Multiple zeta values and regularised multiple Eisenstein series

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Content of this talk

- Multiple zeta values
- q-analogues of multiple zeta values and (bi-)brackets.
- Multiple Eisenstein series
- Shuffle regularised multiple Eisenstein series
- Stuffle regularised multiple Eisenstein series

Multiple zeta values

Definition

For natural numbers $s_1 \geq 2$, $s_2, \ldots, s_l \geq 1$, the multiple zeta value (MZV) of weight $k = s_1 + \cdots + s_l$ and length l is defined by

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}} \,.$$

By $\mathcal{MZ}_k$ we denote the space spanned by all MZV of weight k and by $\mathcal{MZ}$ the space spanned by all MZV.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle product)
- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV, for example

$$\zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2) \cdot \zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \,.$$

Multiple zeta values - Quasi-shuffle algebras

We now want to describe the algebraic setup for the stuffle/shuffle product introduced by Hoffman.

- Let A (the alphabet) be a countable set of letters
- $\mathbb{Q}A$ the $\mathbb{Q}$-vector space generated by these letters
- $\mathbb{Q}\langle A\rangle$ the noncommutative polynomial algebra over $\mathbb{Q}$
- and let $\diamond$ be an associative product on $\mathbb{Q}A$

Definiton

For letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$ we define on $\mathbb{Q}\langle A\rangle$ recursively a product by $1 \odot w = w \odot 1 = w$ and

$$aw \odot bv := a(w \odot bv) + b(aw \odot v) + (a \diamond b)(w \odot v)\,.$$

By a result of Hoffman $(\mathbb{Q}\langle A\rangle, \odot)$ is a commutative $\mathbb{Q}$-algebra which is called a **quasi-shuffle algebra**.

Multiple zeta values - Quasi-shuffle algebras

Some notations:

- We consider the alphabet $A_{xy} := \{x, y\}$ and set $\mathfrak{H} = \mathbb{Q}\langle A_{xy}\rangle$.
- The set of words ending in y will be denoted by $\mathfrak{H}^1 = \mathfrak{H}y + 1 \cdot \mathbb{Q}$.
- $\mathfrak{H}^1$ is generated by the elements $z_j = x^{j-1}y$.
- i.e. $\mathfrak{H}^1 = \mathbb{Q}\langle A_z\rangle$ with the alphabet $A_z := \{z_1, z_2, \dots\}$.
- By $\mathfrak{H}^0 = x\mathfrak{H}y + 1 \cdot \mathbb{Q}$ we denote the set of words starting with an x and ending with an y.
- Words $z_{s_1} \dots z_{s_l} \in \mathfrak{H}^1$ correspond to index sets $(s_1, \dots, s_l) \in \mathbb{N}^l$.
- Words $z_{s_1} \dots z_{s_l} \in \mathfrak{H}^0$ correspond to index sets $(s_1, \dots, s_l) \in \mathbb{N}^l$, where $s_1 \geq 2$, i.e. where the MZV $\zeta(s_1, \dots, s_l)$ is defined .

Stuffle product

The quasi-shuffle product on $\mathfrak{H}^1 = \mathbb{Q}\langle A_z \rangle$ with $z_i \diamond z_j = z_{i+j}$, is called **stuffle product** and we denote it by $*$. We have for $z_i, z_j \in A_z$ and $w, v \in \mathfrak{H}^1$

$$z_i w * z_j v = z_i(w * z_j v) + z_j(z_i w * v) + z_{i+j}(w * v)\,,$$

which gives a $\mathbb{Q}$-algebra $(\mathfrak{H}^1, *)$.

The subspace $\mathfrak{H}^0 \subset \mathfrak{H}^1$ is also closed under $*$, i.e. we have a $\mathbb{Q}$-algebra $(\mathfrak{H}^0, *)$.

Example:

$$z_2 * z_3 = z_2 z_3 + z_3 z_2 + z_5\,.$$

Shuffle product

The quasi-shuffle product on $\mathfrak{H} = \mathbb{Q}\langle A_{xy}\rangle$ with $\diamond \equiv 0$, is called **shuffle product** and we denote it by $\sqcup\!\sqcup$. We have for $a, b \in A_{xy}$ and $w, v \in \mathfrak{H}$

$$aw * bv = a(w * bv) + b(aw * v)\,,$$

which gives a $\mathbb{Q}$-algebra $(\mathfrak{H}, *)$.

Both $\mathfrak{H}^0 \subset \mathfrak{H}$ and $\mathfrak{H}^1 \subset \mathfrak{H}$ are also closed under $\sqcup\!\sqcup$, i.e. we obtain $\mathbb{Q}$-algebras $(\mathfrak{H}^0, \sqcup\!\sqcup)$ and $(\mathfrak{H}^1, \sqcup\!\sqcup)$.

Example:

$$\begin{aligned} z_2 \sqcup\!\sqcup z_3 &= xy \sqcup\!\sqcup xxy \\ &= xyxxy + xxyxy + xxyxy + xxyxy + \dots \\ &= xyxxy + 3xxyxy + 6xxxyy \\ &= z_2 z_3 + 3 z_3 z_2 + 6 z_4 z_1\,. \end{aligned}$$

Multiple zeta values - Quasi-shuffle algebras

By the definition of MZV as a ordered sum and by the iterated integral expression one obtains algebra homomorphisms $\zeta : (\mathfrak{H}^0, *) \to \mathcal{MZ}$ and $\zeta : (\mathfrak{H}^0, ш) \to \mathcal{MZ}$ by sending $w = z_{s_1} \dots z_{s_l}$ to $\zeta(w) = \zeta(s_1, \dots, s_l)$.

These can be extended to $\mathfrak{H}^1$:

Proposition (Ihara, Kaneko, Zagier)

There exist algebra homomorphism

$$\zeta^* : (\mathfrak{H}^1, *) \longrightarrow \mathcal{MZ}\,,$$
$$\zeta^{ш} : (\mathfrak{H}^1, ш) \longrightarrow \mathcal{MZ}\,,$$

which are uniquely determined by $\zeta^*(w) = \zeta^{ш}(w) = \zeta(w)$ for $w \in \mathfrak{H}^0$ and $\zeta^*(z_1) = \zeta^{ш}(z_1) = 0$.

We also write $\zeta^*(z_{s_1} \dots z_{s_l}) = \zeta^*(s_1, \dots, s_l)$ and similar for $\zeta^{ш}$.

These two maps can differ for words $w \in \mathfrak{H}^1 \backslash \mathfrak{H}^0$, for example $\zeta^*(1,1) = -\frac{1}{2}\zeta(2)$ and $\zeta^{ш}(1,1) = 0$.

q-analogues of multiple zeta values

"Roughly speaking, in mathematics, specifically in the areas of combinatorics and special functions, a q-analogue of a theorem, identity or expression is a generalization involving a new parameter q that returns the original theorem, identity or expression in the limit as $q \to 1$. "

— Wikipedia

q-analogues of multiple zeta values

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- There are a lot of different q-analogues for multiple zeta values.
- Often these q-analogues have an analogon for the stuffle product but not for the shuffle product.
- We are interested in a specific model inspired by modular forms.
- Therefore we often view q not just as a parameter but as $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$ beeing an element in the complex upper half-plane.

bi-brackets

For $r_1, \dots, r_l \geq 0$ and $s_1, \dots, s_l \geq 1$ we define the following q-series

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} := \sum_{\substack{u_1 > \dots > u_l > 0 \\ v_1, \dots, v_l > 0}} \frac{u_1^{r_1}}{r_1!} \dots \frac{u_l^{r_l}}{r_1!} \cdot \frac{v_1^{s_1 - 1} \dots v_l^{s_l - 1}}{(s_1 - 1)! \dots (s_l - 1)!} \cdot q^{u_1 v_1 + \dots + u_l v_l} \,,$$

which we call **bi-brackets** of weight $s_1 + \dots + s_l + r_1 + \dots + r_l$ and length l.

For $r_1 = \dots = r_l = 0$ we also write

$$\begin{bmatrix} s_1, \dots, s_l \\ 0, \dots, 0 \end{bmatrix} = [s_1, \dots, s_l]$$

and call these series just **brackets**. In length $l = 1$ they are given by

$$[s_1] = \frac{1}{(s_1 - 1)!} \sum_{n > 0} \sigma_{s_1 - 1}(n) q^n \,, \quad \sigma_k(n) = \sum_{d | n} d^k \,.$$

bi-brackets

For the space spanned by all bi-brackets we write

$$\mathcal{BD} := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \,\middle|\, l \geq 0\,, s_1, \dots, s_l \geq 1, r_1, \dots, r_l \geq 0 \right\rangle_{\mathbb{Q}},$$

where we define the bi-bracket of length $l = 0$ to be 1. By $\mathcal{MD} \subset \mathcal{BD}$ we denote the subspace spanned by the brackets

$$\mathcal{MD} := \left\langle [s_1, \dots, s_l] \in \mathcal{BD} \,\middle|\, l \geq 0\,, s_1, \dots, s_l \geq 1 \right\rangle_{\mathbb{Q}}$$

and for the space of admissible brackets we write

$$\mathrm{q}\mathcal{MZ} := \left\langle [s_1, \dots, s_l] \in \mathcal{MD} \,\middle|\, l \geq 0\,, s_1 \geq 2 \right\rangle_{\mathbb{Q}}.$$

We have the following inclusions

$$\mathrm{q}\mathcal{MZ} \subset \mathcal{MD} \subset \mathcal{BD}\,.$$

(bi-)brackets as q-analogues of multiple zeta values

Define for $k \in \mathbb{N}$ the map $Z_k : \mathbb{Q}[[q]] \to \mathbb{R} \cup \{\infty\}$ by

$$Z_k(f) = \lim_{q \to 1} (1-q)^k f(q)\,.$$

Proposition

Assume that $s_1 > r_1 + 1$ and $s_j \geq r_j + 1$ for $j = 2, \dots, l$, then it holds

$$Z_{s_1 + \dots + s_l}\left(\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \right) = \frac{1}{r_1! \dots r_l!} \zeta(s_1 - r_1, \dots, s_l - r_l)\,.$$

In particular one has for all $[s_1, \dots, s_l] \in \mathrm{q}\mathcal{MZ}$

$$Z_k\left([s_1, \dots, s_l]\right) = \begin{cases} \zeta(s_1, \dots, s_l)\,, & s_1 + \dots + s_l = k, \\ 0\,, & s_1 + \dots + s_l < k\,. \end{cases}$$

bi-brackets - why?

Question

Why do we care about bi-brackets and not just the space $\mathrm{q}\mathcal{MZ}$? We get every MZV by applying the map Z_k and the space $\mathcal{BD}$ seems to be twice as much as we need it to be?!

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- The kernel of the map Z_k contains every relation of MZV in a fixed weight k and is therefore of great interest. It turns out that the bi-brackets are useful to describe this kernel. (Not content of this talk)
- The bi-brackets are necessary to describe the dimorphy structure of the brackets, i.e. an analogue for the stuffle and the shuffle product, and the linear relations between them.
- They are also useful to regularise multiple Eisenstein series.

bi-brackets - quasi-shuffle product

For the alphabet $A_z^{\text{bi}} := \{z_{s,r} \mid s,r \in \mathbb{Z}\,, s \geq 1\,, r \geq 0\}$ we define on $\mathbb{Q}A_z^{\text{bi}}$ the product

$$z_{s_1,r_1} \boxdot z_{s_2,r_2} = \binom{r_1+r_2}{r_1} \sum_{j=1}^{s_1} \lambda^j_{s_1,s_2} z_{j,r_1+r_2} + \binom{r_1+r_2}{r_1} \sum_{j=1}^{s_2} \lambda^j_{s_2,s_1} z_{j,r_1+r_2} + \binom{r_1+r_2}{r_1} z_{s_1+s_2,r_1+r_2}\,,$$

where the numbers $\lambda^j_{a,b} \in \mathbb{Q}$ for $a,b \in \mathbb{N}$ and $1 \leq j \leq a$ are given by

$$\lambda^j_{a,b} = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!}\,.$$

One can show that this product is associative.

bi-brackets - quasi-shuffle product

On $\mathbb{Q}\langle A_z^{\text{bi}}\rangle$ we now define the quasi-shuffle product $\circledast$ by

$$z_{s_1,r_1}w \circledast z_{s_2,r_2}v = z_{s_1,r_1}(w \circledast z_{s_2,r_2}v) + z_{s_2,r_2}(z_{s_1,r_1}w \circledast v) + (z_{s_1,r_1} \boxdot z_{s_2,r_2})(w \circledast v)\,.$$

and obtain a quasi-shuffle algebra $(\mathbb{Q}\langle A_z^{\text{bi}}\rangle, \circledast)$.

Proposition

The map $\left[\cdot\right] : (\mathbb{Q}\langle A_z^{\text{bi}}\rangle, \circledast) \to (\mathcal{BD}, \cdot)$ given by

$$w = z_{s_1,r_1}\dots z_{s_l,r_l} \longmapsto [w] = \begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix}$$

fulfills $[w \circledast v] = [w]\cdot[v]$ and therefore $\mathcal{BD}$ is a $\mathbb{Q}$-algebra.

Here the $\cdot$ is the multiplication of q-series in $\mathbb{Q}[[q]]$.

bi-brackets - quasi-shuffle product - example

Corollary (explicit quasi-shuffle product $\circledast$)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{\circledast}{=} \begin{bmatrix} s_1, s_2 \\ r_1, r_2 \end{bmatrix} + \begin{bmatrix} s_2, s_1 \\ r_2, r_1 \end{bmatrix} + \binom{r_1 + r_2}{r_1} \begin{bmatrix} s_1 + s_2 \\ r_1 + r_2 \end{bmatrix}$$
$$+ \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_1} \frac{(-1)^{s_2 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_1 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}$$
$$+ \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_2} \frac{(-1)^{s_1 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_2 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} \overset{\circledast}{=} \begin{bmatrix} 2, 3 \\ 0, 0 \end{bmatrix} + \begin{bmatrix} 3, 2 \\ 0, 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} - \frac{1}{12} \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \overset{\circledast}{=} \begin{bmatrix} 1, 1 \\ 1, 2 \end{bmatrix} + \begin{bmatrix} 1, 1 \\ 2, 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

bi-brackets - generating series

For the generating function of the bi-brackets we write

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} := \sum_{\substack{s_1,\dots,s_l>0 \\ r_1,\dots,r_l>0}} \begin{bmatrix} s_1, \dots, s_l \\ r_1-1, \dots, r_l-1 \end{bmatrix} X_1^{s_1-1} \dots X_l^{s_l-1} \cdot Y_1^{r_1-1} \dots Y_l^{r_l-1}$$

Lemma

The generating function of the bi-brackets can be written as

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \sum_{u_1>\dots>u_l>0} \prod_{j=1}^{l} e^{u_j Y_j} \frac{e^{X_j} q^{u_j}}{1 - e^{X_j} q^{u_j}} .$$

bi-brackets - partition relation

Theorem (partition relation)

For all $l \geq 1$ we have

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \cdots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}$$

Remark

In the language of moulds this says that the bimould of generating series of bi-brackets is invariant under the **swap** operator.

bi-brackets - partition relation

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Remark

In the language of moulds this says that the bimould of generating series of bi-brackets is invariant under the **swap** operator.

This theorem gives linear relations between bi-brackets in a fixed length, for example

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} r+1 \\ s-1 \end{bmatrix} \quad \text{for all } r, s \in \mathbb{N},$$

$$\begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} = -2\begin{bmatrix} 2,2 \\ 0,2 \end{bmatrix} + \begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} - 4\begin{bmatrix} 3,1 \\ 0,2 \end{bmatrix} + 2\begin{bmatrix} 3,1 \\ 1,1 \end{bmatrix}.$$

Idea of proof: Interpret the sum as a sum over partitions and then use the conjugation of partitions. For this we will now introduce some notation.

bi-brackets - partition relation - idea of proof

By a partititon of a natural number n with l different parts we denote a representation of n as a sum of l different numbers, which are allowed to appear with some multiplicities.

For example

$$\begin{aligned} 15 &= 4+4+3+2+1+1 \\ &= 4\cdot 2 + 3\cdot 1 + 2\cdot 1 + 1\cdot 2 \end{aligned}$$

is a partition of 15 with the 4 different parts $4,3,2,1$ and multiplicities $2,1,1,2$.

We identify a partition of n with l different parts with a tupel $\binom{u}{v}$, with $u,v\in\mathbb{N}^l$.

- The u_j are the l different summands.
- The v_j count their appearence in the sum.

The above partition is therefore given by $\binom{u}{v} = \binom{4,3,2,1}{2,1,1,2}$.

bi-brackets - partition relation - idea of proof

We denote the set of all partition of n with l different parts by $P_l(n)$, i.e. we set

$$P_l(n) := \left\{ \binom{u}{v} \in \mathbb{N}^l \times \mathbb{N}^l \mid n = u_1 v_1 + \dots + u_l v_l \,, \;\; u_1 > \dots > u_l > 0 \right\} .$$

With this the bi-brackets can be written as

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} := c \cdot \sum_{\substack{u_1 > \dots > u_l > 0 \\ v_1, \dots, v_l > 0}} u_1^{r_1} v_1^{s_1 - 1} \dots u_l^{r_l} v_l^{s_l - 1} q^{u_1 v_1 + \dots + u_l v_l}$$

$$= c \cdot \sum_{n > 0} \left(\sum_{\binom{u}{v} \in P_l(n)} u_1^{r_1} v_1^{s_1 - 1} \dots u_l^{r_l} v_l^{s_l - 1} \right) q^n \,,$$

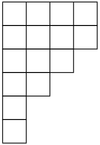
where $c = (r_1!(s_1 - 1)! \dots r_l!(s_l - 1)!)^{-1}$.

bi-brackets - partition relation - idea of proof

On the set $P_l(n)$ we have an involution ρ given by the conjugation of partitions.

To see this one represents an element in $P_l(n)$ by a Young diagram.
In $P_4(15)$ we have for example

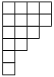
$$\begin{pmatrix}4,3,2,1\\2,1,1,2\end{pmatrix} =$$



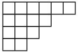
.

The conjugation ρ of this partition is given by

$$\begin{pmatrix}4,3,2,1\\2,1,1,2\end{pmatrix} =$$



$$\xrightarrow{\quad\quad\rho\quad\quad}$$



$$= \begin{pmatrix}6,4,3,2\\1,1,1,1\end{pmatrix}$$

bi-brackets - partition relation - idea of proof

We now can apply the conjugation ρ to the set $P_l(n)$ in the summation as in the following example

$$\begin{bmatrix}2,2\\0,0\end{bmatrix} = \sum_{n>0}\left(\sum_{\binom{u}{v}\in P_2(n)} v_1\cdot v_2\right)q^n = \sum_{n>0}\left(\sum_{\binom{u'}{v'}=\rho\left(\binom{u}{v}\right)\in P_2(n)} v_1'\cdot v_2'\right)q^n$$

$$= \sum_{n>0}\left(\sum_{\binom{u'}{v'}=\rho\left(\binom{u}{v}\right)\in P_2(n)} u_2\cdot(u_1-u_2)\right)q^n$$

$$= \sum_{n>0}\left(\sum_{\binom{u}{v}\in P_2(n)} u_2\cdot u_1\right)q^n - \sum_{n>0}\left(\sum_{\binom{u}{v}\in P_2(n)} u_2^2\right)q^n$$

$$= \begin{bmatrix}1,1\\1,1\end{bmatrix} - 2\begin{bmatrix}1,1\\0,2\end{bmatrix}.$$

bi-brackets - partition relation - idea of proof

In general the conjugation ρ on the partitions $P_l(n)$ is explicitly given by

$$\rho : \begin{pmatrix} u_1, \dots, u_l \\ v_1, \dots, v_l \end{pmatrix} \longmapsto \begin{pmatrix} v_1 + \dots + v_l, \dots, v_1 + v_2, v_1 \\ u_l, u_{l-1} - u_l, \dots, u_1 - u_2 \end{pmatrix}.$$

The partition relation of bi-brackets follows by applying the conjugation ρ to the $P_l(n)$ in the summation of the generating function.

Now we have seen the main idea used in the proof of the partition relation

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \dots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}.$$

This family of relations enables us to obtain a second expression for the product of two bi-brackets.

Using the quasi-shuffle product and the partition relation we obtain

$$\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix} \overset{\circledast}{=} \begin{bmatrix}1,1\\1,2\end{bmatrix}+\begin{bmatrix}1,1\\2,1\end{bmatrix}-3\begin{bmatrix}1\\3\end{bmatrix}+3\begin{bmatrix}2\\3\end{bmatrix},$$

$$\begin{bmatrix}1,1\\1,2\end{bmatrix} \overset{P}{=} \begin{bmatrix}3,2\\0,0\end{bmatrix}+3\begin{bmatrix}4,1\\0,0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}1,1\\2,1\end{bmatrix} \overset{P}{=} \begin{bmatrix}2,3\\0,0\end{bmatrix}+2\begin{bmatrix}3,2\\0,0\end{bmatrix}+3\begin{bmatrix}4,1\\0,0\end{bmatrix},$$

which yields

$$\begin{aligned}\begin{bmatrix}2\\0\end{bmatrix}\cdot\begin{bmatrix}3\\0\end{bmatrix} &\overset{P}{=} \begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}\\ &\overset{\circledast}{=} \begin{bmatrix}1,1\\1,2\end{bmatrix}+\begin{bmatrix}1,1\\2,1\end{bmatrix}-3\begin{bmatrix}1\\3\end{bmatrix}+3\begin{bmatrix}2\\3\end{bmatrix}\\ &\overset{P}{=} \begin{bmatrix}2,3\\0,0\end{bmatrix}+3\begin{bmatrix}3,2\\0,0\end{bmatrix}+6\begin{bmatrix}4,1\\0,0\end{bmatrix}+3\begin{bmatrix}4\\1\end{bmatrix}-3\begin{bmatrix}4\\0\end{bmatrix}.\end{aligned}$$

Compare this to the shuffle product of MZV

$$\zeta(2)\zeta(3)=\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1).$$

bi-brackets - "stuffle" & "shuffle" product

Denote by $P : \mathbb{Q}\langle A_z^{\text{bi}}\rangle \to \mathbb{Q}\langle A_z^{\text{bi}}\rangle$ the linearly extended map which sends a word $w = z_{s_1,r_1} \dots z_{s_l,r_l}$ to the linear combination of words corresponding to the partition relation. With this we get for words $u, v \in \mathbb{Q}\langle A_z^{\text{bi}}\rangle$ two expressions for the product of two bi-brackets:

$$[u] \cdot [v] = [u \circledast v] \, , \qquad [u] \cdot [v] = [P\left(P(u) \circledast P(v)\right)] \, .$$

This yields a large family of linear relations similar to the double shuffle relations of MZV.

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This yields a large family of linear relations similar to the double shuffle relations of MZV.

Question

The first product looks similar to the stuffle product and the second one looks similar to the shuffle product of MZV.
Can we somehow obtain the real stuffle and shuffle product?

Theorem

There are two subalgebras $\mathcal{MD}^{\sqcup\!\sqcup} \subset \mathcal{BD}$ and $\mathcal{MD}^{*} \subset \mathcal{MD}$ spanned by elements $[s_1,\dots,s_l]^{\sqcup\!\sqcup}$ and $[s_1,\dots,s_l]^{*}$, with the following properties

- For $\bullet \in \{*, \sqcup\!\sqcup\}$ the map
$$\begin{aligned}(\mathfrak{H}^1, \bullet) &\longrightarrow \mathcal{MD}^{\bullet}\\ z_{s_1}\dots z_{s_l} &\longmapsto [s_1,\dots,s_l]^{\bullet}\end{aligned}$$
is an algebra homomorphism.
- In length one we have $[s_1]^{\sqcup\!\sqcup} = [s_1]^{*} = [s_1]$.
- For $s_1 \geq 1,\ s_2,\dots,s_l \geq 2$ we have $[s_1,\dots,s_l]^{\sqcup\!\sqcup} = [s_1,\dots,s_l]$.

We are mainly interested in the shuffle brackets $[s_1,\dots,s_l]^{\sqcup\!\sqcup}$.

bi-brackets - shuffle brackets

There are explicit formulas for the shuffle brackets in all length.

Proposition

For $s_1, s_2 \geq 1$ the shuffle brackets in length 2 and 3 are given by

$$[s_1, s_2]^{\sqcup\!\sqcup} = [s_1, s_2] + \delta_{s_2,1} \cdot \frac{1}{2} \left(\begin{bmatrix} s_1 \\ 1 \end{bmatrix} - [s_1] \right) ,$$

$$\begin{aligned} [s_1, s_2, s_3]^{\sqcup\!\sqcup} = [s_1, s_2, s_3] &+ \delta_{s_3,1} \cdot \frac{1}{2} \left(\begin{bmatrix} s_1, s_2 \\ 0, 1 \end{bmatrix} - [s_1, s_2] \right) \\ &+ \delta_{s_2,1} \cdot \frac{1}{2} \left(\begin{bmatrix} s_1, s_3 \\ 1, 0 \end{bmatrix} - \begin{bmatrix} s_1, s_3 \\ 0, 1 \end{bmatrix} - [s_1, s_3] \right) \\ &+ \delta_{s_2 \cdot s_3,1} \cdot \frac{1}{6} \left(\begin{bmatrix} s_1 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} s_1 \\ 1 \end{bmatrix} + [s_1] \right) . \end{aligned}$$

bi-brackets - conjectures

Conjecture

- All linear relations between bi-brackets come from the partition relation and the double shuffle relations.
- Every bi-bracket can be written as a linear combination of brackets, i.e. $\mathcal{BD} = \mathcal{MD}$.
- The dimensions of the weight graded parts $\mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ and $\mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ}^{⧢})$ coincide and they are given by

$$\dim\left(\mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})\right) = \dim\left(\mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ}^{⧢})\right) = d'_k\,,$$

where

$$\sum_{k\geq 0} d'_k X^k := \frac{1 - X^2 + X^4}{1 - 2X^2 - 2X^3}\,.$$

Multiple Eisenstein series

Definition

For $s_1 \geq 3, s_2, \ldots, s_l \geq 2$ we define the *multiple Eisenstein series* of weight $k = s_1 + \cdots + s_l$ and length l by

$$G_{s_1,\ldots,s_l}(\tau) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_l \succ 0 \\ \lambda_i \in \Lambda_\tau}} \frac{1}{\lambda_1^{s_1} \ldots \lambda_l^{s_l}},$$

where $\lambda_i \in \mathbb{Z}\tau + \mathbb{Z}$ are lattice points and the order $\prec$ on $\mathbb{Z} + \mathbb{Z}\tau$ is given by

$$m_1\tau + n_1 \succ m_2\tau + n_2 :\Leftrightarrow (m_1 > m_2 \vee (m_1 = m_2 \wedge n_1 > n_2)) \,.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the stuffle product, i.e. it is for example

$$G_3(\tau) \cdot G_4(\tau) = G_{4,3}(\tau) + G_{3,4}(\tau) + G_7(\tau) \,.$$

Multiple Eisenstein series - Fourier expansion

Remark

The condition $s_1 \geq 3$ is necessary for absolutely convergence of the sum. By choosing a specific way of summation we can also restrict this condition to get a definition of $G_{s_1,\dots,s_l}(\tau)$ with $s_1 = 2$ which also satisfies the stuffle product.

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The connection to the brackets and the MZV is given by the following:

Proposition

For $s_1, \dots, s_l \geq 2$ the $G_{s_1,\dots,s_l}$ have a Fourierexpansion which can be written as a linear combination of products of MZV, powers of $2\pi i$ and brackets by setting $q = e^{2\pi i \tau}$.

In the following we view the multiple Eisenstein series as an element in $\mathbb{C}[[q]]$ and write also $G_{s_1,\dots,s_l}(q)$ instead of $G_{s_1,\dots,s_l}(\tau)$.

$$G_k(q) = \zeta(k) + (-2\pi i)^k [k] = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \,,$$

$$G_{3,2,2}(q) = \zeta(3,2,2) + \left(\frac{54}{5}\zeta(2,3) + \frac{51}{5}\zeta(3,2) \right) (2\pi i)^2 [2] + \frac{16}{3}\zeta(2,2)(2\pi i)^3 [3]$$
$$+ 3\zeta(3)(2\pi i)^4 [2,2] + 4\zeta(2)(2\pi i)^5 [3,2] + (2\pi i)^7 [3,2,2] \,.$$

Multiple Eisenstein series

Due to convergence issues the MES are just defined for $s_1, \ldots, s_l \geq 2$ and therefore there are a lot more MZV than MES. A natural question was therefore the following

Question

What is a "good" definition of a "regularised" multiple Eisenstein series, such that for each multiple zeta value $\zeta(s_1, \ldots, s_l)$ with $s_1 \geq 2$, $s_2, \ldots, s_l \geq 1$ there is a multiple Eisenstein series

$$G^{reg}_{s_1,\ldots,s_l}(q) = \zeta(s_1, \ldots, s_l) + \sum_{n>0} a_n q^n \in \mathbb{C}[[q]]$$

with this multiple zeta values as the constant term in its Fourier expansion and which equals the original multiple Eisenstein series in the case $s_1, \ldots, s_l \geq 2$?

Space of formal iterated integrals

Consider the algebra $\mathcal{I}$ of **formal iterated integrals** generated by the elements $\mathbb{I}(a_0; a_1, \dots, a_N; a_{N+1})$, where $a_i \in \{0, 1\}, N \geq 0$, with the product given by the shuffle product ⧢ together with relations coming from real iterated integrals.

Define

$$\Delta_G\left(\mathbb{I}(a_0; a_1, \dots, a_N; a_{N+1})\right) :=$$

$$\sum \left(\prod_{p=0}^{k} \mathbb{I}(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes \mathbb{I}(a_0; a_{i_1}, \dots, a_{i_k}; a_{N+1}),$$

where the sum runs over all $i_0 = 0 < i_1 < \dots < i_k < i_{k+1} = N+1$ with $0 \leq k \leq N$.

Goncharov

The triple $(\mathcal{I}, ⧢, \Delta_G)$ is a commutative graded Hopf algebra over $\mathbb{Q}$.

Space of formal iterated integrals

For integers $n \geq 0, s_1, \dots, s_l \geq 1$, we set

$$I_n(s_1, \dots, s_l) := I(0; \underbrace{0, \dots, 0}_{n}, \underbrace{1, 0, \dots, 0}_{s_1}, \dots, \underbrace{1, 0, \dots, 0}_{s_l}; 1).$$

In particular, we write $I(s_1, \dots, s_l)$ to denote $I_0(s_1, \dots, s_l)$ and consider quotient space

$$\mathcal{I}^1 = \mathcal{I} / \mathbb{I}(0; 0; 1)\mathcal{I}\,.$$

Proposition

The elements $I(s_1, \dots, s_l)$ form a basis of $\mathcal{I}^1$, i.e. as a $\mathbb{Q}$-algebra the space $(\mathcal{I}^1, ш)$ is isomorphic to $(\mathfrak{H}^1, ш)$.

In the following we therefore consider $(\mathfrak{H}^1, ш, \Delta_G)$ as a Hopf algebra.

Coproduct <-> Multiple Eisenstein series

Example for the coproduct of the word $z_3 z_2 \in \mathfrak{H}^1$:

$$\Delta(z_3 z_2) = z_3 z_2 \otimes 1 + 3 z_3 \otimes z_2 + 2 z_2 \otimes z_3 + 1 \otimes z_3 z_2 \,.$$

Compare this to the Fourier expansion of $G_{2,3}(\tau)$:

$$G_{3,2}(\tau) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2[2] + 2\zeta(2)(-2\pi i)^3[3] + (-2\pi i)^5[3,2]\,.$$

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Since $\Delta(z_{s_1} \dots z_{s_l}) \in \mathfrak{H}^1 \otimes \mathfrak{H}^1$ exists for all $s_1, \dots, s_l \geq 1$ this comparison suggests, that there might be a extended definition of $G_{n_1,\dots,n_r}$ by defining a map

$$\mathfrak{H}^1 \otimes \mathfrak{H}^1 \rightarrow \mathbb{C}[[q]]$$

which sends the first component to the corresponding zeta values and the second component to ($-2\pi i$-multiple of) the bracket.

Define the algebra homomorphism $\mathfrak{g}^{⧢} : (\mathfrak{H}^1, ⧢) \to \mathbb{C}[[q]]$ by

$$\mathfrak{g}^{⧢}(z_{s_1} \dots z_{s_l}) = (-2\pi i)^{s_1+\dots+s_l} [s_1, \dots, s_l]^{⧢} \,.$$

Definition

For integers $s_1, \dots, s_l \geq 1$, we define the **(shuffle) regularised multiple Eisenstein series**, as

$$G^{⧢}_{s_1,\dots,s_l}(q) := m\left((Z^{⧢} \otimes \mathfrak{g}^{⧢}) \circ \Delta_G(z_{s_1} \dots z_{s_l})\right) \,,$$

where m denotes the multiplication given by $m : a \otimes b \mapsto a \cdot b$.

We can view $G^{⧢}$ as an algebra homomorphism $G^{⧢} : (\mathfrak{H}^1_{xy}, ⧢) \to \mathbb{C}[[q]]$ such that the following diagram commutes

$$\begin{array}{ccc} (\mathfrak{H}^1, ⧢) & \xrightarrow{\Delta_G} & (\mathfrak{H}^1, ⧢) \otimes (\mathfrak{H}^1, ⧢) \\ \downarrow G^{⧢} & & \downarrow Z^{⧢} \otimes \mathfrak{g}^{⧢} \\ \mathbb{C}[[q]] & \xleftarrow[m]{} & \mathcal{MZ} \otimes \mathbb{C}[[q]] \end{array}$$

Multiple Eisenstein series - shuffle regularisation

Theorem (B., K. Tasaka 2014)

For all $s_1, \dots, s_l \geq 1$ the shuffle regularised multiple Eisenstein series $G^{ш}_{s_1,\dots,s_l}$ have the following properties:

- They are holomorphic functions on the upper half plane having a Fourier expansion with the shuffle regularised multiple zeta values as the constant term.
- They fulfill the shuffle product.
- For integers $s_1, \dots, s_l \geq 2$ they equal the multiple Eisenstein series

$$G^{ш}_{s_1,\dots,s_l}(q) = G_{s_1,\dots,s_l}(q)$$

  and therefore they fulfill the stuffle product in these cases.

Proof sketch: The first statement follows directly by definition. The second statement follows from the fact that Δ, $Z^{ш}$ and $\mathfrak{g}^{ш}$ are algebra homomorphism and hence $(Z^{ш} \otimes \mathfrak{g}^{ш}) \circ \Delta$ is also an algebra homomorphism. For the third statement we give explicit formulas for the coproduct and the Fourier expansion and then show that they are equal.

Multiple Eisenstein series - shuffle regularisation

The Theorem gives a subset of the double shuffle relations between the $G^{\sqcup\!\sqcup}$, since the stuffle product is just fulfilled for the case $s_1, \dots, s_l \geq 2$.

Question

Do they also fulfill the stuffle product when some indices s_j are equal to 1 ?

Multiple Eisenstein series - shuffle regularisation

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Question

Do they also fulfill the stuffle product when some indices s_j are equal to 1 ?

Yes! For example one can show, by using the quasi-shuffle product of bi-brackets, that

$$G_2^{\sqcup\!\sqcup} \cdot G_{2,1}^{\sqcup\!\sqcup} = G_{2,1,2}^{\sqcup\!\sqcup} + 2G_{2,2,1}^{\sqcup\!\sqcup} + G_{2,3}^{\sqcup\!\sqcup} + G_{4,1}^{\sqcup\!\sqcup} .$$

We want to give another way of proving this by introducing a stuffle regularisation G^* of the multiple Eisenstein series.

Multiple Eisenstein series - stuffle regularisation

Theorem

For $s_1 \geq 2, s_1, \dots, s_l \geq 1$ there exist **stuffle regularised multiple Eisenstein series** $G^*_{s_1,\dots,s_l}(q) \in \mathbb{C}[[q]]$ with the following properties

- They fulfill the stuffle product, i.e. the map $G^* : (\mathfrak{H}^0, *) \to \mathbb{C}[[q]]$ which sends $z_{s_1} \dots z_{s_l}$ to $G^*_{s_1,\dots,s_l}(q)$ is an algebra homomorphism.
- They can be written as a linear combination of MZV, powers of $(-2\pi i)$ and bi-brackets.
- For integers $s_1, \dots, s_l \geq 2$ they equal the multiple Eisenstein series

$$G^*_{s_1,\dots,s_l}(q) = G_{s_1,\dots,s_l}(q) = G^{\sqcup\!\sqcup}_{s_1,\dots,s_l}(q) \,.$$

Notice that in contrast to the $G^{\sqcup\!\sqcup}$ we need $s_1 \geq 2$ for the G^*.

Multiple Eisenstein series - stuffle & shuffle regularisations

These two regularisation coincide in many cases and therefore we can prove the stuffle product for the $G^{⧢}$ in some cases, for example:

Proposition

It is $G^*_{2,1} = G^{⧢}_{2,1}, G^*_{2,1,2} = G^{⧢}_{2,1,2}, G^*_{2,2,1} = G^{⧢}_{2,2,1}$ and $G^*_{4,1} = G^{⧢}_{4,1}$ and therefore

$$G^{⧢}_{2} \cdot G^{⧢}_{2,1} = G^{⧢}_{2,1,2} + 2G^{⧢}_{2,2,1} + G^{⧢}_{2,3} + G^{⧢}_{4,1} \,.$$

But there are $G^*_{s_1,\dots,s_l}$ that differ from $G^{⧢}_{s_1,\dots,s_l}$. For example it is

$$G^{⧢}_{2,1,1} - G^*_{2,1,1} = \frac{5}{2}\zeta(2)(-2\pi i)^2[2] + \frac{1}{8}(-2\pi i)^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \frac{1}{12}(-2\pi i)^4 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq 0$$

It is still an open question for which indices $s_1, \dots, s_l$ we have $G^{⧢}_{s_1,\dots,s_l} = G^*_{s_1,\dots,s_l}$.

Multiple Eisenstein series - relations

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
eds_k	0	0	1	3	6	14	29	60	123	249	503	1012	2032	4075
fds_k	0	0	0	1	2	7	16	40	92	200	429	902	1865	3832
cds_k	0	0	0	1	2	6	14	32	72	156	336	712	1496	3120
rds_k	0	0	0	1	1	3	5	11	19	37	65	120	209	372
d_k	0	1	1	1	2	2	3	4	5	7	9	12	16	21
d'_k	0	1	2	3	6	10	18	32	56	100	176	312	552	976

eds_k = Number of conjectured relations between MZV (extended double shuffle relations),

fds_k = Number of (finite) double shuffle relations,

cds_k = Conjectured number of double shuffle relations between the $G^{\sqcup\!\sqcup}$,

rds_k = Number of (restricted) double shuffle relations, i.e where all indices are ≥ 2.

Summary

- Bi-brackets are q-series with coefficients given by sums over partitions.
- The subspace of brackets give a q-analogue for multiple zeta values.
- Due to the quasi-shuffle product $\circledast$ and the partition relation we have two (different) ways of writing the product of two bi-brackets and therefore have a large family of linear relations.
- They appear also in the Fourier expansion of shuffle and stuffle regularised multiple Eisenstein series.

You can find more details on my homepage:
http://www.math.uni-hamburg.de/home/bachmann/ .