A semi-implicit SPH scheme for the shallow water equations

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***Abstract*—This work focuses on the development of a new semi-implicit SPH scheme for the shallow water equations, following the semi-implicit finite volume and finite difference approach of Casulli [1]. In standard explicit numerical methods, there is often a severe limitation on the time step due to the stability restriction imposed by the CFL condition. This paper proposes, a new semi-implicit SPH scheme, which leads to an unconditionally stable method. To this end, the discrete momentum equation is substituted into the discrete continuity equation to obtain a symmetric positive definite linear system for the free surface elevation. The resulting system can easily be solved by a matrix-free conjugate gradient method. Once the new free surface location is known, the velocity at the new time level can directly be computed and the particle positions can subsequently be updated. A simple and yet non-trivial 1D test problem for the 1D shallow water equation is presented.**

I. Introduction

This paper proposes a novel semi-implicit SPH scheme applied to the shallow water equations. We consider one-dimensional inviscid hydrostatic free surface flows. These flows are governed by the *shallow water equations* which we can derive from the three dimensional Navier-Stokes equations with the assumption of a hydrostatic pressure distribution (see [3], [11]).

A considerable amount of work has been done for both structured and unstructured meshes using finite difference, finite volume and finite element schemes ([3], [11], [17], [18], [19]). A major problem of explicit schemes in numerical methods is their severe time step restriction, where the Courant-Friedrichs-Lewy (CFL) condition imposes the time step size in terms of the wave propagation speed and the mesh size. Hence, the major advantage of a semi-implicit approach is that stable schemes are obtained which allow large time step sizes at a reasonable computational cost. In a staggered mesh approach for finite differences and volumes, discrete variables are often defined at different (staggered) locations. The pressure term, which is the free surface elevation is defined in the cell center while the velocity components are defined at the cell interfaces. In the momentum equation, pressure terms are due to the gradients in the free surface elevations and the velocity in the mass equation (i.e., free surface equation) are both discretized implicitly whereas the nonlinear convective terms are discretized explicitly. The semi-Lagrangian method is one of the techniques to discretize these terms explicitly (see [12], [13], [14]).

In this paper a new semi-implicit *Smoothed Particle Hydrodynamics* (SPH) scheme for the numerical solution of the shallow water equations is proposed and derived. The flow variables in this present study are the particle free surface elevation, particle total water depth and the particle velocity. The discrete momentum equations are substituted into the discretized mass conservation equation to give a discrete equation for the free surface leading to a system in only one single scalar quantity, the free surface elevation location. The system is solved for each time step as a linear algebraic system. The components of the momentum equation at the new time level can be directly computed from the new free surface. This can be conveniently solved by a matrix-free version of the conjugate gradient (CG) algorithm [4]. Consequently, the particle velocities at the new time level are computed and the particle positions are updated. In this semi-implicit SPH method, the stability is independent of the wave celerity. Hence, a relatively large time steps can be permitted to enhance the numerical efficiency [3].

The remainder of this paper is structured as follows: In section II, the numerical models for the one-dimensional shallow water equations and models used for the particle approximations are presented. In section III, the key ideas of the proposed semi-implicit SPH scheme are presented and derived. One dimensional numerical results to validate the scheme are presented in section IV. Section V presents the concluding remarks and an outlook to future research.

II. Numerical Model

This section details the computational models and their accompanying particle approximations. Vectors are defined by reference to Cartesian coordinates. The latin subscript is used to identify particle locations, where subscripts i denotes the focal particle whereas the subscript j denotes the neighbor of particle i. Einstein's summation will be employed for repeated superscripts.

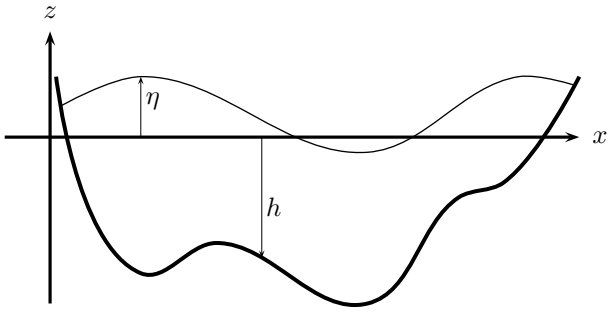

Fig. 1. Flow Domain

A. The Kernel Function

We shall use a regular function W which is a positive non-increasing, axially symmetric shaped function with compact support of the generic form

$$W(r,h) = \frac{1}{h^d} W\left(\frac{\|r\|}{h}\right). \tag{1}$$

In the specific, the classical B-spline kernel function of degree 3 is used in this study given as

$$W(r,h) = W_{ij} = K \times \begin{cases} 1 - \frac{3}{2}\left(\frac{r}{h}\right)^2 + \frac{3}{4}\left(\frac{r}{h}\right)^3 & 0 \le \frac{r}{h} \le 1, \\ \frac{1}{4}\left(2 - \frac{r}{h}\right)^3 & 1 \le \frac{r}{h} \le 2, \\ 0 & \frac{r}{h} > 2, \end{cases} \tag{2}$$

where the normalisation coefficient K takes the value $\frac{2}{3}$, $\frac{10}{7\pi}$, $\frac{1}{\pi}$ according to the dimension of the space for $(d = 1, 2, \text{or } 3)$, respectively. We note that in the function $W \in W^{3,\infty}(\mathbb{R}^d)$, h is the so called smoothing length which is related to the particle spacing Δ_P by the relation $h = 2\Delta_P$ for constant h. The smoothing length h can vary locally according to the relations:

$$h_{ij} = \frac{1}{2}[h_i + h_j] \quad \text{where} \quad h_i = \sigma \sqrt[d]{\frac{m_j}{\rho_j}}. \tag{3}$$

In this study, the smoothing length relation in (3) is used, σ is taken to be 2 which ensures approximately a constant number of neighbors in the compact support of each kernel. A popular and efficient approach based on the Shepard interpolation technique [2]

$$W'_{ij} = \frac{W_{ij}}{\sum_{j=1}^{N} \frac{m_j}{\rho_j} W_{ij}}$$

is used for the kernel function normalisation, especially useful for particles close to free surfaces, this technique remedies problems such as numerical instabilities, partition of unity which affect the convergence of this method.

The gradient of the kernel function is corrected using the formulation proposed by Belytschko et al. [15]. Hereafter by notation, the kernel function W'_{ij} and its gradient $\nabla W'_{ij}$ will be taken as W_{ij} and ∇W_{ij}, respectively.

B. Governing Equations

The governing equations considered in this paper can be written as nonlinear hyperbolic conservation law of the form

$$L_b(\Phi) + \nabla \cdot (F(\Phi, x, t)) = 0, \qquad t \in \mathbb{R}^+, \Phi \in \mathbb{R}, \tag{4}$$

together with the initial condition

$$\Phi(x, 0) = \Phi_0(x), \qquad x \in \Omega \subset \mathbb{R}^d, \Phi_0 \in \mathbb{R}, \tag{5}$$

where L_b is the transport operator given by

$$L_b(\Phi) = \frac{\partial \Phi}{\partial t} + \nabla \cdot ((b\Phi))$$

and

$$x = (x^1, ..., x^d), F = (F^1, ..., F^d), b = (b^1, ..., b^d),$$

where b is a regular vector field in $\mathbb{R}^d$, F is a flux vector in $\mathbb{R}^d$ and x is the position.

Fig. 1 shows the flow domain in the present study. In this configuration, the vertical variation is much smaller when compared to the horizontal variation, typical of rivers flowing over long kilometers. We consider the frictionless, inviscid shallow water equations in Lagrangian derivatives given as

$$\frac{D\eta}{Dt} + \nabla \cdot (Hv) = 0, \tag{6}$$

$$\frac{Dv}{Dt} + g\nabla\eta = 0, \tag{7}$$

$$\frac{Dr}{Dt} = v, \tag{8}$$

where $\eta = \eta(x,t)$ denotes the free surface location, and $H = H(x,t)$ denotes the total water depth which is given as

$$H(x,t) = h(x) + \eta(x,t), \tag{9}$$

where $h(x)$ denotes the bottom bathymetry, $v = v(x,t)$ denotes the particle velocity, $r = r(x,t)$ denotes the particle position, and g denotes the constant of gravity acceleration.

C. Hydrostatic Approximation

In geophysical flows the vertical acceleration is often small when compared to the gravitational acceleration and to the pressure gradient in the vertical direction as in the case of our flow domain in Fig 1. For instance, if we consider tidal flows in the ocean the velocity in the horizontal direction is of the order of $1m/s$, while the velocity in the vertical direction is much smaller of the order of one meter per tidal cycle i.e., $10^{-5}m/s$ [16]. To this end, if the advective and viscous terms are neglected in the vertical momemtum equation of the Navier-Stokes equation, we have the equation for pressure which reads

$$\frac{dp}{dz} = -g. \tag{10}$$

The pressure represents a normalised pressure, that is we mean the pressure is divided by constant density. The solution that satisfies (10) is given by the hydrostatic pressure

$$p(x, y, z, t) = p_0(x, y, t) + g[\eta(x, y, t) - z],$$

where $p_0(x, y, t)$ marks the atmospheric pressure at the free surface which without loss of generality is taken as a constant.

III. Numerical Method

There are several numerical methods that can be employed to solve equations (6) - (7). These methods can be finite differences or finite elements, explicit or implicit, conservative or non-conservative or meshless method. In this section, following the semi-implicit finite volume and finite difference approach of Casulli [1], we will delve into the derivation of the semi-implicit SPH scheme applied to the one dimensional shallow water equations.

In standard explicit numerical methods, there is the severe limitation due to the stability restriction imposed by the CFL condition. The restriction requires a much smaller time step size than permitted by accuracy considerations. Fully implicit discretization often leads to unconditionally stable methods that leads to the solution of simultaneous solution of large number of coupled nonlinear equations. For accuracy, the time step cannot be chosen arbitrary large. To this effect, a stable, efficient, robust and simple semi-implicit SPH numerical method is derived in this section.

A. Classical SPH formulation

The standard SPH formulation discretizes the computational domain $\Omega(t)$ by a finite set of N particles, with positions r_i. According to Gingold and Monaghan [8], the SPH discretization of the shallow water equations (6) - (7) reads:

$$\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}+\sum_{j=1}^{N}\frac{m_j}{\rho_j}H_{ij}v_j\nabla W_{ij}=0, \tag{11}$$

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}+g\sum_{j=1}^{N}\frac{m_j}{\rho_j}\eta_j\nabla W_{ij}=0, \tag{12}$$

$$\frac{Dr_i}{Dt}=v_i, \tag{13}$$

and the particles are moved by (13), where Δt is the time step, m_j denotes the particle mass, ρ_j denotes the particle density, and ∇W_{ij} is the gradient of the interpolation kernel W_{ij} with respect to x_i. In this Gingold and Monaghan [8] scheme v_x, η_x are explicitly computed.

The gradient formulation used in (11) - (12) follows by substituting the flow variable with corresponding derivatives, using integration by parts, the divergence theorem and some trivial transformations.

B. SPH formulation of Vila and Ben Moussa

Towards the derivation of our semi-implicit SPH scheme, the SPH formalism of Vila and ben Moussa ([5], [7]) is used. The basic idea in Vila and Ben Moussa in the scheme comprises of replacing a centered approximation

$$(F(v_i,x_i,t)+F(v_j,x_j,t))\cdot n_{ij}$$

of (4) by a numerical flux of finite difference scheme in conservation form $2G(n_{ij},v_i,v_j)$ which should satisfy

$$G(n(x),v,v)=F(v,x,t)\cdot n(x)$$

$$G(n,v,u)=-G(-n,u,v)$$

With this formalism, the SPH discretization reads

$$\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}+\sum_{j=1}^{N}\frac{m_j}{\rho_j}2H_{ij}v_{ij}\nabla W_{ij}=0, \tag{14}$$

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}+g\sum_{j=1}^{N}\frac{m_j}{\rho_j}2\eta_{ij}\nabla W_{ij}=0, \tag{15}$$

$$\frac{Dr_i}{Dt}=v_i, \tag{16}$$

In this formalism, in Fig. 2, for a pair of particle i and j, we define the free surface elevation η_i, η_j and velocity v_i, v_j at each particle i and j respectively. In our approach, we artificially define a staggered like velocity v_{ij} between two interacting particles i and j as

$$v_{ij}=\frac{1}{2}(v_i+v_j) \tag{17}$$

in the normal direction n_{ij} at the midpoint of the two interacting particles, where n_{ij} is a vector given as

$$n_{ij}=\frac{x_j-x_i}{\|x_j-x_i\|}$$

where we write $\delta_{ij}=\|x_j-x_i\|$ which denotes the distance between pair of particles i and j. Since, we know the value of the velocities at the midpoint of the particles, we use kernel summation to update the velocity at the next location.

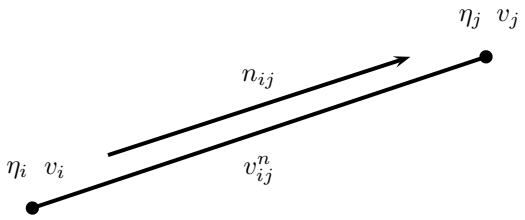

Fig. 2. Staggered velocity defined at the midpoint of two pair of interacting particles i and j

C. Semi-implicit SPH Scheme

To start with, the derivation of the semi-implicit SPH scheme let us consider some characteristic analysis of the governing equations (6) - (7). Writing equations (6) - (7) in a non conservative quasi-linear form by expanding derivatives in the continuity equation and momentum equations (assuming smooth solutions) we obtain

$$v_t+vv_x+g\eta_x=0, \tag{18}$$

$$\eta_t+v\eta_x+Hv_x=0, \tag{19}$$

Writing (18) - (19) in matrix form we obtain

$$\mathbf{Q}_t+\mathbf{A}\mathbf{Q}_x=0 \tag{20}$$

where

$$\mathbf{Q} = \begin{pmatrix} v \\ \eta \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} v & g \\ H & v \end{pmatrix}$$

Equation (20) is a strictly hyperbolic system with eigenvalues been real and distinct. The characteristic equation is given by

$$det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{21}$$

after solving (21), the solution yields

$$\lambda_{1,2} = v \pm \sqrt{gH}$$

When the particle velocity v is far smaller than the particle celerity $\sqrt{gH}$ i.e $|v| \ll \sqrt{gH}$, the particle flow is said to be strictly subcritical and thus the characteristic speeds λ_1 and λ_2 have opposite directions. The maximum wave speed is given as

$$\lambda_{\max} = \max(\sqrt{gH_i}, \sqrt{gH_j}).$$

In this case, $\sqrt{gH}$ represents the dominant term which originates from the off diagonal terms g and H in the matrix A.

Tracking back where the terms $\sqrt{gH}$ originates from in the governing equations. These are the coefficients of the derivative of the free surface elevation η_x in the momentum equation (18), and the coefficient of the derivative of the velocity v_x in the volume conservation (19). Since, we do not want the stability of this method to be dependent on the celerity $\sqrt{gH}$, we discretize the derivatives η_x and v_x implicitly.

Following the characteristic analysis presented above, we want to derive the semi-implicit SPH scheme for the one dimensional shallow water equation. The derivative of the free surface elevation η_x in the momentum equation and the derivative of the velocity in the continuity equation are discretized implicitly. The remaining terms such as the nonlinear advective terms in the momentum equation are discretized explicitly so that the system to be solved eventually will be linear.

Let us consider the continuity equation in the original conservative form given as

$$\eta_t + (Hv)_x = 0 \tag{22}$$

v will be discretized implicitly, H the total water depth is discretized explicitly, for the sake of notation by implicitly and explicitly we mean $n+1$ and n in the superscript respectively:

$$v_t^n + g\eta_x^{n+1} = 0 \tag{23}$$

$$\eta_t^n + (H^n v^{n+1})_x = 0 \tag{24}$$

The general semi-implicit SPH discretization of (23) - (24) assumes the form

$$v_{ij}^{n+1} = Fv_{ij}^n - g\frac{\Delta t}{\delta_{ij}}(\eta_j^{n+1} - \eta_i^{n+1}) \tag{25}$$

$$\eta_i^{n+1} = \eta_i^n - \Delta t \sum_{j=1}^{N} \frac{m_j}{\rho_j}(2H_{ij}^n v_{ij}^{n+1})\nabla W_{ij} \tag{26}$$

where

$$H_{ij}^n = \max(0, h_{ij}^n + \eta_i^n, h_{ij}^n + \eta_j^n) \tag{27}$$

In this formulation, the explicit, nonlinear finite difference operator Fv_{ij}^n in (25) takes the form

$$Fv_{ij}^n = \frac{1}{2}(v_i + v_j) \tag{28}$$

where v_i and v_j denotes the velocity of particle i and j at time t^n. The new velocity is computed through simple kernel summation:

$$v_i^{n+1} = v_i^n + \sum_{j=1}^{N} 2\frac{m_j}{\rho_j}(v_{ij}^{n+1} - v_i^n)W_{ij} \tag{29}$$

We should note that in (25) we have not used the gradient of the kernel function for the discretization of η_x rather we used a finite difference discretization for the pressure gradient this is because this is more accurate, in (25) F corresponds to an explicit spatial discretization of the advective terms. Since SPH is a Lagrangian scheme, the nonlinear convective term is discretized automatically, using the Lagrangian (material) derivative contained in the particle motion in Eqn. (13). Relation (28) is used to interpolate the particle velocities from the particle location to the staggered velocity location.

D. The Free Surface Equation

From the approach of Vila and Ben Moussa ([5], [7]). Let the particle volume ω_i in (26) be given as $\omega_i = \frac{m_i}{\rho_i}$. Irrespective of the form imposed on F, equations (25) - (26) constitute a linear system of equations with unknowns v_i^{n+1} and η_i^{n+1} over the entire particle configuration. We solve this system at each time step for the particle variables from the prescribed initial and boundary conditions. The cardinal feature of this present numerical method from the computational point of view is that the discrete momentum equation is substituted in the discrete continuity equation. The model is reduced into a smaller model in η_i^{n+1} as the only unknowns.

Multiplying (26) by ω_i and inserting (25) into (26) we obtain

$$\omega_i\eta_i^{n+1} - g\frac{\Delta t^2}{\delta_{ij}}\sum_{i=1}^{N}\sum_{j=1}^{N} 2\omega_i\omega_j \left[H_{ij}^n(\eta_j^{n+1} - \eta_i^{n+1})\nabla W_{ij}\right] = b_i^n \tag{30}$$

where the right hand side b_i^n represents the known values at time level t^n given as

$$b_i^n = \omega_i\eta_i^n - \frac{\Delta t}{\delta_{ij}}\sum_{i=1}^{N}\sum_{j=1}^{N} 2\omega_i\omega_j H_{ij}^n F v_{ij}^n \nabla W_{ij} \tag{31}$$

Since H_{ij}^n, ω_i, ω_j are non-negative numbers, equations (30) - (31) constitute a linear system of N equations for η_i^{n+1} unknowns.

The resulting system is symmetric and positive definite (SPD). Because of the SPD property, this system admits a unique solution which can be efficiently obtained by an iterative method. We obtain the new free surface location by (30), equation (25) gives readily and uniquely the new particle velocity v_i^{n+1}.

IV. Numerical Examples

In this section, the semi-implicit SPH scheme that has been derived in Section III will be validated on the one dimensional shallow water equation test problems. In this section, two numerical examples will be validated namely: smooth solution and discontinuous solution. In the subsequent test problems, the acceleration due to gravity constant g is set to $g = 9.81$. In the numerical examples presented, we wish to mention that the particles are not moved.

A. Smooth Surface Wave Propagation

In this example, we consider a smooth free surface wave propagation. We consider the following initial value problem with in the domain $\Omega = [-1, 1]$ with the data

$$\eta(x, 0) = 1 + \frac{1}{2} e^{-\frac{1}{2}(x^2/\sigma^2)},$$
$$v(x, 0) = h(x, 0) = 0,$$

with flat bottom, where $\sigma = 0.1$. The computational domain Ω is discretized with 200 particles. The final time $t = 0.15$ is used and the time step is chosen to be $\Delta t = 0.01$. The numerical solution is given in Fig. 3. The upper profile in Fig. 3 depicts the free surface elevation with a flat bottom bathymetry and the lower profile depicts the particle velocity. We compare our solution with a reference solution obtained by solving the one-dimensional shallow water equation with the finite difference mesh based approach of Casulli on a fine mesh of $10,000$ points. The comparison between our numerical results obtained with semi-implicit SPH scheme and the reference solution is shown. A good agreement between the two solutions is observed in the figure. We attribute the difference in the plot to the low order accurate time integration scheme used.

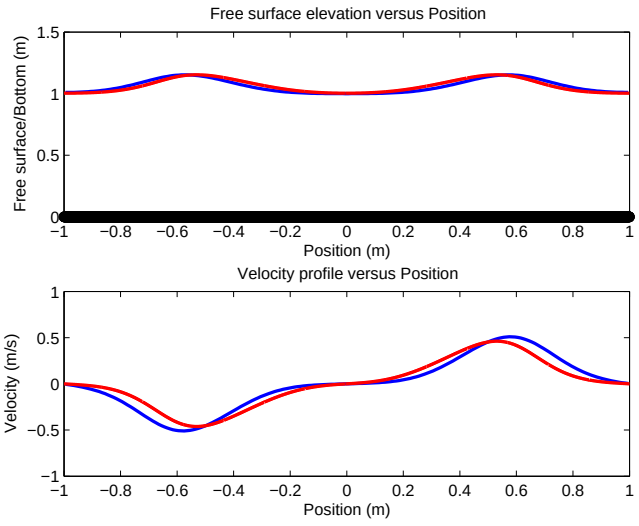

Fig. 3. Semi-implicit SPH scheme solution with 200 particles (solid line - blue) versus reference solution (solid line - red) - staggered finite difference approach with a mesh of 10,000 points.

B. Discontinuous Solution

In this example, we consider the following Riemann problem. Riemann problems are very important cases in initial value problem for PDE systems. The initial data is prescribed by two piecewise constant states often separated by a discontinuity:

$$\mathbf{q}(x, 0) = \begin{cases} \mathbf{q}_l & x < 0, \\ \mathbf{q}_r & x > 0 \end{cases}$$

where $\mathbf{q} = (v(x, 0), \eta(x, 0), h(x))$. The computational domain $\Omega = [x_l, x_r]$ given as $\Omega = [-1, 1]$ is discretized with the semi-implicit SPH scheme using 200 particles. In this example with flat bottom, the exact solution is given by the exact Riemann solver for the shallow water equations [10]. The left state is given as $\mathbf{q}_l = (-1, 1, 0)$ and the right state is given as $\mathbf{q}_r = (1, 1, 0)$. In this present simulation, we used the final time $t = 0.15$, $\Delta t = 0.01$. The rarefaction solution of the one dimensional shallow water equation is presented in Fig. 4, the solution consists of a left moving rarefaction fan and a right moving rarefaction fan solution both moving away from the discontinuity. We compare our semi-implicit SPH solution with the reference solution of the exact riemann solver for the one dimensional shallow water equation. A very good agreement is observed in Fig. 4. The upper profile in Fig. 4 depicts the free surface elevation with a flat bottom bathymetry and the lower profile depicts a rarefaction particle velocity, respectively.

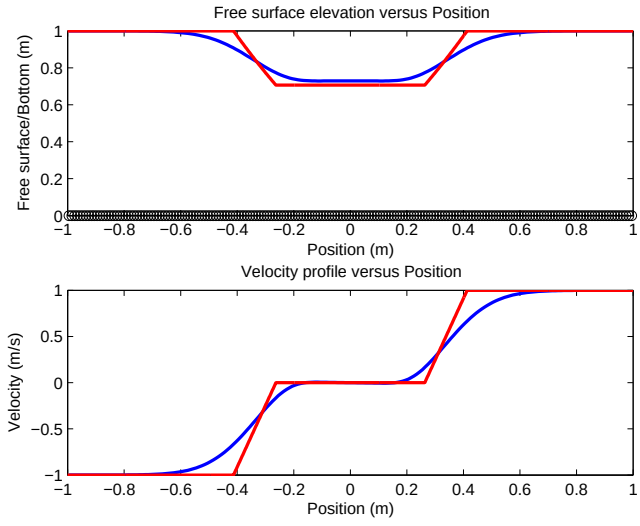

Fig. 4. Semi-implicit SPH scheme rarefaction solution (solid line - blue) versus exact solution (solid line - red). 200 particles is used in the numerical solution.

V. Conclusion

The paper presents a new SPH formulation based on a novel semi-implicit SPH discretization. The semi-implicit algorithm applied to the shallow water equations has been derived and discussed. The momentum equation is discretized by a finite

difference approximation for the gradient of the free surface and SPH appoximation for the mass conservation equation.

Because we substituted the discrete momemtum equations into the discrete mass conservation equations, our scheme reduces to a linear sparse system for the free surface elevation. We therefore have one linear and scalar value for the free surface to be solved, we conviniently solve this with the matrix-free version of the conjugate gradient (CG) algorithm.

This method possesses some key features such as: the method is mass conservative, the time step is not restricted by the stability condition that is dictated by the surface wave speed thus relatively large timesteps are permitted.

Future research will be related to the extension of this scheme to 2D and 3D numerical examples, extension to nonhydrostatic free surface flows.

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