PROJECTIVE AND CONFORMAL FLATNESS

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. These notes are intended to summarize and complete the lectures given by me on Dec. 6 and 10, 2012, as part of the lectures on affine differential geometry by Prof. Dr. V. Cortés.
Feel free to send me by e-mail your remarks and comments.

1. Projective geometry

1.1. **Projective structures.**

Definition 1. *A projective structure on a manifold M of dimension $n > 1$ is a class of torsion-free connections on M that define the same family of unparametrized geodesics.*

A curve $c : I \to M$ is a (parametrized) geodesic for the connection ∇ iff

$$\nabla_{\dot c}\dot c = 0.$$

The curve c is an unparametrized geodesic for ∇ iff there is a diffeoemorphism $\varphi : I_1 \to I$ such that $c \circ \varphi$ is a parametrized geodesic. Equivalently, c is an unparametrized geodesic iff

$$\nabla_{\dot c}\dot c \text{ is colinear to } \dot c. \tag{1}$$

Example 2. $\mathbb{R}\mathrm{P}^n$*, the real projective space of dimension n, is the set of lines (i.e., one-dimesional vector subspaces) of $\mathbb{R}^{n+1}$, or, equivalently, the quotient of $\mathbb{R}^{n+1} \smallsetminus \{0\}$ by the action of $\mathbb{R}^*$.*

The family of unparametrized geodesics on $\mathbb{R}\mathrm{P}^n$ is the family of projective lines*: such a projective line $[E]$ is the set of (vectorial) lines in $\mathbb{R}^{n+1}$ contained in a 2–plane $E \subset \mathbb{R}^{n+1}$ or, equivalently, the set $(E \smallsetminus \{0\})/\mathbb{R}^*$.*

Indeed, on each affine chart $U_\alpha := \{\mathbb{R}x \mid \alpha(x) \neq 0\} \subset \mathbb{R}\mathrm{P}^n$, *(here $\alpha \in (\mathbb{R}^{n+1})^*$) the affine hyperplane*

$$H_\alpha := \{x \in \mathbb{R}^{n+1} \mid \alpha(x) = 1\}$$

is diffeomorphic to U_α. Any affine diffeomorphism of H_α to $\mathbb{R}^n$ induces a torsion-free (flat) connection on U_α whose geodesic are the affine lines in H_α, which correspond to the 2–planes in $\mathbb{R}^{n+1}$ as described above.

Note that $\mathbb{R}\mathrm{P}^n$ is the quotient of the euclidian sphere $S^n \subset R^{n+1}$ under the action of the antipodal transformation. The induced Riemannian connection on S^n has the *big circles* (i.e., intersections of S^n with 2–planes) as geodesics, which cover the projective lines in $\mathbb{R}\mathrm{P}^n$. Therefore, the (non-flat) Levi-Civita connection on the round sphere induce (globally) the same unparametrized geodesics on $\mathbb{R}\mathrm{P}^n$ as the affine charts (which have flat connections).

This suggest the notion of projective equivalence and projective flatness:

Definition 3. *Two connections are* projectively equivalent *iff they induce the same unparametrized geodesics. A connection is* projectively flat *iff it is locally equivalent to a flat connection. The corresponding projective structure is called* flat.

Proposition 1. *For two projectively equivalent connections ∇, ∇' there exists a 1–form η on M such that*

$$\nabla'_X Y = \nabla_X Y + \tilde\eta_X Y;\ \tilde\eta_X = \eta(X)\mathrm{Id} + \eta \otimes X.$$

Proof. Two linear connections are related by

$$\nabla'_X Y = \nabla_X Y + A_X Y,$$

where $A : TM \to \mathrm{End}(TM)$ is a linear map. The connections ∇, ∇' have the same torsion iff $A_X Y = A_Y X$, $\forall X, Y \in TM$.

On the other hand, the condition (1) implies that

$$A_X X \text{ is colinear to } X, \ \forall X \in TM.$$

Equivalently, there is a map $\eta : TM \to R$ such that

$$A_X X = 2\eta(X)X, \ \forall X \in TM. \tag{2}$$

Because $X \mapsto A_X X$ is quadratic, so must be $X \mapsto \eta(X)X$, therefore $\eta : TM \to \mathbb{R}$ has to be linear. Applying (2) to $X + Y$ for any vectors X, Y, and using the symmetry condition $A_X Y = A_Y X$, we obtain

$$A_X Y = \eta(X)Y + \eta(Y)X.$$

□

1.2. **Curvature.** The curvature of a connection ∇ is defined, as usual, by the formula

$$R_{X,Y} Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) Z,$$

and is thus a section R of

$$\Lambda^2 M \otimes \mathrm{End}(TM) = \Lambda^2 M \otimes (T^*M \otimes TM).$$

If the connection is torsion-free, then the first Bianchi identity implies that

$$\sum_{X,Y,Z} R_{X,Y} Z := R_{X,Y} Z + R_{Y,Z} X + R_{Z,X} Y = 0,$$

or, equivalently, that $B_1(R) = 0$, where

$$B_1 : \Lambda^2 M \otimes (T^*M \otimes TM) \to \Lambda^3 M \otimes TM$$

is the corresponding skew-symmetrization of the first 3 factors.

We denote by $\mathcal{R}$ the space of linear curvature tensors R satisfying the Bianchi identity. The dimension of $\mathcal{R}$ is

$$\frac{n(n-1)}{2} n^2 - \frac{n(n-1)(n-2)}{6} n = \frac{n^2(n^2-1)}{3}.$$

This space can be, however, further decomposed, using two independent contractions:

$$F(X,Y) := \frac{1}{n} \mathrm{tr}(Z \mapsto R_{X,Y} Z) \in \Lambda^2 M$$

is called the *Faraday form*, and

$$\mathrm{Ric}(X,Y) := \mathrm{tr}(Z \mapsto R_{Z,X} Y) \in T^*M \otimes T^*M$$

is called the *Ricci tensor* of R. The first Bianchi identity implies that

$$\mathrm{Ric}(X,Y) - \mathrm{Ric}(Y,X) = -nF(X,Y), \tag{3}$$

thus the skew-symmetric part of Ric is

$$\mathrm{Ric}^{sk} = -\frac{n}{2} F \in \Lambda^2 M.$$

There also exist maps from $\Lambda^2 M$, resp from $S^2 M$ to $\mathcal{R}$, constructed by *suspension* with the identity:

Definition 4. *The* suspension *of a bilinear map $A \in T^*M \otimes T^*M$ with the identity is the following element of $\Lambda^2 M \otimes \mathrm{End}(TM)$:*

$$(A \wedge \mathrm{Id})_{X,Y} Z := A(Y,Z)X - A(X,Z)Y.$$

This suspension is also in $\mathcal{R}$ iff $A \in S^2M$ (i.e., A is symmetric). More precisely, for a skew-symmetric form A, the suspension $A \wedge \mathrm{Id}$ needs to be corrected by a Faraday-type term, in order to obtain an element of $\mathcal{R}$:

$$\tilde{A} := \frac{1}{2} A \wedge \mathrm{Id} - A \otimes \mathrm{Id} \in \mathcal{R},\ \forall A \in \Lambda^2 M.$$

The Ricci contraction of $(A \wedge \mathrm{Id})$ is

$$\mathrm{tr}(Z \to (A \wedge \mathrm{Id})_{Z,X} Y) = \mathrm{tr}(Z \to A(X,Y)Z - A(Z,Y)X = (n-1)A(X,Y)$$

(for A symmetric or not), and the Faraday contraction of it is zero iff A is symmetric. The Ricci contraction of $\tilde{F}$ is $\frac{n-1}{2}F + F = \frac{n+1}{2}F$. On the other hand, the Faraday contraction of $\tilde{F}$ is $-\frac{n+1}{n}F$. This implies that, for a symmetric tensor ρ^{sym}, the tensor

$$\rho^{sym} \wedge \mathrm{Id} - \frac{n}{n+1}\tilde{F} \text{ is in } \mathcal{R},$$

its Faraday contraction is F, and its Ricci contraction is $(n-1)\rho^{sym} - \frac{n}{2}F$.

Therefore, if, for a given connection ∇ with curvature R, Ricci tensor Ric and Faraday form F, we define the *normalized Ricci tensor* (or *Schouten tensor*)

$$\rho^R := \frac{1}{n-1}\left(\mathrm{Ric} + \frac{n}{n+1}F\right), \tag{4}$$

then the Faraday and Ricci tensor of

$$(\rho^R \wedge \mathrm{Id}) + \frac{n}{n+1} F \otimes \mathrm{Id} \tag{5}$$

coincide with the ones of R itself. It follows that the *Weyl tensor*, defined by

$$W := R - \rho^R \wedge \mathrm{Id} - \frac{n}{n+1} F \otimes \mathrm{Id} \in \mathcal{R},$$

has vanishing Faraday and Ricci trace (hence it is completely trace-free). Denote by $\mathcal{W} \subset \mathcal{R}$ the space of trace-free (or Weyl) curvature tensors.

We conclude:

Proposition 2. *The curvature of a torsion-free connection decomposes in the following tensors, sections of vector bundles associated to $GL(n)$–irreducible representations:*

$$R = W + \rho^{sym} \wedge \mathrm{Id} + \frac{n}{n+1}\tilde{F},$$

where $W \in \mathcal{W}$, $\rho^{sym} \in S^2M$, $F \in \Lambda^2 M$.

The dimension of $\mathcal{W}$ is thus

$$\dim \mathcal{W} = \frac{n^2(n^2-1)}{3} - n^2 = \frac{n^2(n^2-4)}{3} > 0 \ \Leftrightarrow\ n > 2.$$

The Weyl tensor vanishes thus for $n = 2$ by dimensional reasons.

We will show that, for $n > 2$, the vanishing of W is equivalent to the projective flatness of ∇. One of the claimed implications follows from

Proposition 3. *If $\nabla' = \nabla + \tilde{\eta}$, we have the following relations between the curvature components of R, R':*

$$F' = F + d\eta,\ \rho' = \rho - \nabla\eta + \eta \otimes \eta,\ W' = W.$$

Proof. We make the computations in a point $p \in M$, and we suppose that all the vector fields used for the computations are ∇–parallel at p. Thus

$$R'_{X,Y} = [\nabla'_X, \nabla'_Y] = [\nabla_X + \tilde{\eta}_X, \nabla_Y + \tilde{\eta}_Y].$$

We have $[\nabla_X, \nabla_Y] = R_{X,Y}$, and we need to compute

$$[\nabla_X, \tilde{\eta}_Y] - [\nabla_Y, \tilde{\eta}_X]$$

and $[\tilde{\eta}_X, \tilde{\eta}_Y]$. Recall that $\tilde{\eta}_X = \eta(X)\mathrm{Id} + \eta \otimes X$. Therefore

$$\begin{aligned} [\nabla_X, \tilde{\eta}_Y] - [\nabla_Y, \tilde{\eta}_X] &= (\widetilde{\nabla_X \eta})_Y - (\widetilde{\nabla_Y \eta})_Y = \\ &= d\eta(X,Y)\mathrm{Id} - (\nabla\eta \wedge \mathrm{Id})_{X,Y}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} [\tilde{\eta}_X, \tilde{\eta}_Y] &= [\eta \otimes X, \eta \otimes Y] = \\ &= ((\eta \otimes \eta) \wedge \mathrm{Id})_{X,Y} \,. \end{aligned} \tag{7}$$

$\square$

1.3. **The Cotton tensor.** Before proving the main theorem of projective geometry, we define

Definition 5. *The* Cotton tensor *of a connection ∇ is the following section of $\Lambda^2 M \otimes T^*M$ (whose skew-symmetric part, i.e., the component in $\Lambda^3 M$, is zero):*

$$C(X,Y;Z) = \nabla_X \rho(Y,Z) - \nabla_Y \rho(X,Z).$$

Remark 6. *For any connection, the curvature satisfies the second (or differential) Bianchi identity:*

$$\nabla_X R_{Y,Z} + \nabla_Y R_{Z,X} + \nabla_Z R_{X,Y} = 0.$$

The Faraday form is nothing but $1/n$ times the curvature of the connection induced by ∇ on $\Lambda^n TM$. As such it satisfies itself the differential Bianchi identity, and thus $dF = 0$.

This also menas that $F = 0$ iff the connection ∇ is locally equiaffine, i.e., it admits locally ∇–parallel volume forms. (The local character is essential: even-dimensional real projective space are non-orientable, but the Levi-Civita connection on S^n induces a locally equiaffine connection on $\mathbb{R}\mathrm{P}^n$.

At this point, we note the following fact (not related to our main result):

Proposition 4. *For a projective manifold M of dimension $n \geq 2$, for any given non-vanishing volume form ω, there exists a unique connection ∇ adapted to the projective structure and equiaffine w.r.t. ω, i.e., such that $\nabla\omega$.*

In fact, one can prove, more generally, that every connection on the line bundle $\Lambda^n M$ is induced by a unique connection adapted to the projective structure on M.

Proof. Let ∇ be a connection adapted to the given projective structure on M and denote by $R, W, \mathrm{Ric}, F, \rho$ the corresponding tensors as above. The covariant derivative of the volume form ω is thus given by a 1–form, thus

$$\nabla\omega = \alpha \otimes \omega.$$

We set then

$$\nabla' := \nabla' + a\tilde{\alpha},$$

where $a \in \mathbb{R}$ is a constant to be determined. Then

$$\nabla'_X \omega - \nabla_X \omega = -(n+1)a\alpha(X)\omega.$$

By setting $a := -1/(n+1)$ we obtain that $\nabla'\omega = 0$. The relation above also implies that the connection ∇' with the property that $\nabla'\omega = 0$ is unique. $\square$

Returning to the Cotton tensor, note that it vanishes iff the covariant derivative of ρ (or of Ric) is completely symmetric. For $n > 2$, it turns out that C is determined by the Weyl tensor:

Proposition 5. *If we denote by $\delta W(X, Y; Z) := \mathrm{tr}(V \mapsto \nabla_V W_{X,Y} Z$, we have, for $n > 2$:*

$$\delta W(X, Y; Z) = (n-2)C(X, Y; Z).$$

Proof. We take the trace of the second Bianchi identity:

$$\begin{aligned}
0 &= \delta R(X, Y; Z) + \mathrm{tr}(V \mapsto \nabla_X R_{Y,V} Z + \nabla_Y R_{V,X} Z) \\
&= \delta R(X, Y; Z) - \nabla_X \mathrm{Ric}(Y, Z) + \nabla_Y \mathrm{Ric}(X, Z) \\
&= \delta W(X, Y; Z) + \mathrm{tr}\left(V \mapsto (\nabla_V \rho \wedge \mathrm{Id})_{X,Y} Z + \frac{n}{n+1} \nabla_V F(X, Y) Z\right) - \\
\text{(use (4))} \quad & \quad - (n-1)C(X, Y; Z) + \frac{n}{n+1}\left(\nabla_X F(Y, Z) - \nabla_Y F(X, Z)\right) \\
&= \delta W(X, Y; Z) - (n-2)C(X, Y; Z).
\end{aligned} \tag{8}$$

Here we have used that

$$\mathrm{tr}(V \mapsto A(X, Y, V)Z) = A(X, Y, Z)$$

and that F is closed. □

1.4. **Projective flatness.** The main result of this lecture is:

Theorem 1. *Let M be a projective manifold of dimension $n \geq 2$. Then M is projectively flat iff $W = 0$ and, if $n = 2$, $C = 0$.*

Proof. First, if M is projectively flat, then W and hence C have to vanish for $n \geq 3$. For $n = 2$ (actually for all $n \geq 2$), we have the following

Lemma 1. *If $\nabla' = \nabla + \tilde{\eta}$, then $C' - C = \eta(W)$.*

Proof. We compute

$$\begin{aligned}
\nabla'_X \rho'(Y, Z) &= \nabla_X \rho'(Y, Z) - \rho'(\tilde{\eta}_X Y, Z) - \rho'(Y, \tilde{\eta}_X Z) = \\
&= \nabla_X \rho(Y, Z) - \nabla_X \nabla_Y \eta(Z) + \nabla_X \eta(Y)\eta(Z) + \eta(Y)\nabla_X \eta(Z) - \\
&\quad - \rho'(\tilde{\eta}_X Y, Z) - \rho(Y, \tilde{\eta}_X Z) + \nabla_Y \eta(\tilde{\eta}_X Z) - \eta(Y)\eta(\tilde{\eta}_X Z).
\end{aligned} \tag{9}$$

If we skew-symmetrize the above relation in X and Y (using that $\tilde{\eta}_X Y = \tilde{\eta}_Y X$), we obtain

$$\begin{aligned}
C'(X, Y; Z) - C(X, Y; Z) &= -(R_{X,Y}\eta)(Z) - \eta(X)\rho(Y, Z) + \eta(Y)\rho(X, Z) + \\
&\quad + (\rho(X, Y) - \rho(Y, X))\eta(Z) \\
&= \eta(R_{X,Y} Z) - \eta\left((\rho \wedge \mathrm{Id})_{X,Y} Z\right) - \frac{n}{n+1} F(X, Y)\eta(Z) \\
&= \eta(W_{X,Y} Z).
\end{aligned} \tag{10}$$

□

Therefore, if ∇ is projectively flat, then W (if $n \geq 3$), resp. C (if $n = 2$) must coincide with the ones defined by a flat connection.

Conversely, if W and C vanish (from Proposition 5, $W = 0$ implies, for $n > 2$, that $C = 0$, for any adapted connection ∇), we want to construct a flat connection projectively equivalent to ∇. From $W = 0$ we have, for an adapted connection ∇,

$$R = \rho \wedge \mathrm{Id} + F \otimes \mathrm{Id}.$$

From the Proposition 3, we need to find locally a 1–form η such that $\rho = \nabla\eta - \eta\otimes\eta$. In order to solve (locally) this nonlinear PDE, we use the Frobenius theorem:

Theorem 2. *If $\mathcal{D} \subset TN$ is an* involutive distribution *(subbundle in TN) on a manifold N, then there exists a foliation $\mathcal{F}$ (i.e., a family of submanifolds) tangent to this distribution (i.e. $T\mathcal{F} = \mathcal{D}$).*

A distribution $\mathcal{D}$ is said to be *involutive* iff the Lie bracket of two sections of $\mathcal{D}$ (seen as vector fields on N) is a vector field which is also a section of $\mathcal{D}$.

We will construct a distribution $\mathcal{D}$ on $N := T^*M$ as follows:

The tangent space of $N = T^*M$ decomposes, for a given connection ∇, in the vertical and horizontal components

$$T_\alpha N = T^V_\alpha N \oplus T^H_\alpha N,$$

where $T^V \simeq T^*M$ and $T^H N \simeq TM$. More precisely, for a vector field X on M, we denote by $\bar{X} \in T^H N$ the *horizontal lift* of X to T^*M. For an arbitrary section α of T^*M, the vector $\alpha_*(X)$ (which is the the image of the vector X by the tangent map of α) decomposes in $\nabla_X A$ (the vertical part) and $\bar{X}$ (the horizontal part). These parts will be identified, as shown above, with a 1–form, resp. with a vector on M.

We define

$$\mathcal{D}_\alpha := \{(A^V, A^H) \in T^V N \oplus T^H N \mid A^V = \rho(A^H, \cdot) + \alpha(A^H)\alpha\}. \tag{11}$$

It is easy to see that $\mathcal{D}$ is an n–dimensional subbundle of TN since $\mathcal{D}$ can be identified with the graph of a linear map from TM to T^*M. To check that $\mathcal{D}$ is involutive, we let A, B be sections of $\mathcal{D}$ such that $A^H = \bar{X}$ and $B^H = \bar{Y}$, where X, Y are vector fields on M which are ∇–parallel at a point $p \in M$. We compute

$$[A, B] = [\bar{X}, \bar{Y}] + [A^V, \bar{Y}] + [\bar{X}, B^V] + [A^V, B^V]$$

at the point $\alpha \in T^*_p M$. First, because $[X, Y]$ vanishes at p, the horizontal part of $[\bar{X}, \bar{Y}]$ vanishes at α. The vertical part is given by the curvature of the connection ∇:

$$[\bar{X}, \bar{Y}] = -R_{X,Y}\alpha = \alpha(R_{X,Y}\cdot) = \alpha\left((\rho \wedge \mathrm{Id})_{X,Y}\cdot\right) + \frac{n}{n+1}F(X, Y)\alpha. \tag{12}$$

The bracket $[A^V, \bar{Y}]$ is vertical and coincides with $-\nabla_Y A^V$, thus, from (11) we have

$$[A^V, \bar{Y}] = -\nabla_Y \rho(X, \cdot). \tag{13}$$

$$[\bar{X}, B^V] = \nabla_X \rho(Y, \cdot). \tag{14}$$

And the bracket $[A^V, B^V]$ is a bracket of vector field in the fixed vector space $T^*_p M$:

$$\begin{aligned}
[A^V, B^V] = {} & A^V(Y)\alpha + \alpha(Y)A^V - B^V(X)\alpha - \alpha(X)B^V = \\
& = \rho(X, Y)\alpha + \alpha(X)\alpha(Y)\alpha + \alpha(Y)\rho(X, \cdot) - \alpha(Y)\alpha(X)\alpha \\
& \quad - \rho(Y, X)\alpha - \alpha(Y)\alpha(X)\alpha - \alpha(X)\rho(Y, \cdot) + \alpha(X)\alpha(Y)\alpha \\
& = -\frac{n}{n+1}F(X, Y)\alpha - \alpha\left((\rho \wedge \mathrm{Id})_{X,Y}\cdot\right).
\end{aligned} \tag{15}$$

Therefore

$$[A, B] = \alpha\left(C(X, Y; \cdot)\right)$$

which is zero by the hypothesis $C = 0$. The distribution $\mathcal{D}$ is thus involutive, thus there exists a submanifold containing α and tangent to $\mathcal{D}$. Such a submanifold corresponds to a section (also called α) of T^*M, such that the vertical part of $\alpha_*(X)$ (which is $\nabla_X\alpha$) satisfies

$$\nabla_X \alpha = \rho(X, \cdot) + \alpha(X)\alpha,$$

which means that $\rho = \nabla\alpha - \alpha \otimes \alpha$ and thus the connection $\nabla' = \nabla + \tilde{\alpha}$ has zero Ricci (and Faraday) tensor, hence its curvature vanishes. $\square$

In conclusion, the Weyl tensor and, in dimension 2, the Cotton tensor behave like the curvature of a Riemannian manifold: their vanishing is equivalent to the existence of a local isomorphism with a model space (in the projective case, it is $\mathbb{R}\mathrm{P}^n$).

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