Canonical surfaces in $\mathbb{P}^4$ and Gorenstein algebras in codimension 2

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0 Introduction

There are 2 major circles of problems providing motivational background for this work:
The first centers around the questions in the theory of algebraic surfaces of general type concerning the existence of surfaces with prescribed invariants (the "geography") and, more systematically, the problem of describing their moduli spaces and canonical resp. pluricanonical models; I want to consider this general problem in the special case of surfaces with geometric genus $p_g = 5$, more precisely for canonical surfaces in $\mathbb{P}^4$ (i.e. those for which the canonical map is a birational morphism) with $q = 0$ and $p_g = 5$.
The second type of questions is more algebraic in spirit with principal aim to find a satisfactory structure theorem for Gorenstein algebras in codimension 2; roughly, these are finite $R-$algebras B (R some "nice" base ring) with $B \cong \mathrm{Ext}^2_R(B, R)$ (cf. section 1 below for precise definitions). With regard to a structure theorem, "satisfactory" means that one should be able to tell from practically verifiable and non-tautological conditions how the Hilbert resolution of B over R encodes 1) the "duality" $B \cong \mathrm{Ext}^2_R(B, R)$ and 2) the fact that B has not only an $R-$module structure, but also a ring structure. Whereas 1) is by now fairly well understood, 2) is not.
The link between these two problems is the following: Given a regular surface

S of general type with canonical map $S \to Y \subset \mathbb{P}^4$ a birational morphism, the canonical ring $\mathcal{R} = \bigoplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK))$ is a codimension 2 Gorenstein algebra over $\mathcal{A} = \mathbb{C}[x_0, \ldots, x_4]$, the homogeneous coordinate ring of $\mathbb{P}^4$. And conversely, starting from a Gorenstein algebra $\mathcal{R}$ in codimension 2 over $\mathcal{A}$, one recovers in shape of $X = \mathrm{Proj}(\mathcal{R})$ the canonical model of a surface S as above, provided $\mathrm{Ann}_{\mathcal{A}}(\mathcal{R})$ is prime (i.e. $Y = \mathrm{Supp}(\mathcal{R}) \subset \mathbb{P}^4$ with its reduced induced closed subscheme structure is an irreducible surface), X has only rational double points as singularities, and some weak technical assumption on a presentation matrix of $\mathcal{R}$ as $\mathcal{A}-$module holds (cf. thm. 1.6 below).

Let me give a little history of the development that lead to these ideas. The structure of canonical surfaces in $\mathbb{P}^4$ with $q = 0$, $p_g = 5$, $K^2 = 8$ and 9 was worked out already by F. Enriques (cf. [En], p. 284ff.; they are the complete intersections of type (2,4) and (3,3)). The case $K^2 = 10$ was solved by C. Ciliberto using liaison arguments. However, these techniques could not be utilized for higher K^2. D. Roßberg has constructed examples for the cases $K^2 = 11$ and $= 12$, and gives a partial description of the moduli spaces of these surfaces. He constructs these surfaces as degeneracy loci of morphisms between reflexive sheaves of rank n and $n+1$. The approach taken in this work relies instead on ideas in [Cat2]. In this paper canonical surfaces in $\mathbb{P}^3$ are studied (from a moduli point of view) via a structure theorem proved therein for Gorenstein algebras in codimension 1. It is shown that the duality $B \cong \mathrm{Ext}^1_R(B, R)$ for these algebras translates into the fact that the Hilbert resolution of B can be chosen to be self-dual; moreover, that the presence of a ring structure on B is equivalent to a (closed) condition on the Fitting ideals of a presentation matrix of B as $R-$module (the so-called "ring condition" or "rank condition" or "condition of Rouché-Capelli", abbreviated R.C. in any case) whence the moduli spaces of the surfaces studied in that paper can be parametrized by locally closed sets of matrices. These ideas were developed further and generalized in [M-P] and [dJ-vS] (within the codimension 1 setting). In particular the latter papers show that R.C. can be rephrased in terms of annihilators of elements of B and gives a good structure theorem also in the non-Gorenstein case. Subsequently, M. Grassi isolated in [Gra] the abstract kernel of the problem and proved that also for codimension 2 Gorenstein algebras the duality $B = \mathrm{Ext}^2_R(B, R)$ is equivalent to B having a self-dual Hilbert resolution. He introduced the concept of Koszul modules which provide a nice framework for dealing with Gorenstein algebras and also proposed a structure theorem for the codimension 2 case. Unfortunately, as for the question of how the ring structure of a codimension

2 Gorenstein algebra is encoded in its Hilbert resolution, the conditions he gives are tautological and (therefore) too complicated (although they are necessary and sufficient). More recently, D. Eisenbud and B. Ulrich ([E-U]) re-examined the ring condition and gave a generalization of it which appears to be more natural than the direction in which [Gra] is pointing. But essentially, they only give sufficient conditions for B to be a ring, and these are not fulfilled in the applications to canonical surfaces one has in mind. More information on the development sketched here can be found in [Cat4]. For a deeper study of that part of the story that originates from the duality $B \cong \mathrm{Ext}^2_R(B, R)$ and its effects on the symmetry properties of the Hilbert resolution of B, as well as for a generalization of this to the bundle case cf. [Wal].

The aim of this work is to show that whereas in codimension 1, R.C. and the symmetry coming from the Gorenstein condition can be treated as separate concepts, in codimension 2 they seem to be more intimately linked: For canonical surfaces in $\mathbb{P}^4$ with $p_g = 5$, $q = 0$, $K^2 \geq 10$ the symmetry implies R.C. under some mild extra conditions. Moreover, the ideas in [E-U] can be made to work in the geometric setting with some additional effort. Finally, a slightly stronger statement than in [Gra] can be made concerning the structure of the Hilbert resolution of a codimension 2 Gorenstein algebra.

The single sections are organized as follows:

Section 1 is meant to establish the connection mentioned above between canonical surfaces in $\mathbb{P}^4$ and Gorenstein algebras in codimension 2 and thus contains no new results. Some proofs have been added, partly for completeness, partly because occasionally minor simplifications of known proofs could be made by pulling together results from different sources, and sometimes adaptations to the chosen set-up were necessary.

In section 2 I study how for canonical surfaces with $q = 0$, $p_g = 5$, $K^2 = 11$ the ring structure of the canonical ring $\mathcal{R}$ is encoded in its Hilbert resolution by reduction of a presentation matrix of R to a normal form (modelled on an example in [Roß], p. 115). I show how this can be used to derive information on the moduli space of these surfaces.

In section 3 I use a localization argument to reduce, to a major extent, the study of the case of general K^2 to structural aspects of the case $K^2 = 10$ which is considered in [Cat4]. In particular it should be possible on the ground of this to subsequently decide on the existence of these surfaces and to study their moduli spaces (the problem largely boils down to an understanding of the Fano variety of $\mathbb{P}^4$'s on certain varieties of "Gorenstein-symmetric complexes of length 2"). In particular, the automaticity of the

ring condition for these surfaces is established under the assumption that their images $Y \subset \mathbb{P}^4$ have only improper double points in their nonnormal locus (a fact already hinted at in [Cat4], p. 48). Whether this last condition is really needed, or only veils some algebraic counterpart behind it, is doubtful (cf. the remarks at the end of this section).
Section 4 contains a result that improves on work in [Gra]; I have found no further use of it by now, but it could be embodied as a technical lemma in the attempt of finding a good structure theorem for Gorenstein algebras in codimension 2.
As for standard notation from surface theory (such as p_g for the geometric genus, P_m for the plurigenera etc.) I adhere to [Beau] (except that I write $e(S)$ instead of $\chi_{\text{top}}(S)$ for the topological Euler characteristic of S). My commutative algebra notation agrees largely with [Ei], but the following point (which traditionally seems to cause notational confusion) should be noted: For $I \subset R$ an ideal in a Noetherian ring and M a finite $R-$module, I write $\text{grade}(I, M)$ for the length of a maximal $M-$regular sequence contained in I $(= \min\{i : \text{Ext}^i_R(R/I, M) \neq 0\})$, and also, if there is no risk of confusion, $\text{grade}\, M := \text{grade}(\text{Ann}_R(M), R)$ and $\text{grade}\, I := \text{grade}(I, R)$. Furthermore if $R = (R, \mathfrak{m}, k)$ is a Noetherian local ring or graded ring with $\mathfrak{m}$ a unique maximal element among the graded proper ideals of R (e.g. a positively graded algebra over a field), I write $\text{depth}\, R := \text{grade}(\mathfrak{m}, R)$. This is in accordance with [B-He] and the terminology seems to go back to Rees.

1 Résumé of properties of canonical surfaces in $\mathbb{P}^4$

In this section I want to gather together the results on canonical surfaces in $\mathbb{P}^4$ needed in the sequel and give proofs for the more important of them.

Definition 1.1. *Let S be a smooth surface and $\pi : S \to Y \subseteq \mathbb{P}^4$ a morphism given by a 5-dimensional base-point free linear subspace L of $H^0(S, \mathcal{O}_S(K))$ and such that π is birational onto its image Y in $\mathbb{P}^4$. Then Y is called a canonical surface in $\mathbb{P}^4$ (and π an almost generic canonical projection).*

In the above situation, since K_S is nef, S is automatically a minimal model of a surface of general type.
Henceforth I will make the assumption that S is a regular surface, i.e. $q = h^1(S, \mathcal{O}_S) = 0$, basically because then the canonical ring $\mathcal{R} := = \bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(nK))$ enjoys the following property which makes it convenient to study by homological methods:

Proposition 1.2. *$\mathcal{R}$, viewed as a module over the homogeneous coordinate ring of $\mathbb{P}^4$ $\mathcal{A} = \mathbb{C}[x_0, \dots, x_4]$ via π, is a Cohen-Macaulay (CM) module iff S is a regular surface.*

Proof. Since K_S is nef and big, the Ramanujam vanishing theorem gives $H^1(S, \mathcal{O}_S(lK_S)) = 0$ for $l \in \mathbb{Z},\ l < 0$, which holds also for $l \geq 2$ by Serre duality $H^1(S, \mathcal{O}_S(lK_S)) \cong H^1(S, \mathcal{O}_S((1-l)K_S))$ on S. Therefore S is regular iff $H^1(S, \mathcal{O}_S(lK)) = 0\ \forall l \in \mathbb{Z}$ (taking again into account $H^1(S, \mathcal{O}_S(K_S)) \cong H^1(S, \mathcal{O}_S))$. I will prove that the latter is equivalent to $\mathcal{R}$ being CM.
In fact, if $\mathcal{R}$ is CM, $\mathrm{projdim}_{\mathcal{A}} \mathcal{R} = 2$ and in particular I have that $\mathrm{Ext}^3_{\mathcal{A}}(\mathcal{R}(i), \mathcal{A}(-5)) = 0\ \forall i$. By Serre duality $H^1(\mathbb{P}^4, \widetilde{\mathcal{R}(i)}) = 0\ \forall i$, the tilde denoting the sheaf associated to a graded module. But $\tilde{\mathcal{R}} \cong \pi_* \mathcal{O}_S$, hence $H^1(S, \mathcal{O}_S(lK_S)) = 0\ \forall l \in \mathbb{Z}$.
Conversely, suppose $H^1(S, \mathcal{O}_S(lK_S)) = 0\ \forall l \in \mathbb{Z}$. The idea is now to derive the CM property of $\mathcal{R}$ by looking at C on S, the pullback of a generic hyperplane section H of Y via π: genericity means that C is a nonsingular divisor in L such that $\pi|_C$ is a birational morphism onto H (by Bertini's theorem such C exists). It is known that then $\mathcal{R}' = \bigoplus_{m \geq 0} H^0(C, \mathcal{O}_C(mK_S))$ is CM (cf. [Sern], lemma 1.1). Assume H is cut out on Y by $x_4 = 0$ and consider for each m the cohomology long exact sequence

$$
\begin{aligned}
0 \longrightarrow H^0(S, \mathcal{O}_S((m-1)K_S)) \xrightarrow{\cdot x_4} H^0(S, \mathcal{O}_S(mK_S)) \longrightarrow H^0(C, \mathcal{O}_C(mK_S)) \\
\longrightarrow H^1(S, \mathcal{O}_S((m-1)K_S)) \longrightarrow \dots
\end{aligned}
$$

Since the H^1-terms vanish by hypothesis, I get $\mathcal{R}' = \mathcal{R}/x_4\mathcal{R}$, whence

$$
\begin{aligned}
2 = \dim(\mathcal{R}') = \mathrm{depth}_{\mathbb{C}[x_0,\dots,x_3]}(\mathcal{R}') &= \mathrm{depth}_{\mathbb{C}[x_0,\dots,x_3]}(\mathcal{R}/x_4\mathcal{R}) \\
&\leq \mathrm{depth}_{\mathbb{C}[x_0,\dots,x_4]}(\mathcal{R}) - 1
\end{aligned}
$$

since x_4 is regular on $\mathcal{R}$. But as $\dim(\mathcal{R}) = 3$ and $\mathrm{depth}_{\mathbb{C}[x_0,\dots,x_4]}(\mathcal{R}) \leq \dim(\mathcal{R})$, I get $\mathrm{depth}_{\mathbb{C}[x_0,\dots,x_4]}(\mathcal{R}) = \dim(\mathcal{R})$, i.e. $\mathcal{R}$ is CM. □

On the other hand, the fact that π is an almost generic canonical projection ($\mathcal{O}_S(K) \cong \pi^* \mathcal{O}_{\mathbb{P}^4}(1)$) implies that various remarkable duality statements hold for $\mathcal{R}$, which I shall frequently exploit in the sequel and which can be best expressed in terms of properties of the minimal free resolution of $\mathcal{R}$. Precisely:

Definition 1.3. *Let $R := k[x_1, \dots, x_r]$ be a polynomial ring in r indeterminates over some field k, graded in the usual way, and let B be a graded R-algebra. B is said to be a Gorenstein algebra of codimension c (and with twist $d \in \mathbb{Z}$) over $R :\Longleftrightarrow B \cong \mathrm{Ext}^c_R(B, R(d))$ as B-modules.*

[The B-module structure on $\mathrm{Ext}^c_R(B, R(d))$ is induced from B by functoriality of $\mathrm{Ext}^c_R(\cdot, R(d))$: If $b \in B$ and $m_b : B \to B$ is multiplication by b on B, the map $\mathrm{Ext}^c_R(m_b, R(d))$ is multiplication by b on $\mathrm{Ext}^c_R(B, R(d))$.]

Theorem 1.4. *With the hypotheses and notation of definition 1.1 $\mathcal{R}$ is a Gorenstein algebra of codimension 2 over $\mathcal{A}$ and as such has a minimal graded free resolution of the form:*

$$\mathbf{R}_\bullet : 0 \longrightarrow \bigoplus_{i=1}^{n+1} \mathcal{A}(-6+r_i) \xrightarrow{\binom{-\beta^t}{\alpha^t}} \bigoplus_{j=1}^{n+1} \mathcal{A}(-6+s_j) \oplus \bigoplus_{j=1}^{n+1} \mathcal{A}(-s_j)$$
$$\xrightarrow{(\alpha\,\beta)} \bigoplus_{i=1}^{n+1} \mathcal{A}(-r_i) \longrightarrow \mathcal{R} \longrightarrow 0\,.$$

Proof. (sketch) Setting $X := \mathrm{Proj}(\mathcal{R})$, the canonical model of S, I get that π, being given by a base-point free linear subsystem of $|K_S|$, factors through X as in the picture:

$$\begin{tikzcd} S \arrow[rr, "\pi"] \arrow[dr, "\kappa"'] & & Y \subset \mathbb{P}^4 \\ & X \arrow[ur, "\psi"'] & \end{tikzcd}$$

and ψ is a finite morphism onto Y. Hence by relative duality for finite morphisms (cf. e.g. [Lip], p. 48ff.), $\psi_*\omega_X = \mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y)$, where $\omega_Y = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\mathcal{O}_Y, \omega_{\mathbb{P}^4})$ and ω_X are the Grothendieck dualizing sheaves of Y, X resp.; but $\mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y) = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^4})$ since Y has codimension 2 in $\mathbb{P}^4$ (cf. also [Har], p. 242). Furthermore $\psi_*\omega_X = \tilde{\mathcal{R}}(1)$ (cf. [Cat 2], p.76, prop. 2.7) and $\psi_*\mathcal{O}_X = \tilde{\mathcal{R}}$. Thus I get

$$\tilde{\mathcal{R}} = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6)). \tag{1}$$

Since $\mathcal{R}$ is CM I get a length 2 resolution

$$0 \to F_2 \to F_1 \to F_0 \to \mathcal{R} \to 0, \tag{2}$$

with F_0, , F_1, F_2 graded free $\mathcal{A}-$modules. Taking $\mathrm{Hom}_{\mathcal{A}}(\cdot, \mathcal{A}(-6))$ of (2) I obtain

$$0 \to F_0^\vee(-6) \to F_1^\vee(-6) \to F_2^\vee(-6) \to \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6)) \to 0. \tag{3}$$

From the facts that $\mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6))$ is the sheaf associated to $\mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6))$, and $\tilde{\mathcal{R}}$ the sheaf associated to $\mathcal{R}$, and I have resolutions (2) and (3) of length 2 over $\mathcal{A}$ for these two modules, it follows easily that $\mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6))$ equals the full module of sections of the sheaf $\mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}, \mathcal{O}_{\mathbb{P}^4}(-6))$, and $\mathcal{R}$ the full module of sections of $\tilde{\mathcal{R}}$ (see section 3, lemma 3.3, below, where this argument is made precise); thus from (1) I infer the isomorphism of $\mathcal{A}-$modules

$$\mathcal{R} = \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6)), \tag{4}$$

which is also an isomorphism of $\mathcal{R}-$modules since it is functorial with respect to endomorphisms of $\mathcal{R}$ (which follows from the functoriality of the isomorphisms $\psi_*\omega_X = \mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y)$ and $\mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y) = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^4})$ above). The isomorphism (4) lifts to an isomorphism of minimal graded free resolutions (2) and (3). In particular, $\mathrm{rank}\, F_0 = \mathrm{rank}\, F_2$, and since $\mathrm{Ann}_{\mathcal{A}}(\mathcal{R}) \neq 0$ one has $\mathrm{rank}\, F_0 - \mathrm{rank}\, F_1 + \mathrm{rank}\, F_2 = 0$ whence there exists an integer n such that $\mathrm{rank}\, F_0 = \mathrm{rank}\, F_2 = n + 1$, $\mathrm{rank}\, F_1 = 2n + 2$. For the fact that now (2) can be symmetrized to give a resolution $\mathbf{R}_\bullet$ as in the statement of the theorem I refer to [Gra], p. 938ff., lemma 2.1 and proposition 2.3., whose proof applies in the present situation with minor modifications; for a thorough exposition of the argument that the isomorphism (4) gives rise to a symmetric resolution of $\mathcal{R}$ cf. also [Wal]. $\square$

Next I certainly have $p_g(S) \geq 5$ for surfaces S as in definition 1.1, and for simplicity I assume $p_g(S) = 5$ in what follows. As for K_S^2 of such surfaces, I list here:

- One can only expect to find canonical surfaces in $\mathbb{P}^4$ with $p_g = 5$ and $q = 0$ in the range $8 \leq K^2 \leq 54$. The lower bound follows from Castelnuovo's inequality $K^2 \geq 3p_g + q - 7$. The upper bound follows from the Bogomolov-Miyaoka-Yau inequality $K^2 \leq 3e(S)$ in combination with Noether's formula $K^2 + e(S) = 12(1 - q + p)$, where $e(S)$ is the topological Euler characteristic of S.

- For $K^2 = 8$ resp. $= 9$ the solutions one gets are the complete intersections of type $(2, 4)$ resp. $(3, 3)$ (cf. [En], p. 284ff.).

- Existence is known in cases $K^2 = 10, 11, 12$; the case $K^2 = 10$ is treated in [Cil], subsequently also in [Cat 4] (p. 42ff.) and [Roß] (p. 108ff.), by approaches different in taste each time. Moreover, in the latter case one has a satisfactory picture of the moduli space of these surfaces; for $K^2 = 11, 12$ a partial description of the moduli spaces is in [Roß].

Therefore I will also assume $K^2 \geq 10$ henceforth.
For the case $p_g = 5$, $q = 0$, $K^2 \geq 10$, the numbers n, r_i, s_i, $i = 1, \ldots, n+1$, appearing in the resolution $\mathbf{R}_\bullet$ of theorem 1.4 are readily calculated; this is done in [Cil], p. 302, prop. 5.3 (cf. also [Cat4], p. 41, prop. 6.2):

Theorem 1.5. *For a canonical surface in $\mathbb{P}^4$ with $q = 0$, $p_g = 5$, $K^2 \geq 10$ one has a resolution of the canonical ring $\mathcal{R}$*

$$\begin{aligned} \mathbf{R}_\bullet :\ 0 &\longrightarrow \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n \xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^{2n+2} \\ &\xrightarrow{(\alpha\,\beta)} \mathcal{A} \oplus \mathcal{A}(-2)^n \longrightarrow \mathcal{R} \longrightarrow 0, \end{aligned} \tag{5}$$

where $n := K^2 - 9$.

However, what is important here is that there is a converse to the story told so far, on which rests the analysis of canonical surfaces done in this work:

Theorem 1.6. *Let $\mathcal{R}$ be some algebra (commutative with 1) over $\mathcal{A} = \mathbb{C}[x_0, \ldots, x_4]$ with minimal graded free resolution as in (5), with $1 \in \mathcal{R}$ corresponding to the first row of $(\alpha\,\beta)$ as $\mathcal{A}-$module generator. Write $A := (\alpha\,\beta)$, $A' := A$ with first row erased, $I_n(A')$ = Fitting ideal of $n \times n$ minors of A'.*

Then $\mathcal{R}$ is a Gorenstein algebra, and if one assumes that $\operatorname{Ann}_{\mathcal{A}}(\mathcal{R})$ is a prime ideal, then $Y := \operatorname{Supp}(\mathcal{R}) \subseteq \mathbb{P}^4$ with its reduced induced subscheme structure (thus the ideal of polynomials vanishing on Y is $\mathcal{I}_Y = \operatorname{Ann}_{\mathcal{A}} \mathcal{R}$) is an irreducible surface, and if furthermore one assumes $\operatorname{grade} I_n(A') \geq 3$ and $X := \operatorname{Proj}(\mathcal{R})$ has only rational double points as singularities, then X is the canonical model of a surface S of general type with $q = 0, p_g = 5, K^2 = n+9$. More precisely, writing $\mathcal{A}_Y$ for the homogeneous coordinate ring of Y, one has that the morphism $\psi : X \to Y \subset \mathbb{P}^4$ induced by the inclusion $\mathcal{A}_Y \subset \mathcal{R}$ is a finite birational morphism, and is part of a diagram

$$\begin{tikzcd} S \arrow[rr, "\pi"] \arrow[dr, "\kappa"'] & & Y \subset \mathbb{P}^4 \\ & X \arrow[ur, "\psi"'] & \end{tikzcd}$$

where S is the minimal desingularization of X, κ is the contraction morphism contracting exactly the (-2)-curves of S to rational double points on X, and the composite $\pi := \psi \circ \kappa$ is a birational morphism with $\pi^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_S(K_S)$ (i.e. is 1-canonical for S).

Proof. Taking Hom of (5) into $\mathcal{A}(-6)$ and using the canonical isomorphism from (5) to its dual, one gets an isomorphism $\mathcal{R} = \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{R}, \mathcal{A}(-6))$, functorial with respect to endomorphisms of $\mathcal{R}$, which is therefore an isomorphism of $\mathcal{R}-$modules. Thus $\mathcal{R}$ is a Gorenstein algebra.
Remark that since the ideal of $(n+1)\times(n+1)$ minors of A, $I_{n+1}(A)$ (i.e. the zeroeth Fitting ideal of $\mathcal{R}$), and $\mathrm{Ann}_{\mathcal{A}}\mathcal{R}$ have the same radical, the Eisenbud-Buchsbaum acyclicity criterion (cf. [Ei], thm. 20.9, p. 500) gives $\mathrm{grade}\, I_{n+1}(A) = \mathrm{grade}\,\mathrm{Ann}_{\mathcal{A}}\mathcal{R} = \mathrm{codim}_{\mathcal{A}}\mathrm{Ann}_{\mathcal{A}}\mathcal{R} \geq 2$, whereas also $\mathrm{grade}\,\mathcal{R} \equiv \mathrm{grade}(\mathrm{Ann}_{\mathcal{A}}\mathcal{R}, \mathcal{A}) \leq \mathrm{projdim}_{\mathcal{A}}\mathcal{R} = 2$ (cf. e.g. [B-He], p.25), whence Y, defined by the annihilator ideal $\mathrm{Ann}_{\mathcal{A}}\mathcal{R} \subset \mathcal{A}$, is in fact an irreducible surface.
$\mathcal{R}$ is CM because $\mathrm{grade}(\mathcal{R}) = \mathrm{projdim}_{\mathcal{A}}(\mathcal{R})$ and thus $\mathcal{R}$ is a perfect module (cf. [B-He], p. 59, thm. 2.1.5). Next, the morphism $\psi : X \to Y$ induced by the inclusion $\mathcal{A}_Y \subset \mathcal{R}$ is finite since $\mathcal{R}$ is a finite $\mathcal{A}_Y-$module and thus $\psi_*\mathcal{O}_X = \tilde{\mathcal{R}}$ is a finite $\mathcal{O}_Y-$module over any affine open of Y. Now A' is a presentation matrix of $\mathcal{R}/(\mathcal{A}_Y \cdot 1)$ whence by Fitting's lemma, $I_n(A') \subset \mathrm{Ann}_{\mathcal{A}}(\mathcal{R}/\mathcal{A}_Y)$ and $(I_n(A')\cdot\mathcal{A}_Y)\,\mathcal{R} \subset \mathcal{A}_Y$. Since $(I_n(A')\cdot\mathcal{A}_Y) \neq 0$ ($I_n(A') \not\subset (Y)$ because $\mathrm{grade}\, I_n(A') \geq 3$) and $\mathcal{A}_Y$ is an integral domain, $\exists$ a (homogeneous) nonzerodivisor $d \in \mathcal{A}_Y$ such that $d\cdot\mathcal{R} \subset \mathcal{A}_Y$. Now d is also a nonzerodivisor on $\mathcal{R}$ because $\mathcal{R}$ is a maximal Cohen-Macaulay module over $\mathcal{A}_Y$ (see [Ei], p. 534, prop. 21.9). Thus one gets

$$\mathcal{R}[d^{-1}] = \mathcal{A}_Y[d^{-1}]. \tag{6}$$

(Note incidentally that $\mathcal{R}$ is an integral domain because it is contained in $d^{-1}\mathcal{A}_Y$ and that the algebra structure on $\mathcal{R}$ is uniquely determined since it is a subalgebra of $\mathcal{A}_Y[d^{-1}]$). From (6) one sees that ψ gives an isomorphism of function fields $\mathbb{C}(X) = \mathbb{C}(Y)$, thus is birational.
The fact that X has only rational double points as singularities implies that X is locally Gorenstein and the dualizing sheaf ω_X is invertible, $\omega_X = \mathcal{O}_X(K_X)$, where K_X is an associated (Cartier) divisor. Now one can run the argument given in theorem 1.4 in reverse: The sheafified Gorenstein condition $\tilde{\mathcal{R}} = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^4}(-6))$ together with $\mathcal{H}om_{\mathcal{O}_Y}(\psi_*\mathcal{O}_X, \omega_Y) = \mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X, \omega_{\mathbb{P}^4})$ and relative duality for the finite morphism ψ gives

$$\psi_*\omega_X = \tilde{\mathcal{R}}(1). \tag{7}$$

Therefore one finds $\mathcal{R} \cong \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X))$ where the latter is also equal to $\bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(mK_S))$ with $\kappa : S \to X$ the minimal desingularization of X and κ as in the statement of the theorem. Thus X is the canonical model of a surface of general type (since dim $\mathcal{R} = 3$). Furthermore, since $\psi^*\mathcal{O}_{\mathbb{P}^4}(1) = \mathcal{O}_X(K_X)$ and $\kappa^*\mathcal{O}_X(K_X) = \mathcal{O}_S(K_S)$, it follows that $\pi := \psi \circ \kappa$ is a 1-canonical map for S and clearly a birational morphism, and Y is a canonical surface in $\mathbb{P}^4$.
The invariants $p_g(S)$, $q(S)$, K_S^2 are immediately found from the resolution (5): On the one hand, for the plurigenera one has $P_1 = p_g$, $P_m = \binom{m}{2}K_S^2 + \chi(\mathcal{O}_S)$, $m \geq 2$, by Kodaira's formula (cf. [Bom], p. 185), on the other hand, writing $\mathcal{R}_m$ for the mth graded piece of $\mathcal{R}$, and $\bigoplus_j \mathcal{A}(-a_{0,j}) := \mathcal{A} \oplus \mathcal{A}(-2)^n$, $\bigoplus_j \mathcal{A}(-a_{1,j}) := \mathcal{A}(-3)^{2n+2}$, $\bigoplus_j \mathcal{A}(-a_{2,j}) := \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n$, one has $\dim_{\mathbb{C}} \mathcal{R}_m = \sum_{i=0}^2 (-1)^i \sum_j \binom{m-a_{i,j}+4}{4}$ from the Hilbert resolution of $\mathcal{R}$ ($\binom{k}{l} = 0$ for $k < l$). Comparing these one finds $p_g = 5$, $K^2 + 6 - q = 15 + n$, $3K^2 + 6 - q = 33 + 3n$ whence the invariants are the ones given in the theorem ($q = 0$ is also clear since $\mathcal{R}$ is CM by prop. 1.2). $\square$

Thus one morally sees that the important question remaining is how the algebra structure of $\mathcal{R}$ is reflected in the Hilbert resolution resp. to give necessary and sufficient conditions for the presentation matrix $(\alpha\ \beta)$ such that $\mathcal{R}$ supports the structure of an $\mathcal{A}-$algebra.

Remark 1. Let $\mathcal{R}$ be an $\mathcal{A}-$module with resolution (5) such that $\mathrm{Ann}_{\mathcal{A}}\mathcal{R}$ is prime (defining the surface $Y \subset \mathbb{P}^4$), grade $I_n(A') \geq 3$, and with a distinguished element, call it 1, in $\mathcal{R}$ corresponding to the first row of $(\alpha\,\beta)$. Then it follows from the proof of theorem 1.6 that $\exists$ a nonzerodivisor d on $\mathcal{R}$ such that $d \cdot \mathcal{R} \subset \mathcal{A}_Y$; thus if $\mathcal{R}$ is an algebra, it is a subalgebra of $\mathcal{A}_Y[d^{-1}]$. In particular, in all what follows, an algebra structure on $\mathcal{R}$, if it exists, will be unique. $\mathcal{R}$ is what is called a finite birational module in [E-U]. In other words, it's a fractional ideal of $\mathcal{A}_Y$. This entails for example that $\mathrm{Hom}_{\mathcal{A}_Y}(\mathcal{R}, \mathcal{A}_Y) = (\mathcal{A}_Y :_K \mathcal{R})$ (where K is the quotient field of $\mathcal{A}_Y$) is an ideal of $\mathcal{A}_Y$ (the so-called conductor of $\mathcal{R}$ into $\mathcal{A}_Y$).

Remark 2. With the set-up of theorem 1.6, $V(I_n(A'))$ = nonnormal locus of Y. In fact, $\psi : X \to Y$ is the normalization map, and therefore the sheaf of ideals $\mathcal{A}nn_{\mathcal{O}_{\mathbb{P}^4}}(\psi_*\mathcal{O}_X/\mathcal{O}_Y) = \mathcal{A}nn_{\mathcal{O}_{\mathbb{P}^4}}(\tilde{\mathcal{R}}/\mathcal{O}_Y)$ defines the nonnormal locus of Y. But since A' is a presentation matrix for $\mathcal{R}/\mathcal{A}_Y$, it is $\sqrt{\mathrm{Ann}_{\mathcal{A}}(\mathcal{R}/\mathcal{A}_Y)} = \sqrt{I_n(A')}$ (cf. e.g. [Ei], prop. 20.6, p. 498) and the assertion follows. The assumption grade $I_n(A') \geq 3$ in the theorem anticipates this fact.

Remark 3. Whereas the above theorem is valid without any condition on the singularities of Y, in the sequel it will be sometimes convenient to assume that $Y \subset \mathbb{P}^4$ has only improper double points as singularities (i.e. points with tangent cone consisting of two planes spanning $\mathbb{P}^4$); slightly more generally, the investigation of the ring structure of $\mathcal{R}$ as contained in theorem 3.1 below can be carried out under the assumption that Y is normal off a finite number of improper double points. Such Y is sometimes said to have quasi-ordinary singularities (it is said to have ordinary singularities if it is smooth off the improper double points). I state here (cf. [Cil], p. 306ff.):

Theorem 1.7. *Let $\pi : S \to Y \subset \mathbb{P}^4$ be a canonical surface with $q = 0$, $p_g = 5$. If Y has ordinary singularities, the number $\delta(Y) := \binom{K_S^2-8}{2}$ is the number of improper double points of Y (very special case of the "double point formula of Severi").*

2 Analysis of the case $K^2 = 11$

Let $\pi : S \to Y$ be a canonical surface in $\mathbb{P}^4$ with $q = 0,\ p_g = 5,\ K^2 = 11$.According to theorem 1.5, one has a resolution

$$\begin{aligned} \mathbf{R}_\bullet :\ 0 \longrightarrow \mathcal{A}(-6) \oplus \mathcal{A}(-4)^2 &\xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^6 \\ &\xrightarrow{(\alpha\,\beta)} \mathcal{A} \oplus \mathcal{A}(-2)^2 \longrightarrow \mathcal{R} \longrightarrow 0 \end{aligned} \tag{1}$$

of the canonical ring $\mathcal{R}$ of S. I want to examine how, in this particular case, $\mathbf{R}_\bullet$ encodes the ring structure of $\mathcal{R}$. In the next section I will treat the case of general K^2, but it may be worthwhile considering $K^2 = 11$ separately to get a feeling for the concepts entering the computations and because the results are accidentally slightly stronger in this case. More notation:

$$A := (\alpha\ \beta) =: \left(\begin{array}{ccc|ccc} A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\ a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array} \right), \tag{2}$$

where the $A_i,\ B_i,\ i = 1,2,3$, are cubic forms, the $a_j,\ b_j,\ j = 1,\dots,6$, are linear forms; $A' := A$ with 1^{st} row erased, $I_2(A), I_2(A')$:=Fitting ideals of $2 \times 2-$minors of $A,\ A'$ respectively; $\mathcal{I}_Y$:=ideal of polynomials vanishing on $Y = \mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$, $\mathcal{A}_Y$:=homogeneous coordinate ring of Y. Furthermore I will for simplicity assume that Y has only improper double points as singularities. Severi's double point formula then gives that there are 3 of them for

$K^2 = 11$; and we have: {3 improper double pt.s of Y}$= V(I_2(A'))$.
I claim:

Lemma 2.1. *Acting on the tableau in (2) with elements* $\begin{pmatrix} 1 & 0 \\ 0 & \varphi \end{pmatrix}$, $\varphi \in Gl_2(\mathbb{C})$, *from the left, and elements of* $Sp_6(\mathbb{C})$ *from the right, one can eventually obtain*

$$\tilde{A} = \left(\begin{array}{ccc|ccc} \tilde{A}_1 & \tilde{A}_2 & \tilde{A}_3 & \tilde{B}_1 & \tilde{B}_2 & \tilde{B}_3 \\ 0 & \tilde{a}_2 & \tilde{a}_3 & 0 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{a}_4 & -\tilde{a}_2 & 0 & \tilde{b}_4 & -\tilde{b}_2 & 0 \end{array} \right) =: (\tilde{\alpha}\ \tilde{\beta}) \tag{3}$$

preserving the symmetry: $\tilde{\alpha}\tilde{\beta}^t = \tilde{\beta}\tilde{\alpha}^t$. *The* $\tilde{a}_i$, $\tilde{b}_i$ *are linear forms s.t.*

$$\begin{aligned} V(\tilde{a}_2, \tilde{a}_3, \tilde{b}_2, \tilde{b}_3) &= \{1^{st} \textit{ improper dbl. pt.}\} \\ V(\tilde{a}_4, \tilde{a}_2, \tilde{b}_4, \tilde{b}_2) &= \{2^{nd} \textit{ improper dbl. pt.}\} \\ V(\tilde{a}_4, \tilde{a}_3, \tilde{b}_4, \tilde{b}_3) &= \{3^{rd} \textit{ improper dbl. pt.}\}. \end{aligned}$$

(The $\tilde{A}_j$, $\tilde{B}_j$, $j \in \{1, 2, 3\}$ *are of course cubics, linear combinations of the* A_j, B_j*).*

Proof. Write α_μ resp. β_ν for the $\mu-$th resp. $\nu-$th column of α resp. β. Let's make a list of some useful allowable operations on A, i.e. such that they preserve the symmetry:

(i) Elementary operations on rows: indeed, $\forall g = \begin{pmatrix} 1 & 0 \\ 0 & \varphi \end{pmatrix}$, $\varphi \in Gl_2(\mathbb{C})$: $\alpha\beta^t = \beta\alpha^t \Rightarrow (g\alpha)(g\beta)^t = (g\beta)(g\alpha)^t$

(ii) For $\lambda \in \mathbb{C}$ and μ a fixed but arbitrary column index, adding $\lambda\beta_\mu$ to α_μ: Remark that both sides of each of the equations $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li}$ are just changed by a summand $\lambda\beta_{h\mu}\beta_{l\mu}$. This operation is of course as well applicable with the rôles of α and β interchanged.

(iii) For $\lambda \in \mathbb{C}$ and μ, ν column indices, adding $\lambda\beta_\nu$ to α_μ and at the same time adding $\lambda\beta_\mu$ to α_ν: Both sides of each of the equations $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li}$ change by a summand $\lambda(\beta_{h\nu}\beta_{l\mu} + \beta_{h\mu}\beta l\nu)$. [(ii) is thus a special case of (iii) with $\mu = \nu$]; the same operation also with the rôles of α, β interchanged.

(iv) For $\lambda \in \mathbb{C}$ and $\mu \neq \nu$ column indices, adding $\lambda\alpha_\nu$ to α_μ and simultaneously subtracting $\lambda\beta_\mu$ from β_ν; this is O.K. since it corresponds to changing the left side of $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li}$ by a summand $\lambda(\alpha_{h\nu}\beta_{l\mu} - \alpha_{h\nu}\beta_{l\mu}) = 0$, and the right side by a summand

$\lambda(\beta_{h\mu}\alpha_{l\nu} - \beta_{h\mu}\alpha_{l\nu}) = 0$; the same operation also with the rôles of α, β interchanged.

(v) For $\mu \neq \nu$, interchanging columns α_μ, α_ν and at the same time interchanging columns β_μ, β_ν, which clearly preserves the symmetry.

(vi) For a column index μ, multiplying column α_μ by (-1) and then interchanging columns $-\alpha_\mu$ and β_μ (i.e. the substitution $\alpha_\mu \mapsto \beta_\mu$, $\beta_\mu \mapsto -\alpha_\mu$): Namely, $\sum_i \alpha_{hi}\beta_{li} = \sum_i \beta_{hi}\alpha_{li} \Leftrightarrow \sum_{i\neq\mu} \alpha_{hi}\beta_{li} - \beta_{h\mu}\alpha_{l\mu} = \sum_{i\neq\mu} \beta_{hi}\alpha_{li} - \alpha_{h\mu}\beta_{l\mu}$.

Call these operations (Op). Remark that (Op), (ii)-(vi) correspond to multiplication on A from the right by symplectic 6×6 matrices. In fact, more systematically, one sees that since symplectic matrices $\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \in Gl_{2n+2}(\mathbb{C})$, S_1, S_2, S_3, S_4 $(n+1) \times (n+1)$ matrices, can be characterized by equations $S_1 S_2^t = S_2 S_1^t$, $S_3 S_4^t = S_4 S_3^t$, $S_1 S_4^t - S_2 S_3^t = I_{n+1}$, one has for $A = (\alpha\ \beta)$ an $(n+1) \times (n+1)$ matrix with $\alpha\beta^t$ symmetric (as in thm. 1.5) that also $(\alpha S_1 + \beta S_3)(\alpha S_2 + \beta S_4)^t$ is symmetric (this is also immediate because the symmetry condition can be rephrased as saying that, for each choice of homogeneous coordinate vector $(x_0 : \ldots : x_4)$ in $\mathbb{P}^4$, the rows of $(\alpha\, \beta)$ span an isotropic subspace for the standard symplectic form on $\mathbb{C}^{2n+2}$, and a matrix is symplectic iff its transpose is).

First a general remark: Given a matrix of linear forms, call a generalized row of this matrix an arbitrary linear combination of the rows with not all coefficients zero. Then the locus where the rows are linearly dependent is the union, over all generalized rows, of the linear spaces cut out by the linear forms which are the entries of the generalized row. Therefore I can assume $A' = \left(\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array}\right)$ to be such that one of the improper double points is given by the vanishing of the linear forms in the upper row of A', the second one by the vanishing of the linear forms in the lower row of A', and the third as the zero set of the linear forms gotten by adding the two rows together.

The rest of the proof is a game on the tableau A', using (Op) and the symmetry, and deriving Koszul sequences from the fact that the rows of A' resp. their sum define 3 distinct points. To ease notation, I will treat the a_i, b_i, $i = 1, \ldots, 6$, and A' as dynamical variables. For clarity's sake, I will box certain assumptions in the course of the following argument, especially when they are cumulative.

Using (Op),(v)/(vi), then (iv) and finally (iii) one gets

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & b_6 \end{array} \right); \tag{4}$$

$\boxed{a_4 = 0}$: Then $D := \left\{ \mathrm{rk} \begin{pmatrix} a_2 & a_3 & b_1 & b_2 & b_3 \\ a_5 & a_6 & b_4 & b_5 & b_6 \end{pmatrix} \leq 1 \right\} = \{3 \text{ dbl. pt.s}\}$. But the determinantal locus D has the expected codimension since the generic $2 \times 5-$matrix degenerates in codimension 4. Therefore, by Porteous' formula (cf. [A-C-G-H], p. 90ff.), its degree is also the expected one, namely 5, a contradiction. Therefore quite generally the possibility of a zero column is excluded.

$\boxed{a_4 \neq 0}$: Use (Op), (v), (vi), (iv) , (iii) in this order to put a zero in place of b_6 ($a_4,\ a_5,\ a_6,\ b_5,\ b_6$ are dependent!) :

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & 0 \end{array} \right). \tag{5}$$

I claim that now $\boxed{a_2,\ b_1,\ b_2,\ b_3 \text{ are dependent}}$: For if they are independent I can also assume $a_4,\ a_5,\ a_6,\ b_5$ independent (otherwise interchange rows and use (Op), (v) and (vi)).Symmetry gives: $b_1 a_4 + b_2 a_5 + b_3 a_6 + b_5 \cdot (-a_2) = 0$, which is a Koszul relation saying $\exists$ antisymmetric matrices of scalars $S,\ \tilde{S}$ such that

$$\begin{pmatrix} a_4 \\ a_5 \\ a_6 \\ b_5 \end{pmatrix} = \tilde{S} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ -a_2 \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ -a_2 \end{pmatrix} = S \begin{pmatrix} a_4 \\ a_5 \\ a_6 \\ b_5 \end{pmatrix},$$

$\tilde{S} S = I,\ S,\ \tilde{S}$ are invertible. Now interchange the 4^{th} and 5^{th} columns of A' and multiply by $\left(\begin{array}{c|cc} S^t & \cdots & 0 \\ \hline \vdots & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$ on the right. (This will in general destroy the symmetry but preserve the points that are defined by the rows of A' and their sum; this operation is only used to derive a contradiction). The second row of the transformed matrix is then $(b_1, b_2, b_3, -a_2, b_4, 0)$, and one sees that it either defines $\emptyset$ or the same point as the first row, a contradiction because I assumed the points defined by the rows of A' to be distinct. Therefore $a_2,\ b_1,\ b_2,\ b_3$ are dependent.

I claim further that then $\boxed{a_2,\ b_2,\ b_3 \text{ are independent}}$. Suppose not. Since the possibility of a zero column was excluded for reasons of degree above, I can then use (Op), (iii) and if necessary (vi) to get $A' = \left(\begin{array}{ccc|ccc} 0 & 0 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & 0 \end{array}\right)$. Here $a_3,\ b_1,\ b_2,\ b_3$ are independent. I have 2 cases:

1. $a_4,\ a_5,\ a_6$ are independent. Then the symmetry gives that $\exists$ antisymmetric matrices of scalars $T,\ \tilde{T}$ s.t. $\begin{pmatrix} a_4 \\ a_5 \\ a_6 \end{pmatrix} = T \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \tilde{T} \begin{pmatrix} a_4 \\ a_5 \\ a_6 \end{pmatrix}$; but then $T,\ \tilde{T}$ are invertible contradicting the fact that a skewsymmetric matrix of odd-dimensional format has determinant zero.

2. $a_4,\ a_5,\ a_6$ are dependent. Since no zero column can occur, I can use (Op), (iv) to write $A' = \left(\begin{array}{ccc|ccc} 0 & 0 & a_3 & b_1 & b_2 & b_3 \\ a_4 & a_5 & 0 & b_4 & b_5 & 0 \end{array}\right)$; but the symmetry $a_4 b_1 = -a_5 b_2$ tells me I am left with discussing the case $A' = \left(\begin{array}{ccc|ccc} 0 & 0 & a_3 & -a_5 & a_4 & b_3 \\ a_4 & a_5 & 0 & b_4 & b_5 & 0 \end{array}\right)$. But then the points defined by the second row and the sum of the rows coincide, or the linear forms in the sum of the rows define $\emptyset$, a contradiction.

Using the last two boxed assumptions and (Op), (iii) and then (iv), I can pass from the shape of A' in (5) to

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & a_5 & a_6 & b_4 & b_5 & 0 \end{array}\right). \tag{6}$$

Now I play the game again, but this time it is quicker. I claim: $\boxed{a_5,\ a_6,\ b_5 \text{ dependent}}$: If not, the symmetry $a_5 b_2 + a_6 b_3 + b_5(-a_2) = 0$ gives as above the existence of 3×3 invertible skew-symmetric matrices, a contradiction. But I also claim: $\boxed{a_5,\ b_5 \text{ are independent}}$: Otherwise I get, using (Op), (ii) and possibly (vi), $A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & 0 & a_6 & b_4 & b_5 & 0 \end{array}\right)$, and using the symmetry $a_2 b_5 = a_6 b_3$, I must look at $A' = \left(\begin{array}{ccc|ccc} 0 & a_6 & a_3 & 0 & b_2 & b_5 \\ a_4 & 0 & a_6 & b_4 & b_5 & 0 \end{array}\right)$. But then either the points defined by the second row of A' and the sum of its rows resp. coincide, or the linear forms in the sum of the rows define $\emptyset$, a contradiction. Using the previous 2 boxed assumptions and (Op), (iv) and then (iii), I can pass from (6) to

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & a_5 & 0 & b_4 & b_5 & 0 \end{array}\right). \tag{7}$$

Invoking the symmetry $a_2b_5 = a_5b_2$ a last time, I am through:

$$A' = \left(\begin{array}{ccc|ccc} 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & -a_2 & 0 & b_4 & -b_2 & 0 \end{array} \right).$$

□

The above lemma implies that for $K^2 = 11$ the one half of the "Eisenbud-Ulrich ring condition" on the presentation matrix $A = (\alpha\ \beta)$ of $\mathcal{R}$ is more or less automatical:

Proposition 2.2. *Assume $A = (\alpha\ \beta)$ with $\alpha\beta^t = \beta\alpha^t$ is as in (1) (precisely, I assume A is a $3\times6-$matrix with first row cubic forms and other rows linear forms, and the locus where A drops rank consists of 3 distinct points (the improper double points of Y)). Then*

$$I_2(A) = I_2(A'). \tag{8}$$

Remark. The theorem of Eisenbud & Ulrich says, in this special case: If (8) holds and in addition grade$(I_2(A')) \geq 5$ or $I_2(A')$ is radical, then this suffices to give and determine a ring structure on $\mathcal{R}^{**}$ (double dual with respect to the $\mathcal{A}_Y-$module structure of $\mathcal{R}$). I will show below (cf. lemma 3.3) that in the situation I am in $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}^{**}$ as sheaves, but one remarks that $I_2(A')$ is generated by quadratics and the 3 points it defines trivially lie in a hyperplane in $\mathbb{P}^4$, so $I_2(A')$ is not radical; nor is grade$(I_2(A')) \geq 5$ since here $I_2(A')$ defines a codimension 4 subset. Cf. section 3 below that their theorem does not apply in case of higher K^2 either.

Proof. The idea, if there is any, is simply that I can write A as in Lemma 2.1, and then for each of the 3 points in the degeneracy locus of A I get a bunch of Koszul relations which allow me to check (8) by explicit computation. So write:

$$A = \left(\begin{array}{ccc|ccc} A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\ 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & -a_2 & 0 & b_4 & -b_2 & 0 \end{array} \right),$$

$v_1 := (a_2,\ a_3,\ b_2,\ b_3)^t$, $v_2 := (a_4,\ -a_2,\ b_4,\ -b_2)^t$, $v_3 = (a_4,\ a_3,\ b_4,\ b_3)^t$ regular sequences. Putting $W_1 := (-B_2,\ -B_3, A_2,\ A_3)^t, W_2 := (-B_1,\ -B_2,\ A_1,\ A_2)^t$, $W_3 := (B_1,\ B_3,\ -A_1,\ -A_3)^t$, the symmetry amounts to:

$$W_1^t v_1 = 0,\ W_2^t v_2 = 0,\ W_3^t v_3 = 0,$$

where the last equation is obtained by adding the first two together. These are Koszul relations saying that $\exists$ skew-symmetric matrices $P,\ Q,\ R$ of quadratic forms such that

$$W_1 = Pv_1,\ W_2 = Qv_2,\ W_3 = Rv_3. \tag{9}$$

$I_2(A')$ is generated by all possible products of elements of the first and second row ofA' resp. except $a_2^2,\ a_2b_2,\ b_2^2$. A direct computation using the relations (9) shows $I_2(A) \subseteq I_2(A')$. (To take up an example, look at $A_2b_2 - B_2a_2 \in I_2(A)$. Writing out the second resp. fourth vector component of the equation $W_2 = Qv_2$ in (9) gives

$$\begin{aligned} -B_2 &= Q_{21}a_4 + Q_{23}b_4 - Q_{24}b_2 \\ A_2 &= Q_{41}a_4 + Q_{24}a_2 + Q_{43}b_4 \end{aligned}$$

Multiplying the first by a_2, the second by b_2 and adding gives $A_2b_2 - B_2a_2 \in I_2(A')$ since $a_2a_4,\ a_2b_4,\ b_2a_4,\ b_2b_4 \in I_2(A')$. Here the skew-symmetry of Q is relevant. Similarly for the other minors.) $\square$

In section 3 (cf. p. 25ff. below) I will show that the condition $I_2(A') = I_2(A)$ of proposition 2.2 (or slightly weaker $\overline{I_2(A)} = \overline{I_2(A')}$ where the bar denotes saturation) implies under the assumption that Y has only improper double points as singularities that a ring structure is given to and determined on $\mathcal{R}$ via $\mathcal{R} = \mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_2(A')} \cdot \mathcal{A}_Y, \overline{I_2(A')} \cdot \mathcal{A}_Y)$. I'm sorry I have to refer forward to this result but I don't know any simpler proof in the special case $K^2 = 11$ than the one that applies uniformly to all higher K^2 as well. Assuming this for the moment and combining what has been said so far with theorem 1.6, what one gets out of the above discussion is this:
The datum (D) of

> *a matrix* $A = \left(\begin{array}{ccc|ccc} A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\ 0 & a_2 & a_3 & 0 & b_2 & b_3 \\ a_4 & -a_2 & 0 & b_4 & -b_2 & 0 \end{array}\right)$ *with the* $A_i,\ B_i,\ i = 1, \ldots, 3$ *cubic forms,* $a_2, a_3, a_4, b_2, b_3, b_4$ *linear forms on* $\mathbb{P}^4$ *satisfying the symmetry* $A_2b_2 + A_3b_3 + B_2(-a_2) + B_3(-a_3) = 0$, $A_1b_4 + A_2(-b_2) + B_1(-a_4) + B_2a_2 = 0$, *plus the open conditions that, with* $\mathcal{R} := \operatorname{coker} A$, $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$ *be prime and* $Y = \mathrm{Supp}(\mathcal{R}) \subset \mathbb{P}^4$ *(with its reduced induced closed subscheme structure) be an (irreducible) surface with only singularities* 3 *improper double points given by* $V(a_2, a_3, b_2, b_3)$, $V(a_4, -a_2, b_4, -b_2)$ *and* $V(a_4, a_2, b_4, b_2)$ *and* $X = \mathrm{Proj}\,\mathcal{R}$ *have only rational double points as singularities, modulo graded automorphisms of* $\mathcal{A} \oplus \mathcal{A}(-2)^2$ *resp.* $\mathcal{A}(-3)^6$

> *(acting on A from the right resp. left) which preserve the normal form of A just described, modulo automorphisms of* $\mathbb{P}^4$,

is equivalent to the datum (D') of

> *a canonical surface* $\pi : S \to Y \subset \mathbb{P}^4$ *with* $q = 0$, $p_g = 5$, $K^2 = 11$ *such that* Y *has only improper double points as singularities, modulo isomorphism.*

The benefit of the normal form in (D) to which the presentation matrices $A = (\alpha\ \beta)$ of the canonical rings $\mathcal{R}$ of the afore-mentioned surfaces can be reduced is that the symmetry condition ($\approx$ ring condition) $\alpha\beta^t = \beta\alpha^t$ can be explicitly solved (for $K^2 > 11$ I find no such normal form). Furthermore this can be used to describe the set of isomorphism classes of surfaces in (D') inside their moduli space $\mathfrak{M}_{K^2,\chi} = \mathfrak{M}_{11,6}$:
First one notes that as in the proof of prop. 2.2, the symmetry condition in (D) amounts to the existence of skew-symmetric 4×4 matrices $P = (P_{ij})$ and $Q = (Q_{ij})$ of quadratic forms such that $(-B_2, -B_3, A_2, A_3)^t = P(a_2, a_3, b_2, b_3)^t$ and $(-B_1, -B_2, A_1, A_2)^t = Q(a_4, -a_2, b_4, -b_2)^t$. Of course there is some ambiguity in choice of the (P_{ij}), (Q_{ij}), for the Koszul complexes $\mathbf{K}_\bullet(a_2, a_3, b_2, b_3)$ and $\mathbf{K}_\bullet(a_4, -a_2, b_4, -b_2)$ associated to these regular sequences

$$\mathcal{A}(-4) \overset{d_3}{\rightarrowtail} \mathcal{A}(-3)^4 \xrightarrow{d_2} \mathcal{A}(-2)^6 \xrightarrow{d_1} \mathcal{A}(-1)^4 \xrightarrow{d_0} \mathcal{A} \twoheadrightarrow \mathcal{A}/(a_2, a_3, b_2, b_3),$$
$$\mathcal{A}(-4) \overset{d_3'}{\rightarrowtail} \mathcal{A}(-3)^4 \xrightarrow{d_2'} \mathcal{A}(-2)^6 \xrightarrow{d_1'} \mathcal{A}(-1)^4 \xrightarrow{d_0'} \mathcal{A} \twoheadrightarrow \mathcal{A}/(a_4, -a_2, b_4, -b_2)$$

show that e.g. the vector $(P_{ij})_{i<j} \in \mathcal{A}(-2)^6_4$ is only determined up to addition of $d_2(\underline{l})$ where $\underline{l} \in \mathcal{A}(-3)^4_4$ is a vector of linear forms, and two $\underline{l}$'s give rise to the same $(P_{ij})_{i<j}$ iff they differ by $d_3(s)$ where $s \in \mathcal{A}(-4)_4$ is a complex scalar. In other words, $\dim_{\mathbb{C}}(\ker(d_1)_4) = 19$ and effectively, instead of the $(P_{ij})_{i<j}$, one chooses $(\overline{P}_{ij})_{i<j} \in \mathcal{A}(-2)^6_4/d_2(\mathcal{A}(-3)^4_4/d_3(\mathcal{A}(-4)_4))$. Similarly for the (Q_{ij}).
Next it is clear that whereas now $\boxed{P_{24}}$ and $\boxed{Q_{13}}$ are subject to no further relations, for the $\{(P_{ij})_{i<j}\} - \{P_{24}\}$ and $\{(Q_{ij})_{i<j}\} - \{Q_{13}\}$ the relations

$$A_2 = -P_{13}a_2 - P_{23}a_3 + P_{34}b_3 \qquad B_2 = -P_{12}a_3 - P_{13}b_2 - P_{14}b_3$$
$$A_2 = -Q_{14}a_4 + Q_{24}a_2 - Q_{34}b_4 \qquad B_2 = Q_{12}a_4 - Q_{23}b_4 + Q_{24}b_2$$

imply relations

$$Q_{14}a_4 + (P_{13} + Q_{24})(-a_2) + (-P_{23})a_3 + Q_{34}b_4 + P_{34}b_3 = 0 \tag{10}$$
$$Q_{12}a_4 + P_{12}a_3 + (-Q_{23})b_4 + (P_{13} + Q_{24})b_2 + P_{14}b_3 = 0. \tag{11}$$

I claim that I can assume that the sequences $(a_4, -a_2, a_3, b_4, b_3)$ and $(a_4, a_3, b_4, b_2, b_3)$ are both regular whence (10) and (11) would be Koszul relations. According to the normal form of the matrix A given in (D), a_4, a_3, b_4, b_3 are independent (and define one of the improper double points of Y). Assume both $-a_2$ and b_2 were expressible in terms of the latter. Then $V(a_2, a_3, b_2, b_3)$ and $V(a_4, a_3, b_4, b_3)$ would not give distinct points, contradiction. Therefore at least one of the sequences $(a_4, -a_2, a_3, b_4, b_3)$ and $(a_4, a_3, b_4, b_2, b_3)$ is regular. But if one of them, $(a_4, -a_2, a_3, b_4, b_3)$ say, is not regular, then replacing a_2 with $a_2 + b_2$ (which corresponds to applying once (Op), (ii) to the matrix A) the sequence $(a_4, -(a_2 + b_2), a_3, b_4, b_3)$ will be regular. Similarly if $(a_4, a_3, b_4, b_2, b_3)$ fails to be regular.
Therefore considering (10) and (11) as Koszul relations, one gets two skew-symmetric 5×5 matrices $L = (L_{kl})$ and $M = (M_{kl})$ of linear forms such that

$$(Q_{14}, P_{13} + Q_{24}, -P_{23}, Q_{34}, P_{34})^t = L(a_4, -a_2, a_3, b_4, b_3)^t \tag{12}$$
$$(Q_{12}, P_{12}, -Q_{23}, P_{13} + Q_{24}, P_{14})^t = M(a_4, a_3, b_4, b_2, b_3)^t. \tag{13}$$

Call $\underline{a} := (a_4, -a_2, a_3, b_4, b_3)$, $\underline{a}' := (a_4, a_3, b_4, b_2, b_3)$. Again looking at Koszul complexes

$$\mathcal{A}(-5) \overset{D_4}{\rightarrowtail} \mathcal{A}(-4)^5 \xrightarrow{D_3} \mathcal{A}(-3)^{10} \xrightarrow{D_2} \mathcal{A}(-2)^{10} \xrightarrow{D_1} \mathcal{A}(-1)^5 \xrightarrow{D_0} \mathcal{A} \twoheadrightarrow \mathcal{A}/\underline{a},$$
$$\mathcal{A}(-5) \overset{D_4'}{\rightarrowtail} \mathcal{A}(-4)^5 \xrightarrow{D_3'} \mathcal{A}(-3)^{10} \xrightarrow{D_2'} \mathcal{A}(-2)^{10} \xrightarrow{D_1'} \mathcal{A}(-1)^5 \xrightarrow{D_0'} \mathcal{A} \twoheadrightarrow \mathcal{A}/\underline{a}'$$

one sees that whereas e.g. the (L_{kl}) are not unique, the $(\overline{L}_{kl})_{k<l} \in \mathcal{A}(-2)^{10}_3/D_2(\mathcal{A}(-3)^{10}_3)$ are, and $\dim_{\mathbb{C}} (\ker(D_1))_3 = 10$. Likewise for the (M_{kl}).
Now equations (12) and (13) should be interpreted as saying that after one of P_{13} and Q_{24}, $\boxed{P_{13}}$ say, is chosen freely, the other P's and Q's in (12) and (13) are determined by L, M, $\underline{a}$, $\underline{a}'$.
Furthermore one remarks that then the 6 $\boxed{(L_{kl})_{k<l,k\neq 2,l\neq 2}}$ and the 6 $\boxed{(M_{kl})_{k<l,k\neq 4,l\neq 4}}$ satisfy no further relations, but the other ones enter in the following relation resulting from equating the 2nd resp. 4th vector components of (12) resp. (13):

$$(M_{14} - L_{12})a_4 + (M_{24} + L_{23})a_3 + (M_{34} + L_{24})b_4 + (-M_{45} + L_{25})b_3 = 0. \tag{14}$$

The sequence (a_4, a_3, b_4, b_3) is regular by the characterization of the normal form of A given in (D). One therefore infers the existence of a 4×4 skew-symmetric matrix $S = (S_{rs})$ of complex scalars such that

$$(M_{14} - L_{12}, M_{24} + L_{23}, M_{34} + L_{24}, -M_{45} + L_{25})^t = S(a_4, a_3, b_4, b_3)^t \tag{15}$$

and one notes that the (S_{rs}) are then uniquely determined from equation (14). Moreover upon choosing $\boxed{M_{14},\ M_{24},\ M_{34},\ M_{45}}$ arbitrarily, I can recover $L_{12},\ L_{23},\ L_{24},\ L_{25}$ from S and (a_4, a_3, b_4, b_3) using (15); and the 6 scalars $\boxed{(S_{rs})_{r<s}}$ are not subject to any other relation in the present set-up.

To get back to the study of the moduli space of surfaces in (D'), fit together the $a_t,\ b_t,\ t = 2, \ldots, 4$ and all the boxed objects above into one big affine space of parameters:

$$\mathcal{P} = \left\{ \begin{array}{c|l} & t \in \{2,3,4\};\ k,\ l,\ \kappa,\ \lambda \in \{1,\ldots,5\},\ \kappa < \lambda, \\ (a_t, b_t, P_{24}, Q_{13}, & k < l,\ k \neq 2,\ l \neq 2;\ r,\ s \in \{1,\ldots,4\},\ r < s; \\ P_{13}, L_{kl}, M_{\kappa\lambda}, S_{rs}) & \text{and } P_{24},\ Q_{13},\ P_{13} \text{ quadratic, } a_t,\ b_t,\ L_{kl}, \\ & M_{\kappa\lambda} \text{ linear in the hom. coord. } (x_0 : \ldots : x_4), \\ & S_{rs} \text{ complex scalars.} \end{array} \right\}.$$

Counting one finds that there are 3 quadratic forms, 22 linear forms and 6 scalars in $\mathcal{P}$, depending on 45, 110 and 6 parameters respectively, whence I have $\mathcal{P} = \mathbb{A}^{161}$.

According to the above discussion, for each choice in an open set of $\mathcal{P}$ one gets a matrix A meeting the requirements in (D) and a ring $\mathcal{R}$ which is the canonical ring of a surface of general type S as in (D'). In other words the parameter space for the canonical rings of the surfaces in (D') is a projection of an open set of $\mathcal{P}$. (In fact it would be necessary to show that this open set is non-empty; this is possible, making general choices in $\mathcal{P}$ and verifying that one gets a matrix A fulfilling the open conditions in (D) e.g. with the help of a computer algebra package like MACAULAY; anyway, the existence of surfaces in (D') has been established by Roßberg, cf. [Roß], 112ff., whence I do not carry out what I said before).

In particular, by the preceding remark one finds that the surfaces in (D') form an irreducible open set $\mathfrak{U}$ inside their moduli space, and $\mathfrak{U}$ is unirational (since $\mathcal{P}$ is rational).

To calculate the dimension of $\mathfrak{U}$ I note that I have 3 groups acting on the set of normal forms of matrices A in (D):

1. $G = PGl(5) = Aut(\mathbb{P}^4)$ changing coordinates $(x_0 : \ldots : x_4) \mapsto (y_0 : \ldots : y_4)$, $\dim G = 24$.

2. $H = \left\{ \text{graded auto.'s of } \mathcal{A} \oplus \mathcal{A}(-2)^2 \text{ of the form } \begin{pmatrix} s_1 & q_1 & q_2 \\ 0 & s_2 & 0 \\ 0 & 0 & s_2 \end{pmatrix} \right\}$
with $s_1,\ s_2 \in \mathbb{C}\backslash\{0\}$ and $q_1,\ q_2$ quadratic. Here $\dim H = 32$.

3. $L = \left\{ \text{group of matrices} \begin{pmatrix} \lambda_1 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & \mu_3 \\ \mu_4 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & \mu_5 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & \mu_6 & 0 & 0 & \lambda_6 \end{pmatrix} \right\} \cap Sp_6(\mathbb{C})$,
where $\lambda_i \in \mathbb{C}\backslash\{0\}$, $\mu_i \in \mathbb{C}$, $i = 1, \ldots, 6$, and $Sp_6(\mathbb{C})$ denotes the group of symplectic 6×6 matrices. I have dim $L = 9$.

Thus one can calculate an upper bound for the dimension of $\mathfrak{U}$ as follows:

$$161(\dim \mathcal{P}) - 38(\dim_{\mathbb{C}}(\ker(d_1)_4) + \dim_{\mathbb{C}}(\ker(d_1')_4)) - 20(\dim_{\mathbb{C}}(\ker(D_1)_3) + \dim_{\mathbb{C}}(\ker(D_1')_3)) - 24(\dim G) - 32(\dim H) - 9(\dim L) = 38.$$

On the other hand dim $\mathfrak{U} \geq 10\chi - 2K^2 = 38$ by general principles (see [Cat1b], p. 484). Thus dim $\mathfrak{U} = 38$ and one recovers the following theorem (cf. [Roß], p. 116, thm. 1):

Theorem 2.3. *Regular surfaces of general type with $p_g = 5$, $K^2 = 11$ such that the canonical map is a birational morphism and the image $Y \subset \mathbb{P}^4$ has only improper double points as singularities form an irreducible unirational open set $\mathfrak{U}$ of dimension 38 inside their moduli space.*

3 The case of general K^2

Recall that for a canonical surface $\pi : S \to Y \subseteq \mathbb{P}^4$ with $p_g = 5$, $q = 0$ one has a resolution of the canonical ring

$$\mathbf{R}_\bullet : 0 \longrightarrow \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n \xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^{2n+2} \xrightarrow{(\alpha\,\beta)} \mathcal{A} \oplus \mathcal{A}(-2)^n \longrightarrow \mathcal{R} \longrightarrow 0 \tag{1}$$

where $n := K^2 - 9$. In this section, as sort of a converse to this, I propose to prove

Theorem 3.1. *Let $A = (\alpha\,\beta)$ be an $(n+1) \times (2n+2)$ matrix with entries in the 1^{st} row cubic forms and linear forms otherwise such that $\alpha\beta^t = \beta\alpha^t$ is symmetric. Put $A' :=$ (A with 1^{st} row erased) and*

$$M^s(n, 2n+2) := \{[\mathbf{M}] : \mathbf{M} = (\mathbf{a}\,\mathbf{b}) \in \mathbb{C}^{n\times(2n+2)} \ s.t.\ \mathbf{ab}^t = \mathbf{ba}^t\} \subset \mathbb{P}^{n(2n+2)-1},$$

(where $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n\times(n+1)}$*) and*

$$\Delta := \{[\mathbf{M}] \in M^s(n, 2n+2) : \mathrm{rk}\,\mathbf{M} \leq n-1\} \subset M^s(n, 2n+2).$$

Then the degeneracy locus Δ *sits as an irreducible 4-codimensional subvariety inside the irreducible variety* $M^s(n, 2n+2)$*. Assume that the linear forms in* A' *parametrically define a* $\mathbb{P}^4$ *inside* $M^s(n, 2n+2)$ *that is transverse to the locus* Δ*, the* $(n+1)\times(n+1)$ *minors of* A *have no common factor and that* $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R}$ *is prime, where* $\mathcal{R} := \mathrm{coker}(A : \mathcal{A}(-3)^{2n+2} \to \mathcal{A} \oplus \mathcal{A}(-2)^n)$*. Then* $Y = \mathrm{Supp}\,\mathcal{R} \subset \mathbb{P}^4$*, with closed subscheme structure given by* $\mathrm{Ann}_{\mathcal{A}}\,\mathcal{R} \subset \mathcal{A}$*, is an irreducible surface with isolated nonnormal locus defined by* $I_n(A')$ *as a reduced subscheme; assume further that the latter points are improper double points of* Y*. Then* $\mathcal{R}$ *gets a ring structure via* $\mathcal{R} = \mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')}\cdot\mathcal{A}_Y, \overline{I_n(A')}\cdot\mathcal{A}_Y)$*, where the bar denotes saturation. Moreover,* $X = \mathrm{Proj}(\mathcal{R})$ *is then the canonical model of a surface of general type with* $K^2 = n+9$*,* $p_g = 5$*,* $q = 0$ *provided* X *has only rational double points as singularities. The morphism* $\psi : X \to Y$ *induced by the inclusion* $\mathcal{A}_Y \subset \mathcal{R}$ *is finite and birational, and* $Y \subset \mathbb{P}^4$ *is a canonical surface.*

Proof. To begin with, let's check that $M^s(n, 2n+2)$ resp. Δ are irreducible and $\mathrm{codim}_{M^s(n,,2n+2)}\Delta = 4$ as stated. In fact, I will prove that the irreducible algebraic group $PGl_n(\mathbb{C}) \times PSp_{2n+2}(\mathbb{C})$ acts (morphically) on $M^s(n, 2n+2)$ (via left resp. right multiplication) with orbits $M^s_k(n, 2n+2) - M^s_{k-1}(n, 2n+2)$, $k = 0, \ldots, n$, where $M^s_k(n, 2n+2)$ is the locus of matrices inside $M^s(n, 2n+2)$ of rank $\leq k$. Since $M^s(n, 2n+2) - M^s_{n-1}(n, 2n+2)$ resp. $M^s_{n-1}(n, 2n+2) - M^s_{n-2}(n, 2n+2)$ are clearly dense in $M^s(n, 2n+2)$ resp. Δ, the latter are then irreducible.

Given $(\mathbf{a}, \mathbf{b}) \in M^s(n, 2n+2)$ I will now use the operations (Op) of Lemma 2.1 which belong to $PGl_n(\mathbb{C}) \times PSp_{2n+2}(\mathbb{C})$. Applying (Op),(i) and (iv), possibly (v) one transforms $(\mathbf{a}, \mathbf{b})$ to get $\left(\begin{array}{cc|c} Id_r & 0 & \\ 0 & 0 & \mathbf{b}' \end{array}\right)$, where Id_r is the $r\times r$ identity matrix, r the rank of $\mathbf{a}$ and $\mathbf{b}'$ an $n\times(n+1)$ matrix. Using the symmetry, one finds that one has actually put $(\mathbf{a}, \mathbf{b})$ in the shape

$$\left(\begin{array}{cc|cc} Id_r & 0 & \mathbf{b}'_1 & \mathbf{b}'_2 \\ 0 & 0 & 0 & \mathbf{b}'_3 \end{array}\right),$$

where $\mathbf{b}'_1$, $\mathbf{b}'_2$, $\mathbf{b}'_3$ are $r\times r$, $r\times(n+1-r)$, $(n-r)\times(n+1-r)$ matrices, respectively, and $\mathbf{b}'_1$ is symmetric. Using (Op), (iii) I get

$$\left(\begin{array}{cc|cc} Id_r & 0 & \mathbf{b}'_1 & 0 \\ 0 & 0 & 0 & \mathbf{b}'_3 \end{array}\right),$$

and using (Op), (iii) again and the fact that $\mathbf{b}'_1$ is symmetric:

$$\left(\begin{array}{cc|cc} Id_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{b}'_3 \end{array}\right).$$

Finally, using (Op),(i),(iv) and (v), afterwards (vi), one arrives at $\left(\begin{array}{cc|c} Id_{r+s} & 0 & \multirow{2}{*}{0} \\ 0 & 0 & \end{array}\right)$, where s is the rank of $\mathbf{b}'_3$. Thus the orbits of the action of $PGl_n(\mathbb{C}) \times PSp_{2n+2}(\mathbb{C})$ on $M^s(n, 2n+2)$ are the ones mentioned above.

Denote homogeneous coordinates in $\mathbb{P}^{n(2n+2)-1}$ by $\{\mathbf{a}_{ij}; \mathbf{b}_{ij}\}_{1 \le i \le n, 1 \le j \le n+1}$ and consider the linear subspace

$$\Lambda := \{\mathbf{a}_{ij} = 0, 1 \le i \le n, 1 \le j \le n+1; \ \mathbf{b}_{kl} = 0, 1 \le k \le l \le n+1\}$$

and the restriction of the projection with center Λ $\pi_\Lambda : M^s(n, 2n+2) - \Lambda \to \mathbb{P}^N$, where $N = n(n+1) + (n+1)(n+2)/2 - 2$. Clearly, π_Λ is dominant and generically one-to-one ($\pi_\Lambda|(\pi_\Lambda^{-1}U)$, where $U = \{\det(\mathbf{a}_{ij})_{1 \le i,j \le h} \neq 0, \ h = 1, \dots, n\}$, is one-to-one), hence $\dim M^s(n, 2n+2) = N$. Consider the following incidence correspondence with the indicated two projections:

$$\begin{tikzcd}[column sep=small] & \tilde{M}^s_{n-1}(n, 2n+2) := \{([\mathbf{M}], L) \in M^s(n, 2n+2) \times Grass(n-1, n) : \mathrm{im}(\mathbf{M}) \subset L\} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ M^s(n, 2n+2) & & Grass(n-1, n) \end{tikzcd}$$

Then pr_2 is surjective, and choosing a suitable basis in $\mathbb{C}^n$ one can identify the fibre of pr_2 over a point in $Grass(n-1, n)$ with (the projectivisation of) the set of matrices of the form $\left(\begin{array}{c|c} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} \\ 0 & 0 \end{array}\right)$, where now $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$ are $(n-1) \times (n+1)$ matrices with $\tilde{\mathbf{a}}\tilde{\mathbf{b}}^t$ symmetric. Analogously to the proof of the irreducibility of and computation of the dimension of $M^s(n, 2n+2)$, one therefore finds that the fibres of pr_2 are irreducible and their dimension equals $(n-1)(n+1) + (n+1)(n+2)/2 - 4$. Therefore $\dim \tilde{M}^s_{n-1}(n, 2n+2) = (n-1)(n+2) + (n+1)(n+2)/2 - 4$. Since pr_1 is generically one-to-one onto Δ, one has $\mathrm{codim}_{M^s(n,2n+2)}(\Delta) = n(n+1) + (n+1)(n+2)/2 - 2 - (n-1)(n+2) - (n+1)(n+2)/2 + 4 = 4$.

Now let $A = (\alpha\ \beta)$ be a matrix of forms meeting the requirements of the theorem. Since the linear forms in A' are supposed to define a $\mathbb{P}^4$ inside $M^s(n, 2n+2)$ transverse to the locus Δ, $I_n(A')$ (scheme-theoretically) defines a finite set of reduced points in $\mathbb{P}^4$. I then have the fundamental

Lemma 3.2. *If $A = (\alpha\ \beta)$ is an $(n+1) \times (2n+2)$ matrix with first row cubic forms, other rows linear forms on $\mathbb{P}^4$ with $\alpha\beta^t = \beta\alpha^t$ and such that $I_n(A')$ defines a set of reduced points, then*

$$\overline{I_n(A')} = \overline{I_n(A)}, \tag{2}$$

where the bar denotes saturation.

Proof. Let P be one of the points that $I_n(A')$ defines. I work locally, in the ring of germs of regular functions around P, $\mathcal{O}_{\mathbb{P}^4,P}$. Slightly abusing notation, I write again A for the matrix of the map $\mathcal{O}_{\mathbb{P}^4,P}^{2n+2} \to \mathcal{O}_{\mathbb{P}^4,P}^{n+1}$ induced by A. I write $A = \begin{pmatrix} a_1 \\ a_2 \\ A'' \end{pmatrix}$, where a_1 denotes the first row of A, a_2 the second, and A'' the $(n-1) \times (2n+2)$ residual matrix. Replacing a_2 by a suitable linear combination of the rows in A', I may assume that the vanishing of the entries in a_2 defines P. Since P is reduced, then $\operatorname{rank} A'' = n-1$. At this point I again use the operations (Op) of lemma 2.1 to transform A'' resp. A such that I can easily read off the information I want. During the calculation I will treat A'' resp. A as dynamical variables. I can find a unit among the entries of the first row of A'' and using (Op), (vi) and (v) I can assume A''_{11} is a unit. Using (Op), (iv) I reach

$$A'' = \left(\begin{array}{c|c} 1\,0\ldots0 & * \\ \hline * & * \end{array}\right)$$

and using (Op), (ii) I can assume $A''_{1,n+2}$ is a unit and using (iv) I reach $A'' = \left(\begin{array}{c|c} 1\,0\ldots0 & u_1\,0\ldots0 \\ \hline * & * \end{array}\right)$, where u_1 is a unit. Remarking that by symmetry, $A''_{i,n+2} = u_1 A''_{i,1}$, $i = 2, \ldots, n+1$, I can reach using row operations

$$A'' = \left(\begin{array}{cccc|cccc} 1 & 0 & \ldots & 0 & u_1 & 0 & \ldots & 0 \\ 0 & & & & 0 & & & \\ \vdots & & * & & \vdots & & * & \\ 0 & & & & 0 & & & \end{array}\right),$$

thus inductively

$$A = \left(\begin{array}{ccccc|ccccc} A_{11} & \cdots & A_{1,n-1} & A_{1,n} & A_{1,n+1} & A_{1,n+2} & \cdots & A_{1,2n} & A_{1,2n+1} & A_{1,2n+2} \\ A_{21} & \cdots & A_{2,n-1} & A_{2,n} & A_{2,n+1} & A_{2,n+2} & \cdots & A_{2,2n} & A_{2,2n+1} & A_{2,2n+2} \\ \hline 1 & \cdots & 0 & 0 & 0 & u_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & u_n & 0 & 0 \end{array}\right).$$

Symmetry gives $A_{i,j}u_i = A_{i,n+1+j}$, $i = 1, 2$ $j = 1, \ldots, n-1$; therefore, using (Op), (ii), I get

$$A = \left(\begin{array}{ccccc|ccccc} A_{11} & \cdots & A_{1,n-1} & A_{1,n} & A_{1,n+1} & 0 & \cdots & 0 & A_{1,2n+1} & A_{1,2n+2} \\ A_{21} & \cdots & A_{2,n-1} & A_{2,n} & A_{2,n+1} & 0 & \cdots & 0 & A_{2,2n+1} & A_{2,2n+2} \\ \hline 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

To ease notation, I put $A_{1,n} =: A_1$, $A_{1,n+1} =: A_2$, $A_{1,2n+1} = A_3$, $A_{1,2n+2} =: A_4$ and $A_{2,n} =: a_3$, $A_{2,n+1} =: a_4$, $A_{2,2n+1} =: -a_1$, $A_{2,2n+2} =: -a_2$ and using row operations I get

$$A = \left(\begin{array}{cc|cc} 0 & \begin{matrix} A_1 & A_2 \\ a_3 & a_4 \end{matrix} & 0 & \begin{matrix} A_3 & A_4 \\ -a_1 & -a_2 \end{matrix} \\ I_{n-1} & 0 & 0 & 0 \end{array}\right)$$

and to make A look more symmetric I can use (Op), (ii) to arrive at

$$A = \left(\begin{array}{cc|cc} 0 & \begin{matrix} A_1 & A_2 \\ a_3 & a_4 \end{matrix} & 0 & \begin{matrix} A_3 & A_4 \\ -a_1 & -a_2 \end{matrix} \\ I_{n-1} & 0 & I_{n-1} & 0 \end{array}\right). \tag{3}$$

Therefore near P, A looks like (3); a_1, a_2, a_3, a_4 is a regular sequence in $\mathcal{O}_{\mathbb{P}^4,P}$ since the second row of A was assumed to define P and the property is invariant under the above process. Moreover, $I_n(A') = \langle a_1, a_2, a_3, a_4\rangle$ and $I_n(A) = \langle A_1, A_2, A_3, A_4,\ a_1, a_2, a_3, a_4\rangle$ near P. But the symmetry $\sum\limits_{i=1}^{4} A_i a_i = 0$ is a Koszul relation saying $\exists$ a skewsymmetric matrix of elements $\{Q_{ij}\}_{1\le i,j\le 4}$ such that $A_i = \sum\limits_{j=0}^{4} Q_{ij} a_j$, $i = 1, \ldots, 4$. Therefore

$I_n(A') = I_n(A)$ locally near P.
Let me now again write A resp. A' for the given matrices of forms on $\mathbb{P}^4$ globally. From the above, I find the equality of sheaves $\widetilde{I_n(A')} = \widetilde{I_n(A)}$ (the latter symbols denoting the sheaves associated to $I_n(A')$ resp. $I_n(A)$ on $\mathbb{P}^4$). Translated back into the language of ideals this just says $\overline{I_n(A')} = \overline{I_n(A)}$. $\square$

Now given the existence of a matrix $A = (\alpha\,\beta)$ with the desirable properties as in the statement of the theorem I have a complex

$$\mathbf{R}_\bullet :\ 0 \longrightarrow \mathcal{A}(-6) \oplus \mathcal{A}(-4)^n \xrightarrow{\binom{-\beta^t}{\alpha^t}} \mathcal{A}(-3)^{2n+2}$$
$$\xrightarrow{(\alpha\,\beta)} \mathcal{A} \oplus \mathcal{A}(-2)^n \longrightarrow \mathcal{R} \longrightarrow 0$$

and the requirement that the $(n+1)\times(n+1)$ minors of A have no common factor translates as $\mathrm{grade} I_{n+1}(A) = \mathrm{codim}_{\mathcal{A}} I_{n+1}(A) \geq 2$ whence this complex is exact by the Eisenbud-Buchsbaum acyclicity criterion (see e.g. [Ei], thm. 20.9, p. 500); moreover, since $\sqrt{I_{n+1}(A)} = \sqrt{\mathrm{ann}_{\mathcal{A}}(\mathcal{R})}$ and then $2 \leq \mathrm{grade}(I_{n+1}(A), \mathcal{A}) = \mathrm{grade}(\mathrm{ann}_{\mathcal{A}}(\mathcal{R}), \mathcal{A}) \equiv \mathrm{grade}(\mathcal{R}) \leq \mathrm{projdim}_{\mathcal{A}}(\mathcal{R}) = 2$ (for the latter inequality "projective dimension bounds grade" see e.g. [B-He], p. 25), I have $\mathrm{codim}_{\mathcal{A}} I_{n+1}(A) = 2$ and $Y := \mathrm{Supp}(\mathrm{Proj}(\mathcal{A}/I_{n+1}(A))) = \mathrm{Supp}(\mathcal{R})$ is a surface in $\mathbb{P}^4$, irreducible by assumption.
I now intend to exploit lemma 3.2 to investigate the ring structure of $\mathcal{R}$. First, quite generally I have

$$\mathcal{R} \subseteq \mathrm{Hom}_{\mathcal{A}_Y}(I_n(A') \cdot \mathcal{A}_Y, I_n(A) \cdot \mathcal{A}_Y).$$

For let me write $\{1,\ v_1, \ldots, v_n\}$ for the minimal set of generators of $\mathcal{R}$ corresponding to the standard basis of $\mathcal{A} \oplus \mathcal{A}(-2)^n$ and let $M =$
$= (m_{ij})_{1\leq i\leq n+1, 1\leq j\leq n}$ be an arbitrary $(n+1)\times n$ submatrix of A, $M = \binom{m}{M'}$, where m is the first row of M, $M' = (m'_{ij})_{1\leq i,j\leq n}$ the $n \times n$ residual matrix. Let v_k be one of the $\{v_1, \ldots v_n\}$. In $\mathcal{R}$ I have relations

$$m_{1i} + \sum_{j=1}^{n} m'_{ji} v_j = 0,\ i = 1, \ldots, n$$

hence also

$$(m_{1i} + \sum_{j=1}^{n} m'_{ji} v_j) M'^*_{ik} = 0,\ i = 1, \ldots, n$$

denoting by M'^*_{ik} the $i,k-$entry of the adjoint M'^* of M'. Adding the latter equations up for the various i gives $\det(M')v_k = \pm\det(M$ with

k^{th} row deleted). This shows that the stated inclusion holds (basically as a consequence of Cramer's rule). By lemma 3.2 it follows that $\mathcal{R} \subseteq \mathrm{Hom}_{\mathcal{A}_Y}(I_n(A') \cdot \mathcal{A}_Y, \overline{I_n(A')} \cdot \mathcal{A}_Y)$.
I claim that then also $\mathcal{R} \subseteq \mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')} \cdot \mathcal{A}_Y, \overline{I_n(A')} \cdot \mathcal{A}_Y)$. In fact, $\overline{I_n(A')} = \{p \in \mathcal{A} = \mathbb{C}[x_0, \ldots, x_4] : \text{for each } i = 0, \ldots, 4 \ \exists n \text{ such that } x_i^n p \in I_n(A')\}$. But then for $\bar{p} \in \overline{I_n(A')} \cdot \mathcal{A}_Y,\ v \in \mathcal{R}$, the expression $\bar{p}v$ is again in $\overline{I_n(A')} \cdot \mathcal{A}_Y$: For i among $0, \ldots, 4$ $(x_i^n \bar{p})v = x_i^n(\bar{p}v)$ is in $\overline{I_n(A')} \cdot \mathcal{A}_Y$, therefore there exists an integer m such that $x_i^m x_i^n(\bar{p}v) \in I_n(A') \cdot \mathcal{A}_Y$, i.e. $\bar{p}v \in \overline{I_n(A')} \cdot \mathcal{A}_Y$.
Therefore I get the chain of inclusions

$$\mathcal{R} \subseteq \mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')} \cdot \mathcal{A}_Y, \overline{I_n(A')} \cdot \mathcal{A}_Y) \subseteq \mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')} \cdot \mathcal{A}_Y, \mathcal{A}_Y). \quad (4)$$

To show the reverse inclusion $\mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')} \cdot \mathcal{A}_Y, \mathcal{A}_Y) \subseteq \mathcal{R}$ I need another technical result. Let me introduce the so called conductor $\mathcal{C}$ of $\mathcal{R}$ into $\mathcal{A}_Y$, $\mathcal{C} := \mathrm{Hom}_{\mathcal{A}_Y}(\mathcal{R}, \mathcal{A}_Y)$, and the associated sheaf on $\mathbb{P}^4$, $\tilde{\mathcal{C}} := \mathcal{H}om_{\mathcal{O}_Y}(\tilde{\mathcal{R}}, \mathcal{O}_Y)$.

Lemma 3.3. *$\mathcal{R}$ being as in the statement of the theorem, one has $\mathcal{R} = \Gamma_*(\tilde{\mathcal{R}})$, where $\tilde{\mathcal{R}}$ is the sheaf on $\mathbb{P}^4$ associated to the graded module $\mathcal{R}$, supported on the surface Y. Moreover the fact that locus defined by $I_n(A')$ as a reduced subscheme is a finite number of points which are improper double points on Y, implies that $\tilde{\mathcal{R}}$ is reflexive in the sense that $\tilde{\mathcal{R}} = \mathcal{H}om_{\mathcal{O}_Y}(\tilde{\mathcal{C}}, \mathcal{O}_Y)$.*

Proof. First, $\mathcal{R}$ equals the full module of sections of $\tilde{\mathcal{R}}$, i.e. $\mathcal{R} = \Gamma_*(\tilde{\mathcal{R}})$. For put $\mathcal{F}_0 := \mathcal{O} \oplus \mathcal{O}(-2)^n$, $\mathcal{F}_1 := \mathcal{O}(-3)^{2n+2}$, $\mathcal{F}_2 := \mathcal{O}(-6) \oplus \mathcal{O}(-4)^n$ and sheafify the resolution of $\mathcal{R}$ above to get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_1 \arrow[rr] \arrow[dr] & & \mathcal{F}_0 \arrow[r] & \tilde{\mathcal{R}} \arrow[r] & 0 \\
& & & \mathcal{G} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

From this one gets the exact sequences

$$0 \to H^0(\mathbb{P}^4, \mathcal{G}(j)) \to H^0(\mathbb{P}^4, \mathcal{F}_0(j)) \to H^0(\mathbb{P}^4, \mathcal{R}(j)) \to 0$$

$$0 \to H^0(\mathbb{P}^4, \mathcal{F}_2(j)) \to H^0(\mathbb{P}^4, \mathcal{F}_1(j)) \to H^0(\mathbb{P}^4, \mathcal{G}(j)) \to 0$$

since $\forall\ j\ H^1(\mathbb{P}^4, \mathcal{G}(j)) = 0,\ H^1(\mathbb{P}^4, \mathcal{F}_2(j)) = 0$ $(H^i(\bigoplus \mathcal{O}(d_k)) = 0,\ i \neq 0, 4$ on $\mathbb{P}^4)$. Putting the above two exact sequences together gives

$$\begin{array}{ccccccc}
0 \to \Gamma_j(\mathcal{F}_2) & \longrightarrow & \Gamma_j(\mathcal{F}_1) & \longrightarrow & \Gamma_j(\mathcal{F}_0) & \longrightarrow & \Gamma_j(\tilde{\mathcal{R}}) \to 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\iota \\
0 \to F_2(j) & \longrightarrow & F_1(j) & \longrightarrow & F_0(j) & \longrightarrow & \mathcal{R}(j) \to 0
\end{array}$$

and ι is an isomorphism (sc. F_2, F_1, F_0 the graded free modules appearing in the resolution (1)).
Secondly, $\tilde{\mathcal{R}} = \mathcal{H}om_{\mathcal{O}_Y}(\tilde{\mathcal{C}}, \mathcal{O}_Y)$, where $\tilde{\mathcal{C}} := \mathcal{H}om_{\mathcal{O}_Y}(\tilde{\mathcal{R}}, \mathcal{O}_Y)$ is the sheaf of conductors of $\tilde{\mathcal{R}}$ into $\mathcal{O}_Y$. Namely, for P a point where $I_n(A')$ does not drop rank the natural homomorphism $\tilde{\mathcal{R}}_P \to \mathrm{Hom}_{\mathcal{O}_{Y,P}}(\tilde{\mathcal{C}}_P, \mathcal{O}_{Y,P})$ is clearly an isomorphism because then locally at P A' is surjective, and $\mathcal{R}/\mathcal{A}_Y$ being the cokernel of the matrix A', I have $\tilde{\mathcal{R}}_P = \mathcal{O}_{Y,P}$ and also $\tilde{\mathcal{C}}_P = \mathcal{O}_{Y,P}$. Therefore the interest is in the improper double points of Y.
Therefore let Q be one of the improper double points that $I_n(A')$ defines. Then locally around Q A can be written as

$$A = \left(\begin{array}{ccc|ccc}
 & A_1 & A_2 & & A_3 & A_4 \\
0 & a_3 & a_4 & 0 & -a_1 & -a_2 \\
I_{n-1} & \multicolumn{2}{c|}{0} & I_{n-1} & \multicolumn{2}{c}{0}
\end{array}\right)$$

as was shown above (see (3)). I have that $\tilde{\mathcal{C}}_Q = (a_1, a_2, a_3, a_4)$ because $\varphi \in \mathcal{O}_{Y,Q}$ is in $\tilde{\mathcal{C}}_Q \Leftrightarrow \exists p, q \in \mathcal{O}_{\mathbb{P}^4,Q}$ s.t. $pv + q = 0$ in $\tilde{\mathcal{R}}_Q$ and p is a lift of φ ($1, v$ denoting the minimal set of generators in $\tilde{\mathcal{R}}_Q$ corresponding to the first two rows of A as above). Since a_1, a_2, a_3, a_4 define the improper double point Q as a reduced subscheme, I can (without loss of generality) assume that $a_i = x_i$, $i = 1, \ldots, 4$ are coordinates in $\mathbb{C}[[x_1, \ldots, x_4]]$ and $\mathcal{O}^{an}_{Y,Q} = \mathbb{C}[[x_1, \ldots, x_4]]/(x_1, x_2) \cap (x_3, x_4)$, changing to the analytic category; then $\tilde{\mathcal{C}}^{an}_Q = (x_1, x_2, x_3, x_4)\mathcal{O}_{Y,Q} = (x_1, x_2, x_3, x_4)/(x_1x_3, x_2x_3, x_1x_4, x_2x_4) = (x_1, x_2)\mathbb{C}[[x_1, x_2]] \oplus (x_3, x_4)\mathbb{C}[[x_3, x_4]]$. I have to show $\tilde{\mathcal{R}}^{an}_Q \subset$
$\subset \mathrm{Hom}_{\mathcal{O}^{an}_{Y,Q}}(\tilde{\mathcal{C}}^{an}_Q, \mathcal{O}^{an}_{Y,Q}) = \tilde{\mathcal{R}}^{an**}_Q$ is an isomorphism. Look at

$$\begin{array}{ccccccc}
\mathcal{O}^{an}_{Y,Q} & \overset{i}{\hookrightarrow} & \tilde{\mathcal{R}}^{an}_Q & \longrightarrow & \mathrm{coker}\ i & \longrightarrow & 0 \\
\| & & \cap & & \cap & & \\
\mathcal{O}^{an}_{Y,Q} & \overset{i'}{\hookrightarrow} & \tilde{\mathcal{R}}^{an**}_Q & \longrightarrow & \mathrm{coker}\ i' & \longrightarrow & 0 \\
 & & \| & & & & \\
 & & \multicolumn{3}{c}{\mathrm{Hom}_{\mathcal{O}^{an}_{Y,Q}}(\tilde{\mathcal{C}}^{an}_Q, \mathcal{O}^{an}_{Y,Q}).} & &
\end{array}$$

Here coker $i \neq 0$ since $\tilde{\mathcal{R}}_Q^{an}$ is minimally generated by 2 elements as an $\mathcal{O}_{Y,Q}^{an}$−module ($\tilde{\mathcal{R}}_Q$ is the cokernel of the matrix in (3)). But on the other hand, one computes $\mathrm{Hom}_{\mathcal{O}_{Y,Q}^{an}}(\tilde{\mathcal{C}}_Q^{an}, \mathcal{O}_{Y,Q}^{an}) = \mathbb{C}[[x_1, x_2]] \oplus \mathbb{C}[[x_3, x_4]]$. Namely, $\mathbb{C}[[x_1, x_2]] \oplus \mathbb{C}[[x_3, x_4]] \subset \mathrm{Hom}_{\mathcal{O}_{Y,Q}^{an}}(\tilde{\mathcal{C}}_Q^{an}, \mathcal{O}_{Y,Q}^{an})$ via $(\varphi_1 \oplus \varphi_2)(c_1 \oplus c_2) = \varphi_1 c_1 + \varphi_2 c_2$ for $(\varphi_1 \oplus \varphi_2) \in \mathbb{C}[[x_1, x_2]] \oplus \mathbb{C}[[x_3, x_4]]$ and $(c_1 \oplus c_2) \in (x_1, x_2)\mathbb{C}[[x_1, x_2]] \oplus (x_3, x_4)\mathbb{C}[[x_3, x_4]]$, and conversely, given $\varphi \in \mathrm{Hom}_{\mathcal{O}_{Y,Q}^{an}}(\tilde{\mathcal{C}}_Q^{an}, \mathcal{O}_{Y,Q}^{an})$, then $\varphi(x_{1/2}) \in \mathrm{Ann}(x_3) \cap \mathrm{Ann}(x_4) = \mathbb{C}[[x_1, x_2]]$ and $\varphi(x_1)x_2 = \varphi(x_2)x_1 \in \mathbb{C}[[x_1, x_2]]$ whence $\varphi(x_{1/2}) = \varphi' x_{1/2}$ with $\varphi' \in \mathbb{C}[[x_1, x_2]]$. Similarly, $\varphi(x_{3/4}) = \varphi'' x_{3/4}$ with $\varphi'' \in \mathbb{C}[[x_3, x_4]]$. Now from the exact sequence

$$0 \to \mathbb{C}[[x_1, \dots, x_4]]/(x_1, x_2) \cap (x_3, x_4) \to \mathbb{C}[[x_1, x_2]] \oplus \mathbb{C}[[x_3, x_4]] \to \mathbb{C} \to 0$$

I have coker $i' \cong \mathbb{C}$ whence the righthand inclusion in the above diagram is an isomorphism and therefore also the middle one (by the 5-lemma). $\square$

Now clearly $(I_n(A') \cdot \mathcal{A}_Y)^\sim \cong \tilde{\mathcal{C}}$ as sheaves; for $I_n(A') \subseteq \mathrm{ann}_{\mathcal{A}}(\mathcal{R}/\mathcal{A}_Y)$ (Fitting's lemma), hence $I_n(A') \cdot \mathcal{A}_Y \subseteq \mathcal{C}$ which gives me a morphism of these sheaves which is an isomorphism (in fact, in the proof of the preceding lemma I saw that for P one of the points where $I_n(A')$ drops rank $(I_n(A') \cdot \mathcal{A}_Y)_P^\sim$ and $\tilde{\mathcal{C}}_P$ are both $(a_1, a_2, a_3, a_4)\mathcal{O}_{Y,P}$ and are $\mathcal{O}_{Y,P}$ otherwise). Hence also $\tilde{\mathcal{R}} = \mathcal{H}om_{\mathcal{O}_Y}((I_n(A') \cdot \mathcal{A}_Y)^\sim, \mathcal{O}_Y)$, but the full module of sections of the latter sheaf contains $\mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')} \cdot \mathcal{A}_Y, \mathcal{A}_Y)$ and combined with the fact that $\tilde{\mathcal{R}}$ equals its full module of sections and equation (4), I arrive at the fact that $\mathcal{R}$ is a ring via $\mathcal{R} = \mathrm{Hom}_{\mathcal{A}_Y}(\overline{I_n(A')} \cdot \mathcal{A}_Y, \overline{I_n(A')} \cdot \mathcal{A}_Y)$. Then $X = \mathrm{Proj}(\mathcal{R})$ is the canonical model of a surface of general type S with $q = 0$, $p_g = 5$, $K^2 = n+9$ by theorem 1.6, if X has only rational double points as singularities.
The fact in the statement of theorem 3.1 that $I_n(A')$ gives precisely the nonnormal locus of the surface Y now follows from remark 2 after the proof of theorem 1.6. This completes the proof of theorem 3.1. $\square$

Remark 1. The variety $M^s(n, 2n+2)$ is a complete intersection of $\frac{n(n-1)}{2}$ quadrics; namely by the argument at the beginning of the proof of theorem 3.1, $\mathrm{codim}_{\mathbb{P}^{n(2n+2)-1}}(M^s(n, 2n+2)) = \frac{n(n-1)}{2}$ and $M^s(n, 2n+2)$ is cut out by the $\binom{n}{2} = \frac{n(n-1)}{2}$ quadratic equations given by the symmetry $\mathbf{ab}^t = \mathbf{ba}^t$. Moreover, $\mathbb{P}^4$'s clearly exist on this variety. In fact, $\dim Grass(5, n(2n+2)) = 5(n(2n+2)-5)$, and the Fano variety of 4-planes lying on one of the above quadrics has codimension 15 in $Grass(5, n(2n+2))$ whence one finds an at least $5(n(2n+2)-5) - 15\frac{n(n-1)}{2} = \frac{5}{2}[n^2 + 7n - 10]$ dimensional family of $\mathbb{P}^4$'s on $M^s(n, 2n+2)$. For $n \geq 2$ ($\Leftrightarrow K^2 \geq 11$) this number is positive. However, this says of course nothing as to whether one can find $\mathbb{P}^4$'s transverse to the locus Δ of theorem 3.1. in all cases. Thus the next step towards the construction of canonical surfaces with higher K^2, $K^2 \geq 13$, say, should be a more detailed analysis of the Fano variety of $\mathbb{P}^4$'s on $M^s(n, 2n+2)$.

Remark 2. As a second step towards understanding the afore-mentioned surfaces one can look at the forgetful maps:

$$F_n^{(1)} : \left\{ \begin{array}{l} \text{Parameter space of } (n+1) \times (2n+2) \text{ matrices} \\ (\alpha\,\beta) \text{ with the properties listed in} \\ \text{the hypotheses of theorem 3.1} \end{array} \right\}$$
$$\longrightarrow \left\{ \begin{array}{l} \text{Parameter space of } n \times (2n+2) \text{ matrices} \\ (\alpha'\,\beta') \text{ of linear forms on } \mathbb{P}^4 \text{ with } \alpha'\beta'^t \\ \text{symmetric and with } (\alpha'\,\beta') \text{ degenerating} \\ \text{in a finite number of reduced points in } \mathbb{P}^4 \end{array} \right\}$$

obtained by erasing the first row of a matrix $(\alpha\,\beta)$, and try to understand 1) when $F_n^{(1)}$ is dominant, 2) what its fibres look like. Forgetting even more, one can ask the same questions for the maps

$$F_n^{(2)} : \left\{ \begin{array}{l} \text{Parameter space of } (n+1) \times (2n+2) \text{ matrices} \\ (\alpha\,\beta) \text{ with the properties listed in} \\ \text{the hypotheses of theorem 3.1} \end{array} \right\}$$
$$\longrightarrow \coprod_{\text{const. polynomials } P} Hilb^P_{\mathbb{P}^4}$$

where the latter denotes the Hilbert scheme of points in $\mathbb{P}^4$, and the map is given by sending a matrix $A = (\alpha\,\beta)$ to the points $I_n(A')$ defines.

Remark 3. Finally, it would be interesting to find a purely algebraic proof of lemma 3.3, thus going beyond the assumption that the nonnormal locus of Y consists of improper double points alone; namely, for Q one of the points that $I_n(A')$ defines, I have to prove that the $\mathcal{O}_{Y,Q}$−module $\tilde{\mathcal{R}}_Q$ is reflexive. A possible strategy to see this algebraically is as follows: Locally at Q, A can be written as in (3). Put $B_1 := a_3, B_2 := a_4, B_3 := -a_1, B_4 := -a_2$ and note that $\mathcal{O}_{Y,Q} = \mathcal{O}_{\mathbb{P}^4,Q}/(A_iB_j - A_jB_i)_{1\leq i,j\leq 4}$. For clearly $(A_iB_j - A_jB_i) \subset \mathrm{ann}(\tilde{\mathcal{R}}_Q)$ by Fitting's lemma, and conversely, writing $\{1, v\}$ for the minimal set of generators of $\tilde{\mathcal{R}}_Q$, if $R \cdot 1 = 0$ in $\tilde{\mathcal{R}}_Q$, $R \in \mathcal{O}_{\mathbb{P}^4,Q}$, then, by the symmetry, $\exists \lambda_i, i = 1, \ldots, 4 : \sum \lambda_i B_i = 0$, $\sum \lambda_i A_i = R$. The former relation is a Koszul relation saying $\exists \mu_{ij} = -\mu_{ji}$, $i, j \in \{1, \ldots, 4\} : \lambda_i = \sum \mu_{ij} B_j$. Therefore $R = \sum\limits_{i<j} \mu_{ij}(B_jA_i - B_iA_j)$, whence $(A_iB_j - A_jB_i) = \mathrm{ann}_{\mathcal{O}_{\mathbb{P}^4,Q}}(\tilde{\mathcal{R}}_Q)$.

Next, having the inclusion $\tilde{\mathcal{R}}_Q \subset \mathrm{Hom}_{\mathcal{O}_{Y,Q}}(\tilde{C}_Q, \mathcal{O}_{Y,Q})$, I want to show that every $\varphi \in \mathrm{Hom}_{\mathcal{O}_{Y,Q}}(\tilde{C}_Q, \mathcal{O}_{Y,Q})$ comes from an element in $\tilde{\mathcal{R}}_Q$ (recall $\tilde{\mathcal{C}}_Q = (B_1, B_2, B_3, B_4)\mathcal{O}_{Y,Q}$ and $B_i v = -A_i$, $B_i \cdot 1 = B_i$). Then putting $\varphi(B_i) =: \beta_i$, $A_i\beta_j = \varphi(A_iB_j) = \varphi(A_jB_i) = A_j\beta_i$ in $\mathcal{O}_{Y,Q}$. Therefore I would get what I want if $(A_i\beta_j - A_j\beta_i) \subset (A_iB_j - A_jB_i)$ implied that the vector $(\beta_1, \dots, \beta_4)$ is a linear combination (mod $(A_iB_j - B_iA_j)$) of the vectors $(A_1, \dots, A_4)$ and $(B_1, \dots, B_4)$; in other words, if the complex

$$\mathcal{O}^2_{Y,Q} \xrightarrow{\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \\ A_4 & B_4 \end{pmatrix}} \mathcal{O}^4_{Y,Q} \xrightarrow{\begin{pmatrix} A_2 & -A_1 & 0 & 0 \\ A_3 & 0 & -A_1 & 0 \\ A_4 & 0 & 0 & -A_1 \\ 0 & A_3 & -A_2 & 0 \\ 0 & A_4 & 0 & -A_2 \\ 0 & 0 & A_4 & -A_3 \end{pmatrix}} \mathcal{O}^6_{Y,Q}$$

was exact. If the A_i, B_i are replaced with indeterminates X_i, Y_i, $i = 1, \dots, 4$, over $\mathcal{O}_{\mathbb{P}^4,Q}$, then this is exact as I checked using the computer algebra system MACAULAY2 (one should of course find a theoretically satisfactory reason for this). Thus it would be nice to know exactly which genericity assumptions on the A_i, B_i are needed for the above complex to remain exact when I specialize $X_i \rightsquigarrow A_i$, $Y_i \rightsquigarrow B_i$.

4 A commutative algebra lemma

This section stands somewhat isolated from the rest of the treatise. I included it merely to fix up a fact that slightly improves on a theorem of M. Grassi ([Gra]). To find some amelioration of the structure theorem for Gorenstein algebras in codimension 2 presented in [Gra] was actually the superordinate aim from which this work departed.
I'd like to work in the generality and setting adopted in [Gra], so let:
$(R, \mathfrak{m}, k) :=$a Cohen-Macaulay local ring with $2 \notin \mathfrak{m}$,
$A :=$a codimension 2 Gorenstein algebra over R, i.e. a finite $R-$algebra with $\dim(R) - \dim_R(A) = 2$ and $A \cong \mathrm{Ext}^2_R(A, R)$ as $A-$modules.
Finally, it will be convenient to have the concept of Koszul module available. Whereas the usual Koszul complex is associated with a linear form $f : R^n \to R$, a Koszul module is a module having a resolution similar to the Koszul complex up to the fact that the rôle of f is taken by a family of (vector-valued) maps from R^n to R^n. I'll only make this precise in the relevant special case:

A finite $R-$module M having a length 2 resolution

$$0 \to R^n \xrightarrow{\binom{\rho_1}{\rho_2}} R^{2n} \xrightarrow{(\tau_1\ \tau_2)} R^n \to M \to 0 \tag{1}$$

some $n \in \mathbb{N}$, is a Koszul module iff $\det(\tau_1),\ \det(\tau_2)$ is a regular sequence on R and $\exists$ a unit $\lambda \in R$: $\det(\rho_1) = (-1)^n \lambda \det(\tau_2),\ \det(\rho_2) = \lambda \det(\tau_1)$.
Then Grassi proves in case R is a domain ([Gra], thm. 3.3) that A has a (Gorenstein) symmetric resolution

$$0 \to R^n \xrightarrow{\binom{-\beta^t}{\alpha}} R^{2n} \xrightarrow{(\alpha\ \beta)} R^n \to A \to 0 \tag{2}$$

and a second resolution of the prescribed type (1) for the Koszul module condition, and that these 2 are related by an isomorphism of complexes which is the identity in degrees 0 and 2; firstly, for sake of generality, I will briefly show that the assumption "R a domain" is in fact not needed, and secondly, prove that there is one single resolution of A meeting both requirements, i.e. a resolution as in (2) with $\det(\alpha),\ \det(\beta)$ an $R-$regular sequence. This still gives no indication of how the ring structure of A is encoded in the resolution, but as the concepts of Koszul module and Gorenstein symmetric resolution seem to provide a pleasing setting to investigate this question, it can be useful to have a result linking these two.
For the first part, I note that the only place in [Gra] where the hypothesis that R be a domain enters is at the beginning of the proof of proposition 1.5, page 930: Here one is given a resolution as in (1), but <u>without</u> any additional assumptions on $\det(\tau_1), \det(\tau_2), \det(\rho_1), \det(\rho_2)$ whatsoever, and Grassi wants to conclude that $\exists$ a base change in R^{2n} such that (in the new base) $\det(\tau_1)$ is not a zero divisor on R. But this can be proven by a similar method as Grassi uses in the sequel of the proof of proposition 1.5, without using "R a domain": For let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the associated primes of R which are precisely the minimal elements of $\mathrm{Spec}(R)$ since R is CM. One shows that $\exists$ a base change in R^{2n} such that $\det(\tau_1) \notin \mathfrak{p}_i,\ \forall i = 1, \ldots, r$ (in the new base), more precisely, that $\exists$ a sequence of r base changes such that after the mth base change

$$(*) \qquad \det(\tau_1) \notin \mathfrak{p}_i,\ \forall i \in \{r-m+1, \ldots, r\},$$

$m = 0, \ldots, r$, the assertion being empty for $m = 0$. Therefore, inductively, suppose $(*)$ holds for m to get it for $m+1$.
Denote by $[i_1, \ldots, i_n]$ the maximal minor of $(\tau_1\ \ \tau_2)$ corresponding to the columns $i_1, \ldots, i_n,\ i_j \in \{1, \ldots, 2n\}$. If $[1, \ldots, n] \notin \mathfrak{p}_{r-m}$ I'm already O.K.,

so suppose $[1,\dots,n] \in \mathfrak{p}_{r-m}$. By the Eisenbud-Buchsbaum acyclicity criterion $I_n((\tau_1\ \tau_2))$ cannot consist of zerodivisors on R alone, therefore set

$$l_1 := \min\{c : \exists s_1,\dots,s_{n-1} \text{ with } s_1 < s_2 < \dots < s_{n-1} < c \\ \text{and}[s_1,\dots,s_{n-1},c] \notin \mathfrak{p}_{r-m}\}$$

(then $n < l_1 \le 2n$) and inductively,

$$l_i := \min\{c : \exists s'_1,\dots,s'_{n-i} \text{ with } s'_1 < \dots < s'_{n-i} < c < l_{i-1} < \dots < l_1 \\ \text{and } [s'_1,\dots,s'_{n-i},c,l_{i-1},\dots,l_1] \notin \mathfrak{p}_{r-m}\},$$

$i = 2,\dots,n$. Then $\exists\ J$ such that $n < l_J < l_{J-1} < \dots < l_1 \le 2n$ and for $I > J$ $l_I \in \{1,\dots,n\}$ ($J = n$ might occur and then the set of $l_I \in \{1,\dots,n\}$ is empty; this does not matter).

I have $[l_n,\dots,l_1] \notin \mathfrak{p}_{r-m}$ by construction. Choose $b \in (\bigcap_{i=r-m+1}^{r} \mathfrak{p}_i)\backslash\mathfrak{p}_{r-m}$, which is nonempty since the $\mathfrak{p}_i$'s are the minimal elements of $\mathrm{Spec}(R)$. Denote by $y_1 < \dots < y_J$ the complementary indices of the $l_I \in \{1,\dots,n\}$ inside $\{1,\dots,n\}$ and consider the base change on R^{2n}: $M_{y_1,l_J}(b) \circ M_{y_2,l_{j-1}}(b) \circ \dots \circ M_{y_J,l_1}(b)$, where $M_{y_\nu,l_{J-\nu+1}}(b)$, $\nu = 1,\dots,J$ is addition of b times the $l_{J-\nu+1}$ column to the y_ν column. Then one sees (by the multilinearity of determinants)

$$[1,\dots,n]_{\text{new}} = [1,\dots,n]_{\text{old}} \pm b^J[l_n,\dots,l_1]_{\text{old}} + b\mu,$$

where "new" means after and "old" before the base change and μ is an element in $\mathfrak{p}_{r-m}$ by the defining minimality property of the l's. Therefore, since by the induction hypothesis $[1,\dots,n]_{\text{old}} \notin \mathfrak{p}_i,\ \forall i \in \{r-m+1,\dots,r\}$ and b is chosen appropriately: $[1,\dots,n]_{\text{new}} \notin \mathfrak{p}_i,\ \forall i \in \{r-m,\dots,r\}$. This finally proves $\det(\tau_1) \notin \mathfrak{p}_i\ \forall i = 1,\dots,r$ after the sequence of base changes, i.e. $\det(\tau_1)$ is then R-regular, that what was to be shown.

Secondly, I now want to prove:

Lemma 4.1. *A codimension 2 Gorenstein algebra A over a local CM ring $(R,\mathfrak{m},k)$ with $2 \notin \mathfrak{m}$ has a resolution*

$$0 \to R^n \xrightarrow{\binom{-\beta^t}{\alpha}} R^{2n} \xrightarrow{(\alpha\,\beta)} R^n \to A \to 0$$

which is also of Koszul module type, i.e. $\det(\alpha)$, $\det(\beta)$ *is an $R-$regular sequence.*

Proof. Taking into account the above remark that one can dispose of the assumption "R a domain" the fact that A has a resolution with the symmetry property above is proven in [Gra], thm. 3.3., so I have to show that $\exists$ a base change in R^{2n} which preserves the relation $\alpha\beta^t = \beta\alpha^t$ and in the new base $\det(\alpha)$, $\det(\beta)$ is a regular sequence. The punch line to show this is as in the foregoing argument except that everything is a little harder because one has to keep track of preserving the symmetry: Therefore let again be $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the associated primes of R, and I show that $\exists$ a sequence of r base changes in R^{2n} preserving the symmetry and such that after the mth base change $(*)$ above holds, the case $m = 0$ being trivial. For the inductive step, suppose $\det(\alpha) \in \mathfrak{p}_{r-m}$ to rule out a trivial case; I write $[i_1, \ldots, i_\nu; j_1, \ldots, j_{n-\nu}] \equiv \det(\alpha_{i_1} \ldots \alpha_{i_\nu}\ \beta_{j_1} \ldots \beta_{j_{n-\nu}})$. Call a minor $[i_1, \ldots, i_\nu; j_1, \ldots, j_{n-\nu}]$ <u>good</u> iff $\{i_1, \ldots, i_\nu\} \cap \{j_1, \ldots, j_{n-\nu}\} = \emptyset$.
I want to find a good minor that does not belong to $\mathfrak{p}_{r-m}$ (possibly after a base change in R^{2n}). Therefore suppose all the good minors belong to $\mathfrak{p}_{r-m}$. Since grade $I_n((\alpha\ \beta)) \geq 2$ by Eisenbud-Buchsbaum acyclicity, $\exists$ a minor $\notin \mathfrak{p}_{r-m}$ (which is not good). For $n = 1$ this is a contradiction since all minors are good, and I can suppose $n > 1$ in the process of finding a good minor. Now choose a minor $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}]$ such that

- $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}] \notin \mathfrak{p}_{r-m}$
- $\mathrm{card}(\{I_1, \ldots, I_k\} \cap \{J_1, \ldots, J_{n-k}\}) =: M_0$ is minimal among the minors which do not belong to $\mathfrak{p}_{r-m}$.

I want to perform a base change in R^{2n} not destroying the symmetry such that in the new base $\exists$ a minor $[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}]$ such that

- $[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}] \notin \mathfrak{p}_{r-m}$
- $\mathrm{card}(\{T_1, \ldots, T_{k-1}\} \cap \{S_1, \ldots, S_{n-k+1}\}) = M_0 - 1$.

Continuing this process M_0 steps (i.e. performing M_0 successive base changes) I can find a good minor not contained in $\mathfrak{p}_{r-m}$.
Let now $[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}]$ be given. Choose $H \in \{I_1, \ldots, I_k\} \cap \{J_1, \ldots, J_{n-k}\}$ and $L \in \{1, \ldots, n\} - \{I_1, \ldots, I_k\} \cup \{J_1, \ldots, J_{n-k}\}$ (both of which exist). Now perform the base change in R^{2n} which corresponds to adding α_H to β_L and α_L to β_H (preserving the symmetry), and consider

$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\ \beta_{J_1} \ldots \beta_H + \alpha_L \ldots \beta_L + \alpha_H \ldots \beta_{J_{n-k}}),$$

an $n \times n-$minor of the transformed matrix which I can write as $[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}]$, where $\{T_1, \ldots, T_{k-1}\} = \{I_1, \ldots, I_k\} - \{H\}$,

$\{S_1, \ldots, S_{n-k+1}\} = \{J_1, \ldots, J_{n-k}\} \cup \{L\}$ and obviously, card$(\{T_1, \ldots, T_{k-1}\} \cap \{S_1, \ldots, S_{n-k+1}\}) = M_0 - 1$. I want to prove that this minor does not belong to $\mathfrak{p}_{r-m}$. For this I show that in fact

$$[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}] = \pm[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}] \\ +\text{"residual terms"},$$

where "residual terms"$\in \mathfrak{p}_{r-m}$. Using the additivity of the determinant in each column I find that "residual terms" consists of 3 summands two of which clearly belong to $\mathfrak{p}_{r-m}$ because $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}]$ was chosen such that card$(\{I_1, \ldots, I_k\} \cap \{J_1, \ldots, J_{n-k}\}) =: M_0$ was minimal among the minors of the matrix before the base change which did not belong to $\mathfrak{p}_{r-m}$, whereas the third summand is (up to sign)

$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\, \alpha_L\, \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\, \beta_L).$$

To show that the latter is in $\mathfrak{p}_{r-m}$ I apply the so-called "Plücker relations":

Given an $M \times N-$matrix, $M \leq N$, $a_1, \ldots, a_p, b_q, \ldots, b_M, c_1, \ldots, c_s \in \{1, \ldots, N\}$, $s = M - p + q - 1 > M$, $t = M - p > 0$, one has

$$(P) \qquad \sum_{\substack{i_1<\ldots<i_t \\ i_{t+1}<\ldots<i_s \\ \{i_1,\ldots,i_s\}=\{1,\ldots,s\}}} \sigma(i_1, \ldots, i_s)[a_1, \ldots, a_p\, c_{i_1} \ldots c_{i_t}][c_{i_{t+1}} \ldots c_{i_s}\, b_q \ldots b_M] = 0$$

where $\sigma(i_1, \ldots, i_s)$ is the sign of the permutation $\binom{1,\ldots,s}{i_1,\ldots,i_s}$ (see e.g. [B-He], lemma 7.2.3, p. 308).

In my situation, I let $M := n,\ N := 2n,\ p := n-2,\ q := n+1,\ s := n+1$ and for the columns corresponding to the a's above I choose the $n-2$ columns

$$\alpha_{I_1},\ \alpha_{I_2}, \ldots, \hat{\alpha_H}, \ldots, \alpha_{I_k},\ \beta_{J_1}, \ldots, \hat{\beta_H}, \ldots, \beta_{J_{n-k}}$$

(in this order), for the columns corresponding to the b's I choose the empty set (which is allowable here), and finally for the columns corresponding to the c's the $n + 2$ columns

$$\alpha_H,\ \beta_H,\ \alpha_L,\ \beta_L,\ \alpha_{I_1},\ \alpha_{I_2}, \ldots,\ \hat{\alpha_H}, \ldots,\ \alpha_{I_k},\ \beta_{J_1}, \ldots,\ \hat{\beta_H}, \ldots, \beta_{J_{n-k}}$$

Applying (P) one gets 6 nonvanishing summands, 4 of which (namely

$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\, \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\, \alpha_H\, \alpha_L) \cdot (\text{a second factor}),$$
$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\, \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\, \alpha_H\, \beta_L) \cdot (\text{a second factor}),$$
$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\, \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\, \beta_H\, \alpha_L) \cdot (\text{a second factor}),$$
$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\, \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\, \beta_H\, \beta_L) \cdot (\text{a second factor}))$$

are in $\mathfrak{p}_{r-m}$ by the defining minimality property of $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}]$ above. The remaining 2 summands add up to (watch the signs!)
$\pm 2 \det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\, \alpha_L\, \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\, \beta_L) \cdot [I_1, \ldots, I_k; J_1, \ldots, J_{n-k}]$ which therefore is also in $\mathfrak{p}_{r-m}$. But $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}] \notin \mathfrak{p}_{r-m}$ and 2 is a unit in R, therefore $\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k} \alpha_L \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}} \beta_L) \in \mathfrak{p}_{r-m}$ as desired, since $\mathfrak{p}_{r-m}$ is prime.
Hence inductively, after M_0 base changes in R^{2n}, I can find a good minor of the transformed matrix that is not in $\mathfrak{p}_{r-m}$. I assume $[1, \ldots, n] \in \mathfrak{p}_{r-m}$. I can now define

$$\begin{aligned} l_1 := \min\{c:\ \exists s_1, \ldots, s_{n-1} \text{ with } s_1 < s_2 < \ldots < s_{n-1} < c \\ \text{and } [s_1, \ldots, s_{n-1}, c] \notin \mathfrak{p}_{r-m} \underline{\text{and}}\ [s_1, \ldots, s_{n-1}, c] \text{ is good}\} \end{aligned}$$

(then $n < l_1 \leq 2n$) and inductively,

$$\begin{aligned} l_i := \min\{c:\ \exists s'_1, \ldots, s'_{n-i} \text{ with } s'_1 < \ldots < s'_{n-i} < c < l_{i-1} < \ldots < l_1 \\ \text{and } [s'_1, \ldots, s'_{n-i}, c, l_{i-1}, \ldots, l_1] \text{ is good} \\ \text{and } [s'_1, \ldots, s'_{n-i}, c, l_{i-1}, \ldots, l_1] \notin \mathfrak{p}_{r-m}\}. \end{aligned}$$

Then $[l_n, \ldots, l_1] \notin \mathfrak{p}_{r-m}$ which is good and can therefore be written as $[l_n, \ldots, l_1] = [l_1^\alpha, \ldots, l_h^\alpha; l_1^\beta, \ldots, l_{n-h}^\beta]$ with $\{l_1^\alpha, \ldots, l_h^\alpha\} \cap \{l_1^\beta, \ldots, l_{n-h}^\beta\} = \emptyset$. Choose $b \in (\bigcap_{i=r-m+1}^{r} \mathfrak{p}_i) \backslash \mathfrak{p}_{r-m}$ and perform a base change in R^{2n} (preserving the symmetry) by adding b times the l_i^β column of β to the l_i^β column of α, for $i = 1, \ldots, n-h$. Then

$$[1, \ldots, n]_{\text{new}} = [1, \ldots, n]_{\text{old}} \pm b^{n-h}[l_n, \ldots, l_1]_{\text{old}} + b\mu,$$

where $\mu \in \mathfrak{p}_{r-m}$ by the defining minimality property of the l's. Thus $[1, \ldots, n]_{\text{new}} \notin \mathfrak{p}_i$ for $i = r-m, \ldots, r$, which is the inductive step for the property $(*)$. Therefore after a sequence of base changes that preserve the symmetry $\alpha\beta^t = \beta\alpha^t$, $\det(\alpha)$ can be made an $R-$regular element.

Let's sum up: I have that $\det(\alpha)$ is a nonzerodivisor in R, and want to prove that $\exists$ a base change in R^{2n} preserving the symmetry and leaving α unchanged (i.e. fixing the first n basis vectors of R^{2n}) such that in the new base $\det(\beta)$ is a nonzerodivisor in $R/(\det(\alpha))$. The argument is almost identical to the preceding one. In fact, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ be the associated primes of $R/(\det(\alpha))$ which are exactly the minimal prime ideals containing $(\det(\alpha))$ because $R/(\det(\alpha))$ is CM (R is CM and $\det(\alpha)$ is $R-$regular). Then the part of the above proof starting with "... the symmetry: Therefore let again

be $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the associated primes of R, and I show that $\exists$ a sequence of r base changes in R^{2n} ..." and ending with "... Choose $H \in \{I_1, \ldots, I_k\} \cap \{J_1, \ldots, J_{n-k}\}$ and $L \in \{1, \ldots, n\} - \{I_1, \ldots, I_k\} \cup \{J_1, \ldots, J_{n-k}\}$ ..." goes through verbatim (and has to be inserted here) if throughout one replaces r with s, $\det(\alpha)$ with $\det(\beta)$, and the symbol "$\mathfrak{p}$" with "$\mathfrak{q}$". Thereafter, a slight change is necessary because in the process of finding a good minor, i.e. in the course of the M_0 base changes on R^{2n} that transform $(\alpha\, \beta)$ s.t. in the new base $\exists$ a good minor, the shape of β is changed. This change must preserve the property $\det(\beta) \notin \mathfrak{q}_1, \ldots, \mathfrak{q}_{s-m+1}$ in order not to destroy the induction hypothesis. The way out is as follows:
Choose $\zeta \in (\bigcap\limits_{i=r-m+1}^{r} \mathfrak{q}_i) \backslash \mathfrak{q}_{r-m}$, which is possible since the q's all have height 1. Now perform the base change in R^{2n} which corresponds to adding $\zeta\alpha_H$ to β_L and $\zeta\alpha_L$ to β_H (preserving the symmetry), and consider

$$\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\; \beta_{J_1} \ldots \beta_H + \zeta\alpha_L \ldots \beta_L + \zeta\alpha_H \ldots \beta_{J_{n-k}}),$$

an $n \times n-$minor of the transformed matrix which I can write as $[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}]$, where $\{T_1, \ldots, T_{k-1}\} = \{I_1, \ldots, I_k\} - \{H\}$, $\{S_1, \ldots, S_{n-k+1}\} = \{J_1, \ldots, J_{n-k}\} \cup \{L\}$ and obviously, $\mathrm{card}(\{T_1, \ldots, T_{k-1}\} \cap \{S_1, \ldots, S_{n-k+1}\}) = M_0 - 1$. I want to prove that this minor does not belong to $\mathfrak{q}_{s-m}$ and furthermore that

$$\det(\beta_1 \ldots \beta_H + \zeta\alpha_L \ldots \beta_L + \zeta\alpha_H \ldots \beta_n) \notin \mathfrak{q}_1, \ldots, \mathfrak{q}_{s-m+1}.$$

The latter statement is obvious by the choice of ζ (and multilinearity of determinants). The former one follows if I show

$$[T_1, \ldots, T_{k-1}; S_1, \ldots, S_{n-k+1}] = \pm\zeta[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}] \\ +\text{"residual terms"},$$

where "residual terms"$\in \mathfrak{q}_{s-m}$ because ζ and $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}]$ are both $\notin \mathfrak{q}_{s-m}$ by assumption. Again "residual terms" consists of 3 summands two of which belong to $\mathfrak{q}_{s-m}$ because of the defining minimality property of $[I_1, \ldots, I_k; J_1, \ldots, J_{n-k}]$. The third summand is up to sign

$$\zeta \det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k}\; \alpha_L\; \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}}\; \beta_L),$$

therefore it suffices to show $\det(\alpha_{I_1} \ldots \hat{\alpha_H} \ldots \alpha_{I_k} \alpha_L \beta_{J_1} \ldots \hat{\beta_H} \ldots \beta_{J_{n-k}} \beta_L) \in \mathfrak{q}_{s-m}$. This is done word by word as in the passage of the first part of this proof starting with "... I apply the so-called "Plücker relations":... "

and ending "... since $\mathfrak{p}_{r-m}$ is prime.... ", taking into account the aforementioned changes in notation.
The rest of the proof is as follows: Inductively, I can find a good minor of the transformed matrix that is not in $\mathfrak{q}_{s-m}$. To avoid a trivial case, I assume $[n+1,\ldots,2n] \in \mathfrak{q}_{s-m}$. Now I define

$$\begin{aligned} L_1 := \max\{c :\ &\exists s_2,\ldots,s_n \text{ with } c < s_2 < s_3 < \ldots < s_n \\ &\text{and } [c,s_2,\ldots,s_n] \notin \mathfrak{q}_{s-m} \underline{\text{and}}\ [c,s_2,\ldots,s_n] \text{ is good}\} \end{aligned}$$

(then $1 \le l_1 < n+1$) and inductively,

$$\begin{aligned} L_i := \max\{c :\ \exists s'_{i+1},\ldots,s'_n \text{ with } L_1 < \ldots < L_{i-1} < c < s'_{i+1} < \ldots < s'_n \\ \text{and } [L_1,\ldots,L_{i-1},c,s'_{i+1},\ldots,s'_n] \text{ is good} \\ \text{and } [L_1,\ldots,L_{i-1},c,s'_{i+1},\ldots,s'_n] \notin \mathfrak{q}_{s-m}\}. \end{aligned}$$

Then $[L_1,\ldots,L_n] \notin \mathfrak{q}_{s-m}$ and is good (furthermore $L_n > n$ since $[1,\ldots,n] \in \mathfrak{q}_{s-m}$). I can write $[L_1,\ldots,L_1] = [L^\alpha_1,\ldots,L^\alpha_h; L^\beta_1,\ldots,L^\beta_{n-h}]$ with $\{L^\alpha_1,\ldots,L^\alpha_h\} \cap \{L^\beta_1,\ldots,L^\beta_{n-h}\} = \emptyset$. Choose $b \in (\bigcap_{i=s-m+1}^{s} \mathfrak{q}_i) \backslash \mathfrak{q}_{s-m}$ and perform a base change in R^{2n} (preserving the symmetry) by adding b times the L^α_i column of α to the L^α_i column of β, for $i = 1,\ldots,h$. Then

$$[n+1,\ldots,2n]_{\text{new}} = [n+1,\ldots,2n]_{\text{old}} \pm b^h[L_1,\ldots,L_n]_{\text{old}} + b\mu,$$

where $\mu \in \mathfrak{q}_{s-m}$ by the defining maximality property of the L's. Thus $[n+1,\ldots,2n]_{\text{new}} \notin \mathfrak{q}_i$ for $i = s-m,\ldots,s$, which is the inductive step. Therefore after a sequence of base changes that preserve the symmetry $\alpha\beta^t = \beta\alpha^t$ (and leave $\det(\alpha)$ unaltered) $\det(\beta)$ can be made an $R/(\det(\alpha))-$regular element, i.e. $\det(\alpha)$, $\det(\beta)$ is an $R-$regular sequence, which proves the lemma. □

References

[A-C-G-H] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: *Geometry of Algebraic Curves*, Springer-Verlag, New York, N.Y. (1985)

[Beau] A. Beauville: *Complex algebraic surfaces*, London Mathematical Society Student Texts 34, CUP (1996)

[Bom] E. Bombieri: *Canonical models of surfaces of general type*, I.H.E.S. Publ. Math. 42 (1973), p. 171-219

[B-He] W. Bruns, J. Herzog: *Cohen-Macaulay rings*, CUP (1998)

[B-V] W. Bruns, U. Vetter: *Determinantal rings*, Springer LNM 1327 (1988)

[Cat1] F. Catanese: *Babbage's conjecture, contact of surfaces, symmetric determinantal varieties and applications*, Inv. Math. 63 (1981), pp.433-465

[Cat1b] F. Catanese: *On the moduli spaces of surfaces of general type*, J. Differential Geom., 19 (1984), p. 483-515

[Cat2] F. Catanese: *Commutative algebra methods and equations of regular surfaces*, Algebraic Geometry-Bucharest 1982, Springer LNM 1056 (1984), pp.68-111

[Cat3] F.Catanese: *Equations of pluriregular varieties of general type*, Geometry today-Roma 1984, Progr. in Math. 60, Birkhäuser (1985), pp. 47-67)

[Cat4] F. Catanese: *Homological Algebra and Algebraic surfaces*, Proc. Symp. in Pure Math., Volume 62.1, 1997, pp. 3-56

[Cil] C. Ciliberto: *Canonical surfaces with* $p_g = p_a = 5$ *and* $K^2 = 10$, Ann. Sc. Norm. Sup. s.IV, v.IX, 2 (1982), pp.287-336

[C-E-P] C. de Concini, D. Eisenbud, C. Procesi: *Hodge algebras*, Astérisque 91 (1982)

[C-S] C. de Concini, E. Strickland: *On the variety of complexes*, Adv. in Math. 41 (1981), pp. 57-77

[D-E-S] W. Decker, L. Ein, F.-O. Schreyer: *Construction of surfaces in* $\mathbb{P}^4$, J. Alg. Geom. 2 (1993), pp.185-237

[Ei] D. Eisenbud: *Commutative Algebra with a view towards Algebraic Geometry*, Springer G.T.M. 150, New York (1995)

[E-U] D. Eisenbud, B. Ulrich: *Modules that are Finite Birational Algebras*, Ill. Jour. Math. 41, 1 (1997), pp. 10-15

[En] F. Enriques: *Le superficie algebriche*, Zanichelli, Bologna, 1949

[Gra] M. Grassi: *Koszul modules and Gorenstein algebras*, J. Alg. 180 (1996), pp.918-953

[Gri-Ha] P. Griffiths, J. Harris: *Principles of Algebraic Geometry*, Wiley, New York (1978)

[Ha] J. Harris: *Algebraic Geometry, A First Course*, Springer G.T.M. 133, New York (1993)

[Har] R. Hartshorne: *Algebraic Geometry*, Springer G.T.M. 52, New York (1977) [dJ-vS] T. de Jong, D. van Straten: *Deformation of the normalization of hypersurfaces*, Math. Ann. 288 (1990), S.527-547

[Lip] J. Lipman: *Dualizing sheaves, differentials and residues on algebraic varieties*, Astérisques 117 (1984)

[M-P] D. Mond, R. Pellikaan: *Fitting ideals and multiple points of analytic mappings*, Springer LNM 1414 (1987), pp.107-161

[P-S1] C. Peskine, L. Szpiro: *Dimension projective finie et cohomologie locale*, Publ. Math. IHES 42 (1972)

[P-S2] C. Peskine, L. Szpiro: *Liaison des variétés algébriques*, Inv. Math. 26 (1972), pp. 271-302

[Roß] D. Roßberg: *Kanonische Flächen mit* $p_g = 5$, $q = 0$ *und* $K^2 = 11, 12$, Doktorarbeit Bayreuth, November 1996

[Sern] E. Sernesi: *L'unirazionalità della varietà dei moduli delle curve di genere dodici*, Ann. Scuola Norm. Pisa 8 (1981), pp. 405-439

[Wal] C. Walter: *Symmetric sheaves and Witt groups*, preprint