expansion* of λ by μ of type ν associated to i (in symbols: $\mathfrak{E}(i)$) we mean the following partially labelled configuration of boxes. Add the boxes of μ to the Young diagram of λ to obtain the Young diagram of ν. Then label the added boxes of μ in the following way: in the boxes of μ in the third row one writes first (reading from left to right) a sequence of $s - i$ 1's followed by a sequence of i 2's. The boxes of μ in the first row of ν all receive the label 1. The added boxes of μ in the second row of ν are labelled (reading again from left to right) by a sequence of $i + t$ 1's followed by sequence of 2's until all boxes of μ in the second row are filled.

- Suppose $t < 0$. Let i be a non-negative integer satisfying $\boxed{i \leq d+t}$ and $\boxed{i \leq s+t}$. By the *pseudo-expansion* of λ by μ of type ν associated to i (in symbols: $\mathfrak{E}(i)$) we mean the following partially labelled configuration of boxes. Add again the boxes of μ to the Young diagram of λ to obtain the Young diagram of ν and label the added boxes of μ as follows: the boxes of μ in the second row of ν receive as labels (read from left to right) a sequence of i 1's followed by a sequence of 2's to fill all the remaining boxes of μ in the second row. The boxes of μ in the first row of ν are all labelled with 1, and the boxes of μ in the third row of ν are labelled (reading from left to right) by $s-(i-t)$ 1's followed by $i-t$ 2's to fill the remaining boxes of μ in the last row of ν.

Note first that this definition makes sense because of the inequalities imposed on i in each case: in case $t \geq 0$ we have $0 \leq s-i \leq s$ and $0 \leq i \leq s$ and s is the number of boxes in the last row of ν, so the labelling of the boxes of μ in the third row of ν is well-defined; now the number of boxes of μ in the second row of ν is $f+s-b$, so to be able to carry out the labelling described above we should have $0 \leq i+t \leq f+s-b$. But $f+s-b-t=d$ by definition, so the labelling is well-defined because $i \leq d$ by assumption. It also follows that there are then d 2's in total among the labels of $\mathfrak{E}(i)$ (namely, $f+s-b-(i+t)+i=d$) and consequently $c+d$ 1's as it should be.
For $t<0$, note that we can fill in the boxes of μ in the third row of ν in the way described since $0 \leq s-(i-t) \leq s$ and $0 \leq (i-t) \leq s$ (since $0 \leq i \leq s+t$ and $t<0$); and there are again $f+s-b$ boxes of μ added to the second row of ν, and $0 \leq i \leq f+s-b$ is implied by the inequality $0 \leq i \leq d+t = (f+s-b-t)+t$. In this case we have again that there are d labels 2 in $\mathfrak{E}(i)$ in total ($i-t$ 2's in the last row, $f+s-b-i$ 2's in the second row), and consequently $c+d$ labels 1.

This being said, we will prove Proposition 5.2.2.1 by setting up the bijections as follows:

> if $t \geq 0$ then we associate to i subject to (A.1)-(A.3) the pseudo-expansion $\mathfrak{E}(i)$ of the first part of Definition 5.2.2.2.
>
> If $t<0$, then we let correspond to i subject to (B.1)-(B.3) the pseudo-expansion $\mathfrak{E}(i)$ of the second part of Definition 5.2.2.2.

It remains to see that the inequalities (A.1)-(A.3) resp. (B.1)-(B.3) are exactly equivalent to the pseudo-expansions $\mathfrak{E}(i)$ being μ-expansions of λ of type ν which confirm to the Littlewood Richardson rule. This is proven in the following Lemma.

Lemma 5.2.2.3. *Let $\mathfrak{E}(i)$ be the pseudo-expansion of λ by μ of type ν associated to i. Then $\mathfrak{E}(i)$ is an admissible Littlewood Richardson expansion of λ by μ of type ν if and only if for $t \geq 0$ inequalities (A.1)-(A.3) hold, and for $t < 0$, (B.1)-(B.3) hold. More precisely:*

(A) Suppose $t \geq 0$. The inequalities (A.1)-(A.3) can be interpreted in terms of the pseudo-expansion $\mathfrak{E}(i)$ as follows:

(A.1) The inequality $\boxed{i \leq a}$ is implied by the stronger inequality $i \leq a-t$ (A.3) and will be discussed as item (A.3). $\boxed{i \leq d}$ is used for the notion of pseudo-expansion to be well-defined (see above) and is true for any pseudo-expansion.

(A.2) $\boxed{i \leq s}$ was used for the definition of pseudo-expansion to be well-posed.
$\boxed{s - i \leq b}$ $\iff$ The number of 1's in the third row of $\mathfrak{E}(i)$ is less than or equal to b, the number of boxes in the second row of λ.
$\boxed{s - i \leq c}$ is implied by $\boxed{c - (s - i) \geq t}$ in (A.3).

(A.3) $\boxed{a - i \geq t}$ $\iff$ The number of 1's in the second row of $\mathfrak{E}(i)$ is $\leq a$.
$\boxed{c - (s - i) \geq t}$ $\iff$ The number of 1's in the first row of $\mathfrak{E}(i)$ is greater than or equal to the number of 2's in the second row of $\mathfrak{E}(i)$.

Moreover, one remarks that in case (A) the assertion that the number of 1's in the second row of $\mathfrak{E}(i)$ is greater than or equal to the number of boxes $\mathfrak{b}$ of μ added to the second row of λ for which there exists another box of μ that is added to the third row of λ and is in the same column as $\mathfrak{b}$, is equivalent to the inequality $t + i \geq s - b$ which is implied by the inequality $s - i \leq b$ of (A.2) since $t \geq 0$.

(B) Suppose $t < 0$. The inequalities (B.1)-(B.3) can be interpreted in terms of the pseudo-expansion $\mathfrak{E}(i)$ as follows:

(B.1) The inequality $\boxed{i \le a}$ *means that the number of* 1*'s in the second row of* $\mathfrak{E}(i)$ *is* $\le a$. *The inequality* $\boxed{i \le d}$ *is implied by the inequality* $i \le d + t$ *to be discussed under item (B.3).*

(B.2) $\boxed{i \le s + t}$ *is used for the definition of pseudo-expansion to be well-posed.*
The inequality $\boxed{s + t - i \le b}$ *is implied by the inequality* $b - (s + t - i) \ge -t$ *to be discussed under item (B.3).*
$\boxed{s + t - i \le c}$ $\iff$ *The total number of* 1*'s minus those in the last row of* $\mathfrak{E}(i)$ *is greater than or equal to the total number of* 2*'s in* $\mathfrak{E}(i)$.

(B.3) The inequality $\boxed{b - (s + t - i) \ge -t}$ *is equivalent to saying that the number of* 1*'s in the second row of* $\mathfrak{E}(i)$ *is greater than or equal to the number of boxes* $\mathfrak{b}$ *of* μ *added to the second row of* λ *for which there exists another box of* μ *that is added to the third row of* λ *and is in the same column as* $\mathfrak{b}$.
$\boxed{d - i \ge -t}$ *was used for being able to define the notion of pseudo-expansion.*

Moreover, one remarks that in case (B) the assertion that the number of 1*'s in the first row of* $\mathfrak{E}(i)$ *is greater than or equal to the number of* 2*'s in the second row of* $\mathfrak{E}(i)$ *is equivalent to the inequality* $c \ge s + 2t - i$ *and this is implied by the inequality* $c \ge s + t - i$ *in (B.2) above since* $t < 0$.

Proof. Before proving the assertions under (A) and (B), we show how they imply that a pseudo-expansion $\mathfrak{E}(i)$ is a Littlewood Richardson expansion if and only if (A.1)-(A.3) resp. (B.1)-(B.3) hold. Note that the interpretations of these inequalities given above under (A) resp. (B) obviously hold for a Littlewood Richardson expansion. Conversely, suppose $\mathfrak{E}(i)$ is a pseudo-expansion with i satisfying (A.1)-(A.3) resp. (B.1)-(B.3). First we have to see that the boxes of λ in $\mathfrak{E}(i)$ together with the boxes of μ with labels ≤ 1 form again a Young diagram. In case (A) this follows from the definition of pseudo-expansion and the inequalities $\boxed{s - i \le b}$ and $\boxed{a - i \ge t}$. In case (B) this follows from $\boxed{i \le a}$ and $\boxed{s + t - i \le b}$ since the number of ones in the third row is $s - (i - t)$ in this case. The assertion that the boxes of μ with labels ≤ 2 together with the boxes of λ form again a Young diagram is true by definition of pseudo-expansion in both cases.

Let us see next that no column in $\mathfrak{C}(i)$ can contain boxes with equal labels. In case (A) this follows from $\boxed{s-i \le b}$ together with the remark after item (A.3) and $\boxed{a-i \ge t}$ (no 1's in the same column). To see that there are no 2's in the same column remark that $s \le b+i+t$. In case (B) the asssertion follows from $\boxed{s+t-i \le b}$ together with $\boxed{b-(s+t-i) \ge -t}$ and finally $\boxed{i \le a}$ (no 1's in the same column). The fact that there are no 2's in the same column follows since $s \le b+i$ (since $\boxed{b-(s+t-i) \ge -t}$).
Finally we have to verify that if we read the string of 1's and 2's in $\mathfrak{C}(i)$ from right to left and from top down, then when we stop reading at an arbitrary point, we have always read at least as many 1's as 2's. In case (A) this is implied by $\boxed{c-(s-i) \ge t}$ and the inequality $\boxed{s-i \le c}$; the latter means that the total number of 1's minus the number of 1's in the third row of $\mathfrak{C}(i)$ (which is $= c+d-(s-i)$) is greater than or equal to the total number of 2's in $\mathfrak{C}(i)$ which is d. In case (B) this follows from the remark after (B.3) and $\boxed{s+t-i \le c}$.

It remains to prove the assertions under (A) and (B) of the Lemma. We consider case (A) first. We recall that the number of 1's in the third row of $\mathfrak{C}(i)$ is $s-i$ and the number of 1's in the second row of $\mathfrak{C}(i)$ is $i+t$. Then all the assertions are obvious except the interpretation of the inequality $\boxed{c-(s-i) \ge t}$, and the concluding remark after (A.3). Now the number of 1's in the first row of $\mathfrak{C}(i)$ is $(c+d)-(s-i)-(i+t)$ and the number of 2's in the second row is $s+f-b-(i+t)$. But

$$s+f-b-(i+t) \le (c+d)-(s-i)-(i+t)$$

is certainly equivalent to

$$d-i \le (c+d)-(s-i)-(i+t)$$

which is equivalent to $\boxed{c-(s-i) \ge t}$ as claimed.
The remark after (A.3) is seen by remarking that the number of boxes $\mathfrak{b}$ defined above is $s-b$ in this case.

We turn to the inequalities in case (B). Recall that there are $s-(i-t)$ 1's in the third row of $\mathfrak{C}(i)$ in this case, and i 1's in the second row of $\mathfrak{C}(i)$. Most of the inequalities follow immediately from this. The inequality

$i \geq s - b$ expresses again the property that the number of 1's in the second row is greater than or equal to the number of boxes $\mathfrak{b}$ in item (B.3), and is equivalent to $\boxed{b - (s + t - i) \geq -t}$.
The number of 1's in the first row of $\mathfrak{E}(i)$ is in this case $c + d - i - (s + t - i)$ and the number of 2's in the second row is $f + s - b - i$, and

$$c + d - i - (s + t - i) \geq f + s - b - i$$

is equivalent to (since $f + s - b - t = d$)

$$c + d - s - 2t + i \geq d$$

which proves the assertion in the remark after (B.3).
It remains to consider $\boxed{s + t - i \leq c}$ in (B.2). In fact the total number of 1's in $\mathfrak{E}(i)$ minus the number of 1's in the third row is equal to $c + d - (s - i + t)$ and the condition that this be greater than or equal to the total number of 2's ($= d$) translates exactly into $\boxed{s + t - i \leq c}$. $\square$

With Lemma 5.2.2.3 we have proven Proposition 5.2.2.1 in full.

5.2.3 Linear independence of elementary maps

In this subsection we prove that the maps in Theorem 5.2.1.1 are linearly independent in both cases (A) and (B), thus concluding the proof of the Theorem. Note that the element $p_0 := (e_1^a \otimes x_3^b) \otimes (e_3^c \otimes x_1^d)$ is in the subspace $V(a,\, b) \otimes V(c,\, d) \subset (S^a \otimes D^b) \otimes (S^c \otimes D^d)$ by the definition of Δ in formula 5.1. Note also that the image of the map

$$\delta \,:\, S^{e-1} \otimes D^{f-1} \to S^e \otimes D^f, \quad \delta = \sum_{i=1}^{3} e_i \otimes x_i$$

is a complement to the subspace $V(e,\, f)$ in $S^e \otimes D^f$. It thus suffices to prove the following Lemma.

Lemma 5.2.3.1. *In case (A) the polynomials*

$$(\vartheta^t \circ \beta^{s-i} \circ \alpha^i)(p_0)$$

where i ranges over the set of non-negative integers satisfying (A.1)-(A.3), are linearly independent in $(S^e \otimes D^f)/\mathrm{im}(\delta)$.
In case (B) the polynomials

$$(\omega^{-t} \circ \beta^{s+t-i} \circ \alpha^i)(p_0)$$

where i ranges over the set of non-negative integers satisfying (B.1)-(B.3), are linearly independent in $(S^e \otimes D^f)/\mathrm{im}(\delta)$.

Proof. In case (A) we compute

$$(\vartheta^t \circ \beta^{s-i} \circ \alpha^i)(p_0) = (\text{nonzero constant}) \cdot e_1^{a-i-t} e_3^{c-s+i-t} \otimes x_3^{b-s+i} x_2^t x_1^{d-i}$$

which is in each case a non-zero monomial in $S^e \otimes D^f$ (note also that all exponents are nonnegative), and in case (B) we obtain

$$(\omega^{-t} \circ \beta^{s+t-i} \circ \alpha^i)(p_0) = (\text{nonzero constant}) \cdot e_1^{a-i} e_2^{-t} e_3^{c-s-t+i} \otimes x_3^{b-s+i} x_1^{d-i+t} \,,$$

a non-zero monomial in $S^e \otimes D^f$. Each nonzero bihomogeneous polynomial in the subspace

$$\mathrm{im}(\delta) = (e_1 \otimes x_1 + e_2 \otimes x_2 + e_3 \otimes x_3) \cdot (S^{e-1} \otimes D^{f-1}) \subset S^e \otimes D^f$$

contains monomials (with nonzero coefficient) divisible by $e_2 \otimes x_2$. Since the preceding monomials in cases (A) resp. (B) are not divisible by $e_2 \otimes x_2$, a linear combination of them can be zero modulo $\mathrm{im}(\delta)$ only if this linear combination is already zero as a polynomial in $S^e \otimes D^f$. But in both cases (A) and (B), the degrees of the above monomials with respect to the variable e_1 are pairwise distinct, so they cannot combine to zero nontrivially in $S^e \otimes D^f$. □

5.3 Algorithms related to the double bundle method

5.3.1 Matrix representatives for equivariant projections

To complete the picture, we will give in this section a method to compute the equivariant projection

$$\pi_{e,\, f} \, : \, S^e \otimes D^f \to V(e,\, f) \, .$$

There are several ways of doing this, but we will here describe one that yields matrix representatives for this map explicitly and is most suitable algorithmically for the subsequent applications to curves.

For simplicity, we assume $e \le f$. More generally, we will give a fast method to compute the G-equivariant map

$$p\,:\,V(e,\,0)\otimes V(0,\,f)\to V(e-i_1,\,f-i_1)\oplus\cdots\oplus V(e-i_m,\,f-i_m) \quad (5.6)$$
$$0\le i_1<i_2<\cdots<i_m\le M:=\min(e,\,f)\,.$$

We do this in two steps. First, we compute the projection

$$\begin{aligned}\psi_k\,:\,S^e\otimes S^f &\to S^k\otimes D^{f-e+k}\\ &= V(k,\,f-e+k)\oplus\cdots\oplus V(1,\,f-e+1)\oplus V(0,\,f-e)\,,\end{aligned}$$

for $k=0,\ldots,e$. In terms of the operator Δ in formula 5.1 we have $\psi_k=\Delta^{e-k}$. We choose random linear forms $u_1,\ldots,u_n$ in $e_1,\,e_2,\,e_3$ and random linear forms $v_1,\ldots,v_n$ in $x_1,\,x_2,\,x_3$ so that

$$u_1^e,\ldots,u_n^e\in S^e,\quad v_1^f,\ldots,v_n^f\in D^f$$

span S^e resp. D^f. To compute ψ_k it will thus suffice to compute $\Delta^{e-k}(u_i^e\otimes v_j^f)$. Let first be u and v linear forms in $e_1,\,e_2,\,e_3$ resp. $x_1,\,x_2,\,x_3$. We assume $v(u)\neq 0$ for otherwise $\Delta(u^e\otimes v^f)=0$. We put

$$U_1:=\frac{u}{v(u)}$$

so that $v(U_1)=1$ and complete $V_1:=v$ and U_1 to dual bases $U_1,\,U_2,\,U_3$ in $\mathbb{C}^3$ and $V_1,\,V_2,\,V_3$ in $(\mathbb{C}^3)^\vee$. Then

$$\begin{aligned}\Delta(u^e\otimes v^f) &= \left(\frac{\partial}{\partial U_1}\otimes\frac{\partial}{\partial V_1}+\frac{\partial}{\partial U_2}\otimes\frac{\partial}{\partial V_2}+\frac{\partial}{\partial U_3}\otimes\frac{\partial}{\partial V_3}\right)(u^e\otimes v^f)\\ &= f\cdot\frac{\partial}{\partial U_1}\left((v(u)U_1)^e\right)\otimes v^{f-1}\\ &= v(u)^e e\cdot fU_1^{e-1}\otimes v^{f-1}\\ &= v(u)e\cdot fu^{e-1}\otimes v^{f-1}\,.\end{aligned}$$

Hence

$$\psi_k(u_i^e\otimes v_j^f)=\Delta^{e-k}(u_i^e\otimes v_j^f) \quad (5.7)$$
$$=v_j(u_i)^{e-k}\cdot f\cdot\cdots\cdot(f-(e-k)+1)\cdot e\cdot\cdots\cdot(e-(e-k)+1)\cdot u_i^k\otimes v_j^{f-e+k}\,, \quad (5.8)$$

Now, secondly, we give a method to compute

$$\pi_{a,\,b}\,:\,S^a\otimes D^b\to V(a,\,b)=\ker(\Delta)\subset S^a\otimes D^b\,.$$

Note that -possibly precomposing with some iterates of Δ which can be computed by the preceding procedure- we get in this way a means to calculate maps p of the form given in formula 5.6 quite quickly.

Lemma 5.3.1.1. *One has*

$$\pi_{a,\,b}=\sum_{j=0}^{N}\mu_j\delta^j\Delta^j$$

for some $N\in\mathbb{N}$ and certain $\mu_j\in\mathbb{Q}$.

Proof. Let us denote by $\pi_{a,\,b,\,i}$ the equivariant projection

$$\pi_{a,\,b,\,i}\,:\,S^a\otimes D^b\to V(a-i,\,b-i)\subset S^a\otimes D^b$$

so that $\pi_{a,\,b}=\pi_{a,\,b,\,0}$. Look at the diagram

$$\begin{tikzcd}
S^a\otimes D^b \arrow[r, "\Delta^i"] & S^{a-i}\otimes D^{b-i} \arrow[d, "\pi_{a-i,\,b-i}"] \\
& V(a-i,\,b-i)\subset S^{a-i}\otimes D^{b-i} \arrow[ul, "\delta^i"]
\end{tikzcd}$$

By Schur's lemma,

$$\pi_{a,\,b,\,i}=\lambda_i\delta^i\pi_{a-i,\,b-i}\Delta^i \tag{5.9}$$

for some nonzero constants λ_i. On the other hand,

$$\pi_{a,\,b}=\mathrm{id}-\sum_{i=1}^{\min(a,\,b)}\pi_{a,\,b,\,i}\,.$$

Therefore, since the assertion of the Lemma holds trivially if one of a or b is zero, the general case follows by induction on i. $\square$

Note that to compute the μ_j in the expression of $\pi_{a,\,b}$ in Lemma 5.3.1.1, it suffices to calculate the λ_i in formula 5.9 which can be done by the rule

$$\frac{1}{\lambda_i}(e_1^{a-i}\otimes x_3^{b-i})=\left(\pi_{a-i,\,b-i}\circ\Delta^i\circ\delta^i\right)(e_1^{a-i}\otimes x_3^{b-i})\,.$$

Finally let

$$\chi = \sum_i \xi_i \delta^i \Delta^i$$

be a linear combination as the one in Lemma 5.3.1.1, considered as a map from $S^a \otimes D^b$ to itself. To compute its value $\chi(u^a \otimes v^b)$ fast (where u and v as before are linear forms in e_1, e_2, e_3 resp. x_1, x_2, x_3) we recall that according to formula 5.7

$$\delta^i \Delta^i (u^a \otimes v^b) = \delta^i \left(v(u)^i \frac{a!}{(a-i)!} \frac{b!}{(b-i)!} u^{a-i} \otimes v^{b-i} \right)$$

and we compute the bihomogeneous polynomial $\chi(u^a \otimes v^b)$ by evaluating it in sufficiently many points $p_k \in (\mathbb{C}^3)^\vee$, $q_l \in \mathbb{C}^3$ and interpolation. In fact,

$$\begin{aligned} &\chi(u^a \otimes v^b)(p_k, q_l) \\ = &\sum_i \xi_i \left\{ (\delta(p_k, q_l))^i \cdot \left(v(u)^i \frac{a!}{(a-i)!} \frac{b!}{(b-i)!} u(p_k)^{a-i} v(q_l)^{b-i} \right) \right\} . \end{aligned}$$

Remark 5.3.1.2. If we know how to compute an $\mathrm{SL}_3(\mathbb{C})$-equivariant bilinear map $\psi : U \otimes V \to W$, in the sense say, that upon choosing bases $u_1, \dots, u_r$ in U, $v_1, \dots, v_s$ in V, $w_1, \dots, w_t$ in W, we know the t matrices of size $r \times s$

$$M^1, \dots, M^t$$

given by

$$(M^k)_{ij} := (w_k)^\vee (\psi(u_i,\, v_j)) ,$$

then the map

$$\begin{aligned} \tilde{\psi} &: W^\vee \otimes V \to U^\vee , \\ \tilde{\psi}(l_W,\, v)(u) &= l_W(\psi(u,\, v)) \;\; l_W \in W^\vee,\; v \in V,\; u \in U \end{aligned}$$

induced by ψ has a similar representation by r matrices of size $t \times s$

$$N^1, \dots, N^r$$

in terms of the bases $w_1^\vee, \dots, w_t^\vee$ of $W^\vee$, $v_1, \dots, v_s$ of V, and $u_1^\vee, \dots, u_r^\vee$ of $U^\vee$. In fact,

$$\begin{aligned} (N^i)_{kj} &= (\tilde{\psi}(w_k^\vee,\, v_j))(u_i) \\ &= w_k^\vee(\psi(u_i,\, v_j)) = (M^k)_{ij} . \end{aligned}$$

This remark is sometimes convenient for computational purposes.

5.4 Applications to rationality questions

5.4.1 Results for plane curves

The results on the moduli spaces of plane curves $C(d)$ of degree d that we obtain using the computational methods in this chapter are described below. We organize them according to the method employed.

Double Bundle Method. As we mentioned above, Katsylo obtained in [Kat89] the rationality of $C(d)$, $d \equiv 0 \pmod 3$ and $d \geq 210$. Using the computational scheme of section 5.3, we obtain the rationality of all $C(d)$ with $d \equiv 0 \pmod 3$ and $d \geq 30$ except $d = 48$. Moreover, we obtain rationality for $d = 10$ and $d = 21$ (the latter was known before, since by the results of [Shep], $C(d)$ is rational for $d \equiv 1 \pmod 4$). The cases $d = 27$ and $d = 54$ are special and are discussed below. For $d = 69$ the only candidate for application of the double bundle method is of the form given in Remark 5.3.1.2, but this case is covered by the results of [Shep] since $69 \equiv 1 \pmod 4$).

Method of Covariants. According to Chapter 4, $C(d)$ is rational for $d \equiv 1 \pmod 3$, $d \geq 37$, and $d \equiv 2 \pmod 3$, $d \geq 65$ (for $d \equiv 1 \pmod 9$, $d \geq 19$, rationality was proven before in [Shep]). By the method of Chapter 4, we improve this and obtain that $C(d)$ is rational for $d \equiv 1 \pmod 3$, $d \geq 19$, which uses the covariants S_d of Chapter 4, and rational for $d \equiv 2 \pmod 3$, $d \geq 35$, which uses the family of covariants T_d of Chapter 4.
Let us describe briefly how we obtain the above mentioned improvements algorithmically in this case and for this, we recall some facts from Chapter 4. As usual G is $\mathrm{SL}_3(\mathbb{C})$.

- For $d = 3n + 1$, $n \in \mathbb{N}$, and $V = V(0,\, d) = \mathrm{Sym}^d(\mathbb{C}^3)^\vee$, we took $W = V(0,\, 4)$ and produced covariants
$$S_d \,:\, V(0,\, d) \to V(0,\, 4)$$
of degree 4. We showed that property (b) of Theorem 2.2.3.2 holds for the space
$$L_S = x_1^{2n+3} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-2} \subset V(0,\, d)\,.$$
(cf. Proposition 4.2.3.1). Moreover, $\mathbb{P}(V(0,\, 4))/G$ is stably rational of level 8. So for particular values of d, it suffices to check property (c) of

Theorem 2.2.3.2 by explicit computation. We give the details how this is done below.

- For $d = 3n+2$, $n \in \mathbb{N}$, and $V = V(0,\, d) = \mathrm{Sym}^d(\mathbb{C}^3)^\vee$, we took $W = V(0,\, 8)$ and produced covariants

$$T_d \,:\, V(0,\, d) \to V(0,\, 8)$$

again of degree 4. In this case, property (b) of Theorem 2.2.3.2 can be shown to be true for the subspace

$$L_T = x_1^{2n+5} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-3} \subset V(0,\, d)\,.$$

(cf. again Proposition 4.2.3.1). $\mathbb{P}(V(0,\, 8))/G$ is stably rational of level 8, too, hence again everything comes down to checking property (c) of Theorem 2.2.3.2.

We recall from Chapter 4 how some elements of L_S (resp. L_T) can be written as sums of powers of linear forms which is very useful for evaluating S_d resp. T_d easily. Let K be a positive integer. Then Definition 4.2.4.1 was:

Definition 5.4.1.1. Let $\mathbf{b} = (b_1, \dots, b_K) \in \mathbb{C}^K$ be given. Then we denote by

$$p_i^{\mathbf{b}}(c) := \prod_{\substack{j \neq i \\ 1 \leq j \leq K}} \frac{c - b_j}{b_i - b_j} \tag{5.10}$$

for $i = 1, \dots, K$ the interpolation polynomials of degree $K-1$ w.r.t. $\mathbf{b}$ in the one variable c.

We have the following easy Lemma (see Lemma 4.2.4.2)

Lemma 5.4.1.2. *Let $\mathbf{b} = (b_1, \dots, b_K) \in \mathbb{C}^K$, $b_i \neq b_j$ for $i \neq j$, and set $x = x_1$, $y = \lambda x_2 + \mu x_3$, $(\lambda,\, \mu) \neq (0,\, 0)$. Suppose $d > K$ and put $l_i := b_i x + y$. Then for each $c \in \mathbb{C}$ with $c \neq b_i$, $\forall i$,*

$$f(c) = p_1^{\mathbf{b}}(c) l_1^d + \dots + p_K^{\mathbf{b}}(c) l_K^d - (cx+y)^d \tag{5.11}$$

is nonzero and divisible by x^K.

So for $K = 2n + 3$ we obtain elements $f(c) \in L_S$ and for $K = 2n + 5$ elements $f(c) \in L_T$. We now check property (c) of Theorem 2.2.3.2 computationally in the following way. We choose a fixed $g \in V(0,\, d)$ which we write as a sum of powers of linear forms

$$g = m_1^d + \cdots + m_{\mathrm{const}}^d$$

where const is a positive integer. We choose a random vector $\mathbf{b}$, random λ and μ, and a random c, and use formula (4.29) which reads

$$\begin{aligned}
S_d(f(c) + \epsilon g) &= S_d\Big(p_1^{\mathbf{b}}(c) l_1^d + \cdots + p_K^{\mathbf{b}}(c) l_K^d - (cx+y)^d \\
&\qquad + \epsilon m_1^d + \cdots + \epsilon m_{\mathrm{const}}^d\Big) \\
&= 24\Big(\epsilon^3 \sum_{\substack{i \\ j<k<p}} p_i^{\mathbf{b}}(c) I(l_i,\, m_j,\, m_k,\, m_p)^n l_i m_j m_k m_p \\
&\qquad - \epsilon^3 \sum_{j<k<p} I(cx+y,\, m_j,\, m_k,\, m_p)^n (cx+y) m_j m_k m_p \\
&\qquad + \epsilon^4 \sum_{i<j<k<p} I(m_i,\, m_j,\, m_k,\, m_p)^n m_i m_j m_k m_p\Big)
\end{aligned} \tag{5.12}$$

to evaluate S_d. Here I was a function on quadruples of linear forms to $\mathbb{C}$: if in coordinates

$$L_\alpha = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

and $L_\beta,\ L_\gamma,\ L_\delta$ are linear forms defined analogously, and if we moreover abbreviate

$$(\alpha\,\beta\,\gamma) := \det \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \quad \text{etc.},$$

then

$$I(L_\alpha,\, L_\beta,\, L_\gamma,\, L_\delta) := (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta)\,.$$

For T_d we have by an entirely analogous computation

$$\begin{aligned} T_d(f(c)+\epsilon g) &= T_d\Big(p_1^{\mathbf{b}}(c)l_1^d + \cdots + p_K^{\mathbf{b}}(c)l_K^d - (cx+y)^d \\ &\qquad + \epsilon m_1^d + \cdots + \epsilon m_{\text{const}}^d\Big) \\ &= 24\Big(\epsilon^3 \sum_{\substack{i \\ j<k<p}} p_i^{\mathbf{b}}(c) I(l_i,\, m_j,\, m_k,\, m_p)^n l_i^2 m_j^2 m_k^2 m_p^2 \\ &\qquad - \epsilon^3 \sum_{j<k<p} I(cx+y,\, m_j,\, m_k,\, m_p)^n (cx+y)^2 m_j^2 m_k^2 m_p^2 \\ &\qquad + \epsilon^4 \sum_{i<j<k<p} I(m_i,\, m_j,\, m_k,\, m_p)^n m_i^2 m_j^2 m_k^2 m_p^2\Big) \end{aligned} \tag{5.13}$$

So we can evaluate T_d similarly. Thus for each particular value of d we can produce points in $\mathbb{P}(V(0,\, 4))$, for $d = 3n+1$, or $\mathbb{P}(V(0,\, 8))$, for $d = 3n+2$, which are in the image of the restriction of S_d to a fibre of π_{L_S} resp. in the image of the restriction of T_d to a fibre of π_{L_T}. We then check that these span $\mathbb{P}(V(0,\, 4))$ resp. $\mathbb{P}(V(0,\, 8))$ to check condition (c) of Theorem 2.2.3.2.

The case $d = 27$**.** We establish the rationality of $C(27)$ as follows: there is a bilinear, $\mathrm{SL}_3(\mathbb{C})$-equivariant map

$$\psi\,:\, V(0,\, 27) \times (V(11,\, 2) \oplus V(15,\, 0)) \to V(2,\, 14)$$

and

$$\begin{aligned} &\dim V(0,\, 27) = 406,\ \dim V(11,\, 2) = 270, \\ &\dim V(15,\, 0) = 136,\ \dim V(2,\, 14) = 405\,. \end{aligned}$$

We compute ψ by the method of Theorem 5.2.1.1 and find that $\psi = \omega^2\beta^{11} \oplus \beta^{13}$ in the notation used there. For a random $x_0 \in V(0,\, 27)$, the kernel of $\psi(x_0, \cdot)$ turns out to be one-dimensional, generated by y_0 say, and $\psi(\cdot,\, y_0)$ has likewise one-dimensional kernel generated by x_0 (See[**?**], `degree27.m2` for a Macaulay script doing this calculation). It follows that the map induced by ψ

$$\mathbb{P}(V(0,\, 27)) \dashrightarrow \mathbb{P}(V(11,\, 2) \oplus V(15,\, 0))$$

is birational, and it is sufficient to prove rationality of $\mathbb{P}(V(11,2) \oplus V(15,0))/\mathrm{SL}_3(\mathbb{C})$. But $\mathbb{P}(V(11,\, 2) \oplus V(15,\, 0))$ is birationally a vector bundle over $\mathbb{P}(V(15,\, 0))$, and $\mathbb{P}(V(15,0))/\mathrm{SL}_3(\mathbb{C})$ is stably rational of level 19, so $\mathbb{P}(V(11,2)\oplus V(15,0))/\mathrm{SL}_3(\mathbb{C})$ is rational by the no-name lemma 2.2.2.1.

The case $d = 54$. We establish the rationality of $C(54)$ as follows: there is a bilinear, $\mathrm{SL}_3(\mathbb{C})$-equivariant map

$$\psi\colon V(0,54)\times\big(V(11,8)\oplus V(6,3)\oplus V(5,2)\oplus V(3,0)\big)\to V(0,51)$$

with

$$\dim V(0,\ 54) = 1540,\ \dim V(11,\ 8) = 1134, \dim V(6,\ 3) = 154,$$
$$\dim V(5,\ 2) = 81, \dim V(3,\ 0) = 10, \dim V(0,51) = 1378$$

Since $1134 + 154 + 81 + 10 = 1379 = 1378 + 1$ and $1540 - 1379 > 19$ we only need to check the genericity condition of Theorem 2.2.2.2 to prove rationality. For this we compute ψ by the method of Theorem 5.2.1.1 and find that $\psi = \beta^{11}\oplus\beta^6\oplus\beta^5\oplus\beta^3$ in the notation used there.

For a random $x_0 \in V(0,\ 54)$, the kernel of $\psi(x_0,\cdot)$ turns out to be one-dimensional, generated by y_0 say, and $\psi(\cdot, y_0)$ has full rank 1378 and therefore $\psi(V(0,54),\ y_0) = V(0,51)$ as required. See [?], `degree54.m2` for a Macaulay script doing this calculation.

Combining what was said above with the known rationality results for $C(d)$ for small values of d, we can summarize the current knowledge in the following table:

Degree d of curves	Result and method of proof/reference
1	rational (trivial)
2	rational (trivial)
3	rational (moduli space affine j-line)
4	rational, [Kat92/2], [Kat96]
5	rational, two-form trick [Shep]
6	rationality unknown
7	rationality unknown
8	rationality unknown
9	rational, two-form trick [Shep]
10	rational, double bundle method, this Chapter
11	rationality unknown
12	rationality unknown
13	rational, two-form trick [Shep]
14	rationality unknown
15	rationality unknown
16	rationality unknown
17	rational, two-form trick [Shep]
18	rationality unknown
19	Covariants, [Shep] and this Chapter
20	rationality unknown
21	rational, two-form trick [Shep]
22	Covariants, this Chapter
23	rationality unknown
24	rationality unknown
25	rational, two-form trick [Shep]
26	rationality unknown
27	rational, this Chapter (method cf. above)
28	Covariants, [Shep] and this Chapter
29	rational, two-form trick [Shep]
30	double bundle method, this Chapter
31	Covariants, this Chapter
32	rationality unknown
≥ 33 (excl. 48)	rational, this Chapter, Chapter 4, [Kat89]

Thus we obtain our main theorem.

Theorem 5.4.1.3. *The moduli space $C(d)$ of plane curves of degree d is rational except possibly for one of the values in the following list:*

$$d = 6,\ 7,\ 8,\ 11,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20,\ 23,\ 24,\ 26,\ 32,\ 48\,.$$

We want to mention that a decisive speed-up of the calculations for curve degrees divisible by 3 was obtained using the FFPACK-Library for linear algebra over finite fields and the skills of Jakob Kröker who carried out the algorithms with it. The code necessary to do the computations has been made available at a webpage [**?**].

5.4.2 Results for mixed tensors

In the following example we write down explicit matrix representatives for the maps given in 5.2.1.1 in one special case.

Example 5.4.2.1. In the decomposition of $V(1,\,1) \otimes V(1,\,1)$, the representation $V(1,\,1)$ occurs with multiplicity 2, corresponding to a two dimensional space

$$V(1,\,1) \otimes V(1,\,1) \to V(1,\,1)$$

of $\mathrm{SL}_3(\mathbb{C})$-equivariant maps. Here $a = b = c = d = e = f = s = 1$ and $t = 0$. Therefore a basis for this space of equivariant homomorphisms is given by α and β.

To give matrix representatives of α and β we use the vectors

$$\begin{aligned} q_{12} = e_1x_2,\ q_{13} = e_1x_3 \\ q_{21} = e_2x_1,\ q_{23} = e_2x_3 \\ q_{31} = e_3x_1,\ q_{32} = e_3x_2 \\ q_{22} = e_1x_1 - e_2x_2,\ q_{33} = e_1x_1 - e_3x_3 \end{aligned}$$

(in this order) as a basis of the 8-dimensional space $V(1,\,1)$. Using the definition of α and β we obtain:

$$\alpha = \begin{pmatrix} 0 & 0 & -\tfrac{2}{3}q_{22}+\tfrac{1}{3}q_{33} & 0 & q_{32} & 0 & q_{12} & q_{12} \\ 0 & 0 & q_{23} & 0 & \tfrac{1}{3}q_{22}-\tfrac{2}{3}q_{33} & 0 & q_{13} & q_{13} \\ \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} & 0 & 0 & 0 & 0 & q_{31} & -q_{21} & 0 \\ q_{13} & 0 & 0 & 0 & 0 & \tfrac{1}{3}q_{22}-\tfrac{2}{3}q_{33} & -q_{23} & 0 \\ 0 & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} & 0 & q_{21} & 0 & 0 & 0 & -q_{31} \\ 0 & q_{12} & 0 & -\tfrac{2}{3}q_{22}+\tfrac{1}{3}q_{33} & 0 & 0 & 0 & -q_{32} \\ -q_{12} & 0 & q_{21} & 0 & q_{31} & -q_{32} & -\tfrac{1}{3}q_{22}+\tfrac{2}{3}q_{33} & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} \\ 0 & -q_{13} & q_{21} & -q_{23} & q_{31} & 0 & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} & \tfrac{2}{3}q_{22}-\tfrac{1}{3}q_{33} \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} & q_{13} & 0 & 0 & -q_{12} & 0 \\ 0 & 0 & 0 & 0 & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} & q_{12} & 0 & -q_{13} \\ -\tfrac{2}{3}q_{22}+\tfrac{1}{3}q_{33} & q_{23} & 0 & 0 & 0 & 0 & q_{21} & q_{21} \\ 0 & 0 & 0 & 0 & q_{21} & -\tfrac{2}{3}q_{22}+\tfrac{1}{3}q_{33} & 0 & -q_{23} \\ q_{32} & \tfrac{1}{3}q_{22}-\tfrac{2}{3}q_{33} & 0 & 0 & 0 & 0 & q_{31} & q_{31} \\ 0 & 0 & q_{31} & \tfrac{1}{3}q_{22}-\tfrac{2}{3}q_{33} & 0 & 0 & -q_{32} & 0 \\ q_{12} & q_{13} & -q_{21} & -q_{23} & 0 & 0 & -\tfrac{1}{3}q_{22}+\tfrac{2}{3}q_{33} & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} \\ q_{12} & q_{13} & 0 & 0 & -q_{31} & -q_{32} & \tfrac{1}{3}q_{22}+\tfrac{1}{3}q_{33} & \tfrac{2}{3}q_{22}-\tfrac{1}{3}q_{33} \end{pmatrix}$$

Notice that $\alpha = \beta^t$.

Theorem 5.2.1.1 in conjunction with Theorem 1.3.2.7 yields rationality results for spaces of mixed tensors of which the following is a sample:

Theorem 5.4.2.2. *The space* $\mathbb{P}(V(4,\,4))/\mathrm{SL}_3(\mathbb{C})$ *is rational.*

Proof. In fact

$$V(1,\,7) \subset V(4,\,4) \otimes V(2,\,5)\,,$$
$$\dim V(4,\,4) = 125,\ \dim V(2,\,5) = 81,\ \dim V(1,\,7) = 80\,,$$

and the multiplicity of $V(1,\,7)$ in $V(4,\,4) \otimes V(2,\,5)$ is 2. More precisely here $s = 3$ and $t = 1$. By Theorem 5.2.1.1 $\psi = \vartheta \circ \beta \circ \alpha^2$ and $\phi = \vartheta \circ \alpha^3$ are independent equivariant projections to $V(1,\,7)$. We will use ψ in this argument.

We now consider the induced map

$$\Psi\,:\,\mathbb{P}(V(4,\,4)) \dashrightarrow \mathbb{P}(V(2,\,5))\,.$$

There are stable vectors in $\mathbb{P}(V(2,\,5))$ and on $\mathbb{P}(V(4,\,4)) \times \mathbb{P}(V(2,\,5))$ we can use $\mathcal{L} = \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ as $\mathrm{PGL}_3(\mathbb{C})$-linearized line bundle. Moreover, $\mathbb{P}(V(2,5))/\mathrm{PGL}_3(\mathbb{C})$ is stably rational of level 19 since the action of $\mathrm{PGL}_3(\mathbb{C})$

on pairs of 3×3 matrices by simultaneous conjugation is almost free, and the quotient is known to be rational.

Now consider a point $x_0 \in V(4,\ 4)$. If the map

$$\psi(x_0, \cdot)\ : V(2,\ 5) \to V(1,\ 7)$$

has maximal rank 80, Ψ is well defined. In this situation let y_0 be a generator of $\ker \psi(x_0, \cdot)$. If the map

$$\psi(\cdot,\ y_0)\ :\ V(4,\ 4) \to V(1,\ 7)$$

has also rank 80 we obtain that the fibre $\Psi^{-1}(\Psi([x_0]))$ has the expected dimension. For a random x_0 it is straight forward to check all of this using a computer algebra program. See [**?**] for a Macaulay2-script. We can therefore apply Theorem 1.3.2.7 in the way described in Example 1.3.2.9 and obtain that $\mathbb{P}(V(4,\ 4))/\mathrm{SL}_3(\mathbb{C})$ is rational. $\square$

Bibliography

[Adams] Adams, J.F., *Lectures on Exceptional Lie Groups*, ed. by Zafer Mahmud and Mamoru Mimura, Chicago Lectures in Mathematics Series, The University of Chicago Press, Chicago and London (1996)

[A-M] Artin, M. & Mumford, D., *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75-95

[B-CT-S-SwD] Beauville, A., Colliot-Thélène, J.-L., Sansuc, J.-J., Swinnerton-Dyer, Sir Peter, *Variétés stablement rationnelles non-rationnelles*, Annals of Math. **121** (1985), 283-318

[Beau00] Beauville, A., *Determinantal hypersurfaces*, Michigan Math. J., Vol. **48**, Issue 1 (2000), 39-64

[Be] Beklemishev, N., *Invariants of cubic forms in four variables*, Vestnik Moscov. Univ. Ser. I Mat. Mekh. **2** (1982), 42-49

[BB] Bialynicki-Birula, A., *Quotients by Actions of Groups*, in: Algebraic Quotients. Torus Actions and Cohomology. The Adjoint Representation and the Adjoint Action. Encyclopaedia of Mathematical Sciences, Invariant Theory and Algebraic Transformation Groups II (R.V. Gamkrelidze, V.L. Popov eds.), Springer-Verlag Berlin Heidelberg (2002)

[Bogo79] Bogomolov, F.A., *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR Izvestija, Vol. **13**, no. 3 (1979), 499-555

[Bogo1] Bogomolov, F., *Stable rationality of quotient varieties by simply connected groups*, Mat. Sbornik **130** (1986), 3-17

[Bogo2] Bogomolov, F., *Rationality of the moduli of hyperelliptic curves of arbitrary genus*, (Conf. Alg. Geom., Vancouver 1984), CMS Conf. Proceedings vol. **6**, Amer. Math. Soc., Providence, R.I. (1986), 17-37

[Bogo3] Bogomolov, F., *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. **51**, no. 3, (1987), 485-516

[Bo-Ka] Bogomolov, F. & Katsylo, P., *Rationality of some quotient varieties*, Mat. Sbornik **126** (1985), 584-589

[BPT-1] Bogomolov, F., Petrov, T. & Tschinkel, Y., *Rationality of moduli of elliptic fibrations with fixed monodromy*, GAFA, **12**, no. 6, p. 1105-1160 (2002)

[BPT-2] Bogomolov, F., Petrov, T. & Tschinkel, Y., *Unramified Cohomology of Finite Groups of Lie Type*, preprint (2008), arXiv:0812.0615 (2002)

[BvB08-1] Böhning, Chr. & Graf v. Bothmer, H.-Chr., *The rationality of the moduli spaces of plane curves of sufficiently large degree*, preprint (2008), arXiv:0804.1503

[BvB08-1a] Böhning, Chr. & Graf v. Bothmer, H.-Chr., Macaulay2 scripts to check the surjectivity of the Scorza and Octa maps. Available at http://www.uni-math.gwdg.de/bothmer/rationality, 2008.

[BvB08-2] Böhning, Chr. & Graf v. Bothmer, H.-Chr., *A Clebsch-Gordan formula for* $\mathrm{SL}_3(\mathbb{C})$ *and applications to rationality*, preprint (2008), arXiv:0812.3278

[BvB08-2a] Böhning, Chr. & Graf v. Bothmer, H.-Chr., Macaulay2 scripts for "A Clebsch-Gordan formula for $\mathrm{SL}_3(\mathbb{C})$ and applications to rationality". Available at http://www.uni-math.gwdg.de/bothmer/ClebschGordan, 2008.

[BvB09-1] Böhning, Chr., Graf v. Bothmer, H.-Chr. & Kröker, Jakob, *Rationality of moduli spaces of plane curves of small degree*, preprint (2009)

[B-S] Brion, M. & Schwarz, G.W., *Théorie des invariants & Géométrie des variétés quotients*, Collection Travaux en Cours, Hermann Éditeurs des Sciences et des Arts, Paris (2000)

[Brown] Brown, R.B., *Groups of type E_7*, Journal für die reine und angewandte Mathematik **236** (1969), 79-102

[Ch-G-R] Chernousov, V., Gille, Ph., Reichstein, Z., *Resolving G-torsors by abelian base extensions*, J. Algebra **296** (2006), 561-581

[Cl] Clebsch, A., *Theorie der binären algebraischen Formen*, B. G. Teubner, Leipzig (1872)

[C-L] Clebsch, A. & Lindemann, F., *Vorlesungen über Geometrie*, Erster Band. Geometrie der Ebene. B. G. Teubner, Leipzig (1876)

[C-G] Clemens, H. & Griffiths, P., *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95** (1972), 281-356

[CT95] Colliot-Thélène, J.-L., *Birational Invariants, Purity and the Gersten Conjecture*, in: "K-Theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras", Summer Research Institute on Quadratic Forms and Division Algebras, July 6-24, 1992, University of California, Santa Barbara. Bill Jacob, Alex Rosenberg eds. Proceedings of Symposia in Pure Mathematics Volume **58.1**, AMS (1995)

[CT-O] Colliot-Thélène, J.-L. & Ojanguren, M., *Variétés unirationelles non rationelles: au-delà de l'exemple d'Artin et Mumford*, Inv. Math. **97** (1989), 141-158

[CT-S] Colliot-Thélène, J.-L. & Sansuc, J.-J., *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*. Algebraic groups and homogeneous spaces, 113–186, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007.

[Dix] Dixmier, J., *On the Projective Invariants of Quartic Plane Curves*, Adv. in Math. **64** (1987), 279-304

[Dolg1] Dolgachev, I., *Rationality of fields of invariants*, in Algebraic Geometry, Bowdoin, Proc. Symp. Pure Math. vol. **46** (1987), 3-16

[Dolg2] Dolgachev, I., *Lectures on Invariant Theory*, London Mathematical Society Lecture Note Series **296**, Cambridge Univ. Press (2003)

[Dolg3] Dolgachev, I., *Topics in Classical Algebraic Geometry. Part I*, available at `http://www.math.lsa.umich.edu/~idolga/lecturenotes.html`

[D-K] Dolgachev, I. & Kanev, V., *Polar Covariants of Plane Cubics and Quartics*, Advances in Mathematics **98** (1993), 216-301

[D-O] Dolgachev, I., Ortland, D., *Point Sets in Projective Space and Theta Functions*, Astérisque, vol. **165** (1989)

[For79] Formanek, E., *The center of the ring of* 3×3 *generic matrices*, Linear and Multilinear Algebra **7** (1979), 203-212

[For80] Formanek, E., *The center of the ring of* 4×4 *generic matrices*, J. Algebra **62** (1980), 304-319

[For84] Formanek, E., *Rational Function Fields. Noether's Problem and Related Questions*, J. Pure and Applied Algebra **31** (1984), 28-36

[For02] Formanek, E., *The ring of generic matrices*, Journal of Algebra **258** (2002), 310-320

[Fu-Ha] Fulton, W. & Harris, J., *Representation Theory. A First Course*, Springer G.T.M. **129**, Springer-Verlag (1991)

[GKZ] Gelfand, I.M., Kapranov, M.M. & Zelevinsky, A.V., *Discriminants, Resultants and Multidimensional Determinants*, Mathematics: Theory and Applications, Birkhäuser Boston (1994)

[G-Y] Grace, J.H. & Young, W.H., *The Algebra of Invariants*, Cambridge Univ. Press (1903); reprinted by Chelsea Publ. Co. New York (1965)

[Gross] Grosshans, F.D., *Algebraic Homogeneous Spaces and Invariant Theory*, Springer LNM **1678**, Springer-Verlag Berlin Heidelberg (1997)

[Groth58] Grothendieck, A., *Torsion homologique et sections rationelles*, Séminaire Claude Chevalley, tome **3** (1958), exp. no. 5, 1-29

[Hess79] Hesselink, W.H., *Desingularizations of Varieties of Nullforms*, Invent. Math. **55** (1979), 141-163

[Hil93] Hilbert, D., *Über die vollen Invariantensysteme*, Math. Annalen **42** (1893), 313-373

[I-K] Iarrobino, A. & Kanev, V., *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Springer Lecture Notes in Mathematics **1721**, Springer-Verlag (1999)

[Is-Pr] Iskovskikh, V.A. & Prokhorov, Yu.G., *Fano Varieties*, Encyclopedia of Math. Sciences Vol. **47**, Algebraic Geometry V, A.N. Parshin, I.R. Shafarevich (Eds.), Springer-Verlag Berlin Heidelberg (1999)

[Jac59] Jacobson, N., *Some Groups of Transformations defined by Jordan Algebras. I.*, Journal für die Reine und Angewandte Mathematik **201** (1959), 178-195

[Jac60] Jacobson, N., *Some Groups of Transformations defined by Jordan Algebras. II. Groups of Type* F_4, Journal für die Reine und Angewandte Mathematik **204** (1960), 74-98

[Jac61] Jacobson, N., *Some Groups of Transformations defined by Jordan Algebras. III. Groups of Type* E_{6I}, Journal für die Reine und Angewandte Mathematik **207** (1961), 61-85

[Kat83] Katsylo, P.I., *Rationality of orbit spaces of irreducible representations of* SL_2, Izv. Akad. Nauk SSSR, Ser. Mat. **47**, No. 1 (1983), 26-36; English Transl.: Math USSR Izv. **22** (1984), 23-32

[Kat84] Katsylo, P.I., *Rationality of the moduli spaces of hyperelliptic curves*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), 705-710

[Kat89] Katsylo, P.I., *Rationality of moduli varieties of plane curves of degree* $3k$, Math. USSR Sbornik, Vol. **64**, no. 2 (1989)

[Kat91] Katsylo, P.I., *Birational geometry of moduli varieties of vector bundles over* $\mathbb{P}^2$, Math. USSR Izvestiya **38** (1992) no. 2, 419-428

[Kat92/1] Katsylo, P. I., *Rationality of the moduli variety of curves of genus* 5, Math. USSR Sbornik, vol. **72** (1992), no. 2, 439-445

[Kat92/2] Katsylo, P.I., *On the birational geometry of the space of ternary quartics*, Advances in Soviet Math. **8** (1992), 95-103

[Kat94] Katsylo, P.I., *On the birational geometry of* $(\mathbb{P}^n)^{(m)}/\mathrm{GL}_{n+1}$, Max-Planck Institut Preprint, MPI/94-**144** (1994)

[Kat96] Katsylo, P.I., *Rationality of the moduli variety of curves of genus* 3, Comment. Math. Helvetici **71** (1996), 507-524

[Ki-Pa] Kim, B. & Pandharipande, R., *The connectedness of the moduli space of maps to homogeneous spaces*, preprint (2000), arXiv:math/0003168v1

[Koll] Kollár, J., *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete **32**, Springer-Verlag New York (1996)

[Kord] Kordonskii, V.E., *Stable rationality of the group* Spin_{10}, Uspekhi Mat. Nauk **55** (2000), 171-172

[LeBr] Le Bruyn, L., *Centers of Generic Division Algebras, the Rationality Problem 1965-1990*, Israel Journal of Math. **76** (1991), 97-111

[Luna] Luna, D., *Slices étales*, Bull. Soc. Math. Fr., Suppl., Mém. **33** (1973), 81-105

[Mer] Merkurjev, A.S., *Unramified cohomology of classifying varieties for classical simply connected groups*, Ann. Sci. École Norm. Sup. (4) **35** (2002), 445-476

[Me-Ra] Mestrano, N., Ramanan, S., *Poincaré bundles for families of curves*, Journal für die reine und angewandte Mathematik **362** (1985), 169-178

[Mi] Miyata, T., *Invariants of certain groups, I*, Nagoya Math. J. **41** (1971), 69-73

[Muk1] Mukai, S., *Counterexamples to Hilbert's Fourteenth Problem for the 3-dimensional additive group*, preprint RIMS-1343, Kyoto Univ. (2001)

[Muk] Mukai, S., *An Introduction to Invariants and Moduli*, Cambridge studies in advanced mathematics **81**, Cambridge Univ. Press (2003)

[Mum] Mumford, D., Fogarty, J., Kirwan, F., *Geometric Invariant Theory*, Third Enlarged Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete **34**, Springer-Verlag (1994)

[Mum2] Mumford, D., *Lectures on curves on an algebraic surface*, Annals of Math. Studies **59**, Princeton University Press, Princeton (1966)

[Nag] Nagata, M., *On the fourteenth problem of Hilbert*, International Cong. Math., Edinburgh (1958)

[Ol] Olver, P. J., *Classical Invariant Theory*, London Mathematical Society Student Texts **44**, Cambridge University Press (1999)

[Pey] Peyre, E., *Unramified cohomology of degree* 3 *and Noether's problem*, Inv. Math. **171**, no. 1, (2008), 191-225

[Po92] Popov, V.L., *Groups, Generators, Syzygies, and Orbits in Invariant Theory*, Translations of Mathematical Monographs Vol. **100**, AMS Providence, Rhode Island (1992)

[Po03] Popov, V.L., *The Cone of Hilbert Nullforms*, Proc. of the Steklov Institute of Mathematics, Vol. **241** (2003), 177-194

[Po-Vi] Popov, V.L. & Vinberg, E.B., *Invariant Theory*, in: Algebraic Geometry IV, A.N. Parshin, I.R. Shafarevich (eds.), Encyclopedia of Mathematical Sciences, vol. **55**, Springer Verlag (1994)

[Post] Postnikov, M., *Lectures in Geometry. Semester V. Lie Groups and Lie Algebras*, Mir Publishers, Moscow (1986)

[Pro67] Procesi, C., *Noncommutative affine rings*, Atti Accad. Naz. Lincei, Ser. VIII **8** (1967), 239-255

[Pro76] Procesi, C., *The invariant theory of $n \times n$ matrices*, Adv. in Math. **19** (1976), 306-381

[Pro] Procesi, C., *Lie Groups. An Approach through Invariants and Representations*, Springer Universitext, Springer-Verlag (2007)

[R-S] Ranestad, K. & Schreyer, F., *Varieties of sums of powers*, J. Reine Angew. Math. **525** (2000), 147-181

[Ros] Rosenlicht, M., *Some basic theorems on algebraic groups*, American Journal of Mathematics **78** no. 2, 401-443

[Sal] Salmon, G., *A Treatise on the Higher Plane Curves*, Hodges, Foster and Figgis, (1879); reprinted by Chelsea Publ. Co. (1960)

[Sa] Saltman, D., *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), 71-84

[Sa2] Saltman, D., *Retract rational fields and cyclic Galois extensions*, Israel J. Math. **47** (1984), 165-215

[Sa3] Saltman, D., *Brauer Groups of Invariant Fields, Geometrically Negligible Classes, an Equivariant Chow Group, and Unramified H^3*, in: "K-Theory and Algebraic Geometry: Connections with Quadratic Forms and Division Algebras", Summer Research Institute on Quadratic Forms and Division Algebras, July 6-24, 1992, University of California, Santa Barbara. Bill Jacob, Alex Rosenberg eds. Proceedings of Symposia in Pure Mathematics Volume **58.1**, AMS (1995)

[Se56] Serre, J.-P., *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier Grenoble **6** (1956), 1-42

[Se58] Serre, J.-P., *Espaces fibrés algébriques*, Séminaire Claude Chevalley, tome **3** (1958), exp. no. 1, 1-37

[Se] Serre, J.-P., *Linear Representations of Finite Groups*, Springer GTM **42**, Springer-Verlag (1977)

[Se97] Serre, J.-P., *Galois Cohomology*, (translation of "Cohomologie Galoisienne") Springer Monographs in Mathematics, Springer (2002)

[Sesh] Seshadri, C.S., *Some results on the quotient space by an algebraic group of automorphisms*, Math. Ann. **149** (1963), 286-301

[Sesh72] Seshadri, C.S., *Quotient spaces modulo reductive algebraic groups*, Ann. of Math. **95** (1972), 511-556

[Shep] Shepherd-Barron, N.I., *The rationality of some moduli spaces of plane curves*, Compositio Mathematica **67** (1988), 51-88

[Shep89] Shepherd-Barron, N.I., *Rationality of moduli spaces via invariant theory*, Topological Methods in Algebraic Transformation Groups (New Brunswick, NJ, 1988), Progr. Math., vol. **80**, Birkhäuser Boston, Boston, MA (1989), 153-164

[Sp-Veld] Springer, T.A. & Veldkamp, F.D., *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monographs in Mathematics, Springer-Verlag (2000)

[Stein] Steinberg, R., *Nagata's example*, in: Algebraic Groups and Lie Groups, Cambridge Univ. Press (1997), 375-384

[Swan] Swan, R.G., *Invariant rational functions and a problem of Steenrod*, Invent. Math. **7** (1969), 148-158

[Wey] Weyman, J., *The Equations of Strata for Binary Forms*, Journal of Algebra **122** (1989), 244-249

[vdBer] Van den Bergh, M., *The Center of the Generic Division Algebra*, Journal of Algebra **127** (1989), 106-126