CONNECTIONS ON VECTOR BUNDLES AND CHARACTERISTIC CLASSES

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. These notes are intended to summarize and complete the lectures given by me on June 6-8, 2011, as part of the lectures on complex geometry by Prof. Dr. V. Cortés. The main reference is the book of R.O. Wells, but the formalism is more modern. Please contact me for any questions or remarks.

1. Connections on vector bundles

Let $E \xrightarrow{p} M$ be a vector bundle on a manifold M, with fiber $\mathbb{V}$. By definition, around each point of M there exist local frames $f : U \times \mathbb{V} \xrightarrow{\sim} E|_U$ such that $p(f(x, v)) = x$ and $f(x, .)$ is linear. (in general f is smooth; if the bundle is holomorphic, f can be holomorphic as well – see below)

Equivalently, f is an isomorphism between the trivial vector bundle $U \times \mathbb{V} \xrightarrow{p_1} U$ and $E|_U \xrightarrow{p} U$ (here $p_1 : U \times \mathbb{V} \to U$ is the projection). We write $f(x, .) = f_x : \mathbb{V} \xrightarrow{\sim} E_x$, for $x \in U$.

A *section* s in a vector bundle $E \xrightarrow{p} M$ is a smooth map $s : M \to E$ such that $p \circ s = Id_M$. We denote by $C^\infty(E)$ the space of sections of E. For the particular case $E = \Lambda^k M$ (the bundle of k-forms on M), we denote by Ω^k_M the space of (global) k-forms on M. Similarly for forms of type (p, q).

Remark 1.1. The sections of a vector bundle E form a sheaf, denoted by $\mathcal{E}$. The sections of the vector bundle $E \otimes F$ (the tensor product of the vector bundles E and F) form a sheaf isomorphic to $\mathcal{E} \otimes \mathcal{F}$, in particular $\Omega^*(E) := C^\infty(\Lambda^* M \otimes E)$ is isomorphic to the tensor product $\Omega^*_M \otimes C^\infty(E)$, and the forms with values in E form a module on the algebra of differential forms on M. We also denote by $\Omega^{p,q}(E)$ the space of (global) p, q–forms on M with values in E.

Definition 1.2. *A connection ∇ on a vector bundle $E \xrightarrow{p} M$ is a first order linear differential operator $\nabla : C^\infty(E) \longrightarrow C^\infty(T^*M \otimes E)$ with symbol $\sigma^\nabla = Id_{T^*M\otimes E}$.*

Equivalently, a connection ∇ on E is a *covariant derivative*

$$\nabla : C^\infty(TM) \otimes C^\infty(E) \longrightarrow C^\infty(E)$$

satisfying

1. Tensoriality in the TM component: $\nabla_{\varphi X} s = \varphi \nabla_X s$, $\forall \varphi : M \to \mathbb{R}$
2. Leibniz identity: $\nabla_X(\varphi s) = \varphi \nabla_X s + X.\varphi s$, $\forall \varphi : M \to \mathbb{R}$ or $\mathbb{C}$, if E is a complex vector bundle.

Example. On the trivial vector bundle $M \times \mathbb{V}$, shortly denoted by $\mathbb{V}$, the usual (componentwise) derivative $d : C^\infty(\mathbb{V}) \to \Omega^1_M \otimes \mathbb{V}$ is a (trivial) connection. In particular, each frame f on U induces a trivial connection on $E|_U$.

Let ∇, ∇' be connections on E. Because they have the same symbol, their difference is a differential operator of a lower order, therefore it induces a bundle map

$$\nabla - \nabla' =: \theta : E \longrightarrow T^*M \otimes E$$

θ can also be seen as a 1–form on M with values in $End(E) \simeq E^* \otimes E$. If ∇' is the trivial connection assciated to a frame f on U, then θ is the *connection form* of ∇ with respect to the frame f.

2. HOLOMORPHIC CONNECTIONS

Let $E \xrightarrow{p} M$ be a *holomorphic vector bundle*, i.e. E and M are complex manifolds, p is a holomorphic map, and there exist local holomorphic frames $f : U \times \mathbb{V} \xrightarrow{\sim} E|_U$ ($\mathbb{V}$ is a complex vector space).

Proposition 2.1. *On a holomorphic vector bundle* $E \xrightarrow{p} M$ *there is a* canonical *differential operator*

$$\bar{\partial}^E : C^\infty(E) \longrightarrow \Omega^{0,1}(E)$$

with symbol $\sigma^{\bar{\partial}^E} = pr^{o,1} \otimes Id_E$*, that vanishes on holomorphic sections. Here* $pr^{0,1} : \Lambda^1 M \otimes \mathbb{C} \to \Lambda^{0,1} M$ *is the projection on the space of* $(0,1)$ *forms.*

Remark 2.2.(1) *canonical* means that the operator $\bar{\partial}^E$ is determined by the (holomorphic) structure of E alone, without any further choices.

(2) As in the case of a connection, an operator $\bar{\partial}$ as above can be equivalently described as a linear map

$$\bar{\partial}^E : C^\infty(TM \otimes \mathbb{C}) \otimes C^\infty(E) \longrightarrow C^\infty(E)$$

which is tensorial in the TM–component and satisfies the following Leibniz identity:

$$\bar{\partial}^E_X(\varphi s) = \varphi \bar{\partial}^E_X s + (X^{0,1}.\varphi)s,$$

where $X^{0,1}.\varphi = \bar{\partial}_X \varphi = d\varphi(X^{0,1})$ is the derivative of φ in the direction of the $(0,1)$–component of X.

Proof. Let $f : \mathbb{V} \xrightarrow{\sim} E|_U$ be a holomorphic frame, considered as a holomorphic isomorphism between the trivial bundle $\mathbb{V}$ over U and $E|_U$. Then any section s of E induces, by restriction, a section $f^{-1} \circ s$ of the trivial bundle $U \times \mathbb{V}$. Let us denote by s^f the $\mathbb{V}$ component of $f^{-1} \circ s$. It is a $\mathbb{V}$–valued function on U, therefore we can consider

$$\bar{\partial} s^f \in \Omega^{0,1}_U \otimes \mathbb{V} \simeq \Omega^{0,1}(U \times \mathbb{V}).$$

Let

$$\bar{\partial}^f s := f(\bar{\partial} s^f) \in \Omega^{0,1}(E|_U).$$

Here, we have applied to $\bar{\partial} s^f$ the isomorphism (denoted shortly by f)

$$Id_{\Lambda^{0,1}M} \otimes f : \Lambda^{0,1} U \otimes \mathbb{V} \longrightarrow (\Lambda^{0,1} M \otimes E)|_U.$$

We want to show that $\bar{\partial}^f s$ is independent of f. If f' is another holomorphic frame, then

$$s^{f'} = g s^f,$$

where $g : U \to GL(\mathbb{V})$ is a holomorphic map (we denote by $GL(\mathbb{V})$ the group of invertible linear maps from $\mathbb{V}$ to itself; it is an open set in $End(\mathbb{V}) \simeq \mathbb{V}^* \otimes \mathbb{V}$, therefore a complex manifold).

We have $\bar{\partial} s^{f'} = g\bar{\partial} s^f$, because g is holomorphic and we get

$$\bar{\partial}^{f'} s = f'(g\bar{\partial} s^f) = f(\bar{\partial} s^f) = \bar{\partial}^f s.$$

This proves that $\bar{\partial}^f$ is determined by the holomorphic structure of E alone. We note that $\bar{\partial}^E s = 0$ iff $\bar{\partial} s^f = 0$ in each local holomorphic frame, i.e. iff s is a holomorphic section of E. □

Let now ∇ be a connection in the vector bundle E. ∇ induces, by complexification of the (co)-tangent bundle of M, an operator $\nabla : C^\infty(E) \longrightarrow \Omega^1(E)$ that splits into two components: $\nabla^{1,0}$, with values in $\Omega^{1,0}(E)$, and $\nabla^{0,1}$, with values in $\Omega^{0,1}(E)$.

Definition 2.3. *A connection ∇ on the holomorphic bundle $E \xrightarrow{p} M$ is* holomorphic *iff the $0,1$ component of ∇ is equal to $\bar{\partial}^E$.*

A trivial connection induced by a holomorphic frame is, of course, holomorphic.

For any holomorphic connection ∇ on E, the connection form θ^f with respect to a holomorphic frame f has vanishing $(0,1)$-part, therefore $\theta^f \in \Omega^{1,0}(U \times \mathrm{End}(\mathbb{V}))$.

3. The conjugated vector bundle

Let $E \xrightarrow{p} M$ be a complex vector bundle. We denote by $\overline{E} \xrightarrow{\overline{p}} M$ the same real vector bundle, but with the conjugated complex structure on the fibers (the multiplication by $i \in \mathbb{C}$, which is a real endomorphism $J^E : E \to E$, is replaced on $\overline{E}$ by $-J^E$).

A connection ∇ on E induces (the same) connection $\overline{\nabla}$ on $\overline{E}$, but the type decomposition (if M is a complex manifold) changes:

$$\overline{\nabla}^{0,1} = \nabla^{1,0} \text{ and } \overline{\nabla}^{1,0} = \nabla^{0,1}.$$

In the particular case where $E \xrightarrow{p} M$ is holomorphic, we can see this also in the following way: $\overline{E} \xrightarrow{\overline{p}} \overline{M}$ is again a holomorphic vector bundle on $\overline{M}$, which is the same real manifold as M, but with conjugated complex structure (equivalently, all holomorphic charts of M are composed with a conjugation in $\mathbb{C}^n$ and become holomorphic charts for $\overline{M}$). Here, we consider $\overline{p} : \overline{E} \to \overline{M}$ as a holomorphic map between complex manifolds ($\overline{E}$ is the conjugated complex manifold to E).

We have $T^{0,1}\overline{M} = T^{1,0}M$ and $T^{1,0}\overline{M} = T^{0,1}M$ hence the relation between type decompositions of ∇ and $\overline{\nabla}$.

Note that, $\overline{E}$ being a holomorphic bundle over $\overline{M}$, has a canonical $\bar{\partial}^{\overline{E}}$ operator. As a vector bundle over M, this operator has symbol

$$\sigma^{\bar{\partial}^{\overline{E}}} = pr^{1,0} \otimes Id_{\overline{E}}.$$

4. Induced connections, hermitian connections

If ∇ is a connection in a vector bundle $E \xrightarrow{p} M$, then it induces a connection (still denoted by ∇) on E^*, $End(E)$ and, in general, on any tensor bundle $E \otimes \ldots \otimes E \otimes E^* \otimes \ldots \otimes E^*$ (we can also add tensor powers of $\overline{E}$ and of $\overline{E}^*$, as we have seen above).

For example, if $\alpha \in C^\infty(E^*)$, then

$$(\nabla_X \alpha)(s) := X.(\alpha(s)) - \alpha(\nabla_X s),$$

or, if $A \in C^\infty(End(E))$, then

$$(\nabla_X A)(s) := \nabla_X(A(s)) - A(\nabla_X s).$$

Definition 4.1. *Let $E \xrightarrow{p} M$ be a complex vector bundle. A hermitian metric h on E is a $\mathbb{C}$–linear bundle map*

$$h : E \otimes \overline{E} \to \mathbb{C}$$

from the complex vector bundle $E \otimes \overline{E}$ and the trivial bundle $M \times \mathbb{C}$ such that:

(1) $h(s, \bar{t}) = \overline{h(t, \bar{s})}$, $\forall s, t \in E$
(2) $h(s, \bar{s}) > 0$, $\forall s \neq 0$.

Remark 4.2. Because h is positive definite, it is non-degenerate, i.e. it induces an isomorphism of complex vector bundles

$$h : E^* \longrightarrow \overline{E}.$$

Note also that, with our convention, h is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$–antilinear in the second. The use of the conjugation bar in the second argument makes this more visible.

Otherwise, $\bar{s}$ still means s itself (the "conjugation map" from E to $\overline{E}$ is the identity of the corresponding real vector space, but it is $\mathbb{C}$-antilinear).

Theorem 4.3. *Let $E \xrightarrow{p} M$ be a holomorphic vector bundle and let h be a hermitian metric on it. There is a unique connection ∇^h on E which is*

(1) *hermitian, i.e.* $\nabla h = 0$
(2) *holomorphic, i.e.* $\nabla^{0,1} = \bar{\partial}^E$.

This connection is called the canonical connection *or the* Chern connection *of the holomorphic hermitian vector bundle $(E, h) \xrightarrow{p} M$.*

Proof. $\overline{E}^* \xrightarrow{\bar{p}} \overline{M}$ is a holomorphic vector bundle on $\overline{M}$, therefore it has a canonical $\bar{\partial}^{\overline{E}^*}$ operator. Using $h : E \xrightarrow{\sim} \overline{E}^*$, we get thus an operator

$$\partial^h := h^{-1} \circ \bar{\partial}^{\overline{E}^*} \circ h : C^\infty(E) \longrightarrow \Omega^{0,1}_{\overline{M}}(E) = \Omega^{1,0}_M(E),$$

whose symbol is $\sigma^{\partial^h} = pr^{1,0} \otimes Id_E$.

Let $\nabla^h := \partial^h + \bar{\partial}^E$. Then ∇^h is holomorphic. Let's check that it is hermitian:

By definition of ∂^h, we have that $h(\partial^h s) = \bar{\partial}^{\overline{E}^*}(h(s))$, i.e.

$$h(\partial^h_X s, \bar{t}) = \bar{\partial}^{\overline{M}}_X (h(s, \bar{t})) - h(s, \bar{\partial}^{\overline{E}}_X \bar{t}) = \partial_X(h(s, \bar{t})) - h(s, \overline{\bar{\partial}^E_{\overline{X}} t}). \tag{1}$$

On the other hand, by conjugating (1) with s and t interchanged, we get

$$\bar{\partial}_X(h(s, \bar{t})) = \bar{\partial}_X \overline{h(t, \bar{s})} = \overline{\partial_X(h(t, \bar{s}))} = h(s, \overline{\partial^h_{\overline{X}} t}) + h(\bar{\partial}^E_X s, \bar{t}),$$

which, together with (1), yields

$$X.h(s, \bar{t}) = h(\nabla^h_X s, \bar{t}) + h(s, \overline{\nabla}^h_{\overline{X}} \bar{t}) = h(\nabla^h_X s, \bar{t}) + h(s, \overline{\nabla_{\overline{X}} t}),$$

so ∇^h is hermitian. On the other hand, if ∇ is hermitian, then, for $X \in T^{1,0}M$ we have

$$X.h(s,\bar{t}) = h(\nabla_X s, \bar{t}) + h(s, \overline{\nabla_{\overline{X}} t}) = h(\nabla^{1,0}_X s, \bar{t}) + h(s, \nabla^{\overline{E}}_{\overline{X}} \bar{t}),$$

which implies

$$h(\nabla^{1,0}_X s)(t) = h(\nabla^{1,0}_X s, \bar{t}) = \partial_X (h(s,\bar{t})) - h(s, \bar\partial^{\overline{E}}_{\overline{X}} \bar{t}) = \bar\partial^{\overline{E}^*}_{\overline{X}} \left(h(s)\right)(t),$$

thus $\nabla^{1,0} = \partial^h$, hence ∇^h is uniquely determined by the holomorphic and the hermitian structure of E. □

If f is a holomorphic frame, then the connection form of the Chern connection has only a $(1,0)$–part. To compute it, we write h^f for the matrix corresponding to h in the frame f. Then h^f is a hermitian matrix, i.e.

$$h^f = {}^t\overline{h^f}.$$

The $(1,0)$–part of ∇^h is then

$$\theta(X)s = \theta^{1,0}(X)s = (h^f)^{-1} \partial_X h^f \, s. \tag{2}$$

Remark 4.4. If $E \xrightarrow{p} M$ is a complex line bundle, then h is (locally) determined by its value $h(s,\bar{s})$ on a non-vanishing (local) section s. The covariant derivative of s is also determined by its hermitian product with s. In particular:

$$X.h(s,\bar{s}) = h(\nabla^h_X s, \bar{s}) + h(s, \overline{\nabla_{\overline{X}} s}) = h(\nabla_X s, \bar{s}) + \overline{h(\nabla_{\overline{X}} s, \bar{s})}.$$

If $X \in T^{1,0}M$, denoting $\nabla^{1,0}s$ by $\theta(.)s$, and using that $\nabla^{0,1}s = \bar\partial^E s = 0$, we get

$$\partial_X (h(s,\bar{s})) = h(\theta(X)s, \bar{s}) = \theta(X) h(s,\bar{s}).$$

If we denote by $\|s\|^2 := h(s,\bar{s})$, then we get

$$\theta(X) = \partial_X \left(\ln(\|s\|^2 \right). \tag{3}$$

5. Exterior covariant derivatives and curvature

Let $E \xrightarrow{p} M$ be a vector bundle (real or complex) with a connection ∇. The operator $\nabla : \Omega^0(E) \longrightarrow \Omega^1(E)$ induces, by the following formula (based on the usual formula for the exterior differential), the following operators:

$$d^\nabla : \Omega^k(E) \longrightarrow \Omega^{k+1}(E),$$

$$\begin{aligned} d^\nabla(\alpha)(X_0, \ldots, X_k) \; := \; & \sum_{j=0}^{k} (-1)^j \nabla_{X_j} \Big(\alpha(X_0, \ldots, \hat{X}_j, \ldots, X_k) \Big) + \\ & + \sum_{0 \le j < l \le k} (-1)^{j+l} \alpha([X_j, X_l], X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_l, \ldots, X_k), \end{aligned} \tag{4}$$

where the hat ˆ indicates a missing term. The proof of the following proposition is straightforward (by induction):

Proposition 5.1. *Let $\alpha \in \Omega^k(E)$ and $\beta \in \Omega^l_M$. Then*

$$d^\nabla(\beta \wedge \alpha) = d\beta \wedge \alpha + (-1)^l \beta \wedge d^\nabla \alpha.$$

Moreover, this property together with the fact that $d^\nabla = \nabla$ on $\Omega^0(E)$ uniquely determines the operator $d^\nabla : \omega^*(E) \longrightarrow \Omega^*(E)$.

Remark 5.2. If ∇ is a trivial connection (for example, given by a (local) frame), then d^∇ is the usual exterior derivative of forms with values in a (fixed) vector space, therefore $(d^\nabla)^2 = 0$.

In general, $(d^\nabla)^2 \neq 0$.

Definition 5.3. *Let ∇ be a connection on a vector bundle $E \xrightarrow{p} M$. Its* curvature R^∇ *is a 2–form with values in $End(E)$, defined by $R^\nabla s := (d^\nabla)^2 s$, more precisely*

$$R^\nabla_{X,Y} s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \;\; X, Y \in TM, \;\; s \in C^\infty(E).$$

Remark 5.4. Even if $(d^\nabla)^2$ is not zero, it is a zero-order differential operator (i.e. a tensor).

Using Proposition 5.1 and the fact that the sheaf $\Omega^*(E)$ is isomorphic to the tensor product $\Omega^*_M \otimes \mathcal{E}$, we get

Proposition 5.5. *Let $\alpha \in \Omega^*(E)$. Then $(d^\nabla)^2 \alpha = R^\nabla \wedge \alpha$.*

Here we have used the following notation: for a k-form α with values in E, and a l-form Q with values in $End(E)$, we define their exterior product by taking the tensor product of the following maps:

$$\begin{array}{rcl} \Lambda^k M \otimes \Lambda^l M & \xrightarrow{\wedge} & \Lambda^{k+l} M \\ \eta \otimes \xi & \mapsto & \eta \wedge \xi, \end{array}$$

$$\begin{array}{rcl} E \otimes End(E) & \longrightarrow & E \\ s \otimes A & \mapsto & A(s). \end{array}$$

If we consider two forms with values in $End(E)$, $A \in \Omega^k(End(E))$ and $B \in \Omega^l(End(E))$, we can consider their commutator

$$[A, B] \in \Omega^{k+l}(End(E))$$

by tensoring the wedge product of forms on M with the following linear map:

$$\begin{array}{lcl} End(E) \otimes End(E) & \longrightarrow & End(E) \\ A \otimes B & \mapsto & AB - BA. \end{array}$$

We have the following *differential Bianchi identity* (sometimes called the *second Bianchi identity* in the case of connections on the tangent bundle):

Proposition 5.6. *The curvature tensor $R^\nabla \in \Omega^2(End(E))$ of a connection ∇ on $E \xrightarrow{p} M$ is d^∇-closed.*

Proof. Let us compute $(d^\nabla)^3 s$, for a section s in E:

$$(d^\nabla)^3 s = d^\nabla((\partial^\nabla)^2 s) = d^\nabla \left(R^\nabla s \right) = d^\nabla R^\nabla s + R^\nabla \wedge \nabla s.$$

On the other hand,

$$(d^\nabla)^3 s = (d^\nabla)^2 \nabla s = R^\nabla \wedge \nabla s,$$

therefore $d^\nabla R^\nabla = 0$. □

Let us compute the change of the curvature tensor when the connection changes: Let $\nabla' := \nabla + \theta$ be another connection on E, where $\theta \in \Omega^1(E)$. Then $d^{\nabla'} = d^\nabla + \theta \wedge \cdot$, therefore

$$(d^{\nabla'})^2 s = d^\nabla(d^\nabla s + \theta s) + \theta \wedge (d^\nabla s + \theta s) = (d^\nabla)^2 s + d^\nabla \theta s - \theta \wedge \nabla s + \theta \wedge \nabla s + \theta \wedge \theta s.$$

the last term is not zero, even if θ is a 1–form with values in $End(E)$; this is because $End(E)$ is not commutative. We get

$$R^{\nabla'} = R^\nabla + d^\nabla \theta + \frac{1}{2}[\theta, \theta]. \tag{5}$$

Remark 5.7. As we have done for d^∇, we can extend the operator $\bar\partial^E$ of a holomorphic vector bundle to an operator $\bar\partial^E : \Omega^{o,q}(E) \longrightarrow \Omega^{0,q+1}(E)$. Its square is zero, as easily checked in a holomorphic frame. Therefore, the curvature of a holomorphic connection has its $(0,2)$–part defined by $(\partial^E)^2 = 0$.

It is possible to define a holomorphic structure on a complex vector bundle as a $\bar\partial$ operator (with the appropriate symbol) with square zero.

Remark 5.8. Because $\nabla^{1,0}$ of the Chern connection is defined by a $\bar\partial$ operator on $\overline{E}$, it turns out that the $(2,0)$–part of the curvature vanishes as well. Therefore, for the Chern connection, $R^\nabla \in \Omega^{1,1}(End(E))$.

6. Chern-Weil theory

Definition 6.1. *Let* $\mathbb{V}$ *be a vector space and denote by* $\mathfrak{g} := End(\mathbb{V})$*, and by* G *the group of linear automorphisms of* $\mathbb{V}$*. An (ad-)invariant polynomial* $P : \mathfrak{g} \to \mathbb{C}$*, resp. an (ad-)invariant symmetric multilinear form* $S : \mathfrak{g} \otimes \ldots \otimes \mathfrak{g} \longrightarrow \mathbb{C}$*, such that*

$$P(gAg^{-1}) = P(A), \ \forall A \in \mathfrak{g}, \ g \in G,$$

resp. that

$$S(gA_1g^{-1}, \ldots, gA_kg^{-1}) = S(A_1, \ldots, A_k), \ \forall A_i \in \mathfrak{g}, \ g \in G.$$

From a symmetric multilinear form S we get a polynomial $P^S(A) := S(A, \ldots, A)$ and, conversely, from a polynomial P on $\mathfrak{g}$, homogeneous of degree k, we get the symmetric multilinear form

$$S^P(A_1, \ldots, A_k) := \frac{1}{k!} \frac{d^k}{dt_1 \ldots dt_k}|_{t_1 = \cdots = t_k = 0} P(t_1A_1 + \cdots + t_kA_k).$$

It is clear that $P^{S^P} = P$ and that P is invariant iff S^P is.

Example. The trace, determinant and, more generally, the homogeneous components of $A \mapsto \det(Id + A)$ are invariant polynomials. Note that these polynomials have real coefficients, i.e., they have real values on real matrices.

Remark 6.2. The notation $\mathfrak{g}$, resp. G and the term *ad*-invariant are reminiscent of the theory of Lie groups (groups that have a differential structure), that is strongly connected to the topic of characteristic classes: G is a Lie group (here $GL(\mathbb{V})$), $\mathfrak{g}$ its Lie algebra (here $\mathrm{End}(\mathbb{V})$) and the action of G on $\mathfrak{g}$ (here $(g, A) \mapsto gAg^{-1}$)) is the *adjoint* action of G on its Lie algebra. Another Lie groups commonly occuring in the theory are the group $O(n)$ of orthogonal real $n \times n$ matrices, with Lie algebra $\mathfrak{o}(n)$ (skew-symmetric real matrices), and the group $U(n)$ of unitary (complex) $n \times n$ matrices, with Lie algebra $\mathfrak{u}(n)$ of anti-hermitian matrices ($\overline{A} = -{}^tA$).

Proposition 6.3. *A symmetric multilinear form* $S : \mathfrak{g}^{\otimes k} \to \mathbb{C}$ *is invariant iff*

$$\sum_{j=1}^{k} S(A_1, \ldots, [B, A_j], \ldots, A_k) = 0, \ \forall B, A_i \in \mathfrak{g}. \tag{6}$$

Proof. Let $g_t := \exp(tB) \in G$, for $t \in \mathbb{R}$, $B \in \mathfrak{g}$ (this holds for the other examples of Lie groups above). All elements in the neighborhood of the identity of G can be written like this. Fix $A_1, \ldots, A_k \in \mathfrak{g}$ and consider $F(t) := S(g_t A_1 g_t^{-1}, \ldots, g_t A_k g_t^{-1})$. Then note that

$$\frac{d}{dt}\exp(tB) = B\exp(tB), \text{ thus } \frac{d}{dt}g_t = Bg_t \text{ and } \frac{d}{dt}g_t^{-1} = -g_t^{-1}B.$$

If S is invariant, then F is constant, therefore $F'(0) = 0$, which implies (6). Conversely, if (6) holds for every A_j and $B \in \mathfrak{g}$, in particular for $g_t A_j g_t^{-1}$ and B, it follows that $F'(t) = 0 \ \forall t \in \mathbb{R}$, therefore

$$F(t) = F(0) = S(A_1, \ldots, A_k).$$

This implies that the map $g \mapsto S(gA_1g^{-1}, \ldots, gA_kg^{-1})$ is constant on a neighborhood of $Id \in G$ (for fixed $A_1, \ldots, A_k \in \mathfrak{g}$). But G is connected and the considered map is analytic (even polynomial), thus it has to be constant on whole G. ☐

Let $E \xrightarrow{p} M$ be a vector bundle with fiber $\mathbb{V}$ and let S be an invariant symmetric multilinear form on $End(\mathbb{V})$. Then S induces a multilinear bundle map

$$S^E : End(E) \otimes \ldots \otimes \mathrm{End}(E) \longrightarrow \mathbb{C},$$

given, in a frame f, by

$$S^f(A_1, \ldots, A_k) := S(A_1^f, \ldots, A_k^f),$$

where A_j^f is the element in $End(\mathbb{V})$ defined by $A_j \in End(E)$ and the frame f. The invariance of S ensures that S^f is independent of the frame and hence S^E is well-defined.

By using again the sheaf isomorphism

$$\Omega^*(End(E)) \simeq \Omega^*_M \otimes End(\mathcal{E})$$

we extend S^E to a multilinear map

$$S^E : \Lambda^*(End(E))^{\otimes k} \longrightarrow \Lambda^* M,$$

$$S^E(\alpha_1 \otimes A_1, \ldots, \alpha_k \otimes A_k) := \alpha_1 \wedge \ldots \wedge \alpha_k \cdot S^E(A_1, \ldots . A_k),$$

forall $A_j \in End(E)$, $\alpha_j \in \Lambda^* M$.

We can also define $P^E(\alpha) := S^E(\alpha, \ldots, \alpha) \in \Lambda^{kp} M$, $\alpha \in \Lambda^p(End(E))$, where P is the polynomial associated to the symmetric multilinear map S.

Theorem 6.4. (Chern-Weil) *Let* $E \xrightarrow{p} M$ *be a vector bundle with fiber* $\mathbb{V}$*,* P *an invariant polynomial on* $End(\mathbb{V})$ *of degree* k *and* ∇ *a connection on* E*, with curvature* R^∇*. Then the* $2k$*-form* $P^E(R^\nabla) \in \Omega^{2k}_M$ *is closed, and its class in the de Rham cohomology group* $H^{2k}(M, \mathbb{C})$ *is independent of* ∇*.*

Proof. First we show that $P^E(R^\nabla)$ is d^∇–closed. This is an immediate consequence of the following

Lemma 6.5. *Let $\alpha_j \in \Omega^{p_j}(End(E))$ and S and invariant symmetric multilinear form of degree k. Denote by $\epsilon_j := p_1 + \cdots + p_{j-1}$. We have then*

$$d\left(S(\alpha_1, \dots \alpha_k)\right) = \sum_{j=1}^{k} S(\alpha_1, \dots, (-1)^{\epsilon_j} d^\nabla \alpha_j, \dots, \alpha_k).$$

Proof. We proceed by induction over the sum ϵ of all degrees of the forms α_j, that we suppose to be arranged such that $p_1 \leq \cdots \leq p_k$. For all $p_j = 0$ the claimed formula is just the covariant derivative of

$$S^E(A_1, \dots, A_k),$$

for $A_j \in End(E)$, where we note that S^E is induced by a *constant* element of $Hom(\mathfrak{g}^{\otimes k}, \mathbb{C})$, therefore its derivative vanishes.

Suppose now the claim is true for any $\alpha_j \in \Omega^{p_j}(End(E))$ such that the sum of the degrees is $\sum p_j = \epsilon \geq 0$ and let now $\alpha'_j := \alpha_j$ for $1 \leq j \leq k$ and $\alpha'_k = \alpha_k \wedge \beta \in \Omega^{p_k+1}(End(E))$, where β is a 1–frm on M. If we show that the claim holds for S and for all $\alpha_1, \dots \alpha_{k-1}, \alpha'_k$ as above and every 1–form β, then (again using that $\Omega^{p+1}(End(E)) = \Omega^1_M \otimes \Omega^p(End(E))$ as a sheaf), the claim will be proven for $\epsilon + 1$.

Note that

$$S^E(\alpha_1, \dots, \alpha'_k) = S^E(\alpha_1, \dots, \alpha_k) \wedge \beta,$$

thus

$$\begin{aligned} & dS^E(\alpha_1, \dots, \alpha'_k) = \\ = \; & \sum_{j=1}^{k} (-1)^{\epsilon_j} S^E(\alpha_1, \dots, d^\nabla \alpha_j, \dots, \alpha_k) \wedge \beta + (-1)^{\epsilon - 1} S^E(\alpha_1, \dots, \alpha_k) \wedge d\beta = \\ = \; & \sum_{j=1}^{k-1} (-1)^{\epsilon_j} S^E(\alpha_1, \dots, d^\nabla \alpha_j, \dots, \alpha'_k) + (-1)^{\epsilon_k} S^E(\alpha_1, \dots, d^\nabla \alpha'_k). \end{aligned}$$

□

To show that the cohomology class defined by $P^E(R^\nabla)$ is independent of ∇, we need to compute the difference between $P^E(R^{\nabla'})$ and $P^E(R^\nabla)$, for a new connection $\nabla' = \nabla + \theta$ and show that it is an exact form.

Actually, we will consider a path of connections ∇^t between ∇ and ∇', for example $\nabla^t := \nabla + t\theta$. Then $\nabla^0 = \nabla$ and $\nabla^1 = \nabla'$. Denote by $R^t := r^{\nabla^t}$. In order to show that

$$P^E(R^1) = P^E(R^0) + d\beta, \tag{7}$$

we will show that

$$\frac{d}{dt} P^E(R^t) = d\beta^t,$$

and conclude by integration that the class of cohomology of $P(R^t)$ is constant. More precisely, (7) holds with

$$\beta = \int_0^1 \beta^t dt$$

Let us compute the derivative in t at $t = t_0$ of $P^E(R^t)$: first note that

$$\frac{d}{dt}|_{t=t_0} R^t = \lim_{t\to t_0} R^t - R^{t_0} = \lim_{t\to t_0}\left(d^{\nabla^{t_0}}(t-t_0)\theta + \frac{1}{2}[(t-t_0)\theta,(t-t_0)\theta]\right) = d^{\nabla^{t_0}}\theta.$$

Therefore

$$\frac{d}{dt}|_{t=t_0} P^E(R^t) = \sum_{j=1}^{k} S^E(R^{t_0},\ldots,\overset{(j)}{d^{\nabla^{t_0}}\theta},\ldots,R^{t_0}),$$

but this is, according to the previous lemma (using that $d^{\nabla^{t_0}}R^{t_0} = 0$), equal to $d\beta^{t_0}$, where

$$\beta^{t_o} := \sum_{j=1}^{k} S^E(R^{t_0},\ldots,\overset{(j)}{\theta},\ldots,R^{t_0}).$$

We get thus $P^E(R^1) - P^E(R^0) = d\beta$, where β is the integral in t of the forms β^t found above. ☐

Corollary 6.6. (Chern classes) *Let $E \xrightarrow{p} M$ be a complex vector bundle and let C_k be the homogeneous polynomial of order k in the invariant (non-homogeneous) polynomial expression $A \mapsto \det(Id + A)$. Let ∇ be a connection on E. Then the cohomology classes*

$$c_k^{\mathbb{R}}(E) := \left[C_k^E(\frac{1}{2\pi i}R^\nabla)\right]$$

are real, i.e. the above forms represent cohmology classes $c_k^{\mathbb{R}}(E) \in H^{2k}(M,\mathbb{R})$, called the (real) Chern classes *of E.*

Remark 6.7. The classes $c_k^{\mathbb{R}}(E)$ defined above are actually induced by the *integer Chern classes $c_k(E) \in H^{2k}(M,\mathbb{Z})$*, defined in algebraic topology.

Proof. It suffices to consider a particular connection and show that the resulting form $C_k^E(\frac{1}{2\pi i}R^\nabla)$ is a real $2k$–form.

We consider a hermitian metric on E and take ∇ a hermitian connection. Then its curvature will be a 2–form with values in $End(E)$, actually in the subspace (or rather, Lie subalgebra) of antihermitian endomorphisms of E. This space is isomorphic, via an *unitary frame* $f : U \times \mathbb{V} \xrightarrow{\sim} E|_U$ (a frame that is a unitary isomorphism on the fibers), to the space of antihermitian matrices $\mathfrak{u}(\mathbb{V}) \subset End(\mathbb{V})$.

What we need to show is that $C_k(iA)\imath n\mathbb{R}$, for any antihermitian matrix A. But iA is hermitian and $Id + iA$ too, therefore $\det(Id + iA)$ is real, and so must be all its homogeneous components. ☐

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