Multiple Eisenstein series

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Motivation

"classical"

Numbers

$$\zeta(s)$$

↑ const. term in Fourierexp.

Hol. Functions

$$G_k(\tau)$$

$$= \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

"classical" "multiple"

Numbers

$\zeta(s) \longrightarrow \zeta(s_1, \ldots, s_l)$

const. term
in Fourierexp.

Hol. Functions

$G_k(\tau)$

$$= \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Motivation

	"classical"	"multiple"
Numbers	$\zeta(s)$	$\zeta(s_1, \ldots, s_l)$
Hol. Functions	$G_k(\tau)$	$G_{s_1,\ldots,s_l}(\tau)$ = ???

const. term
in Fourierexp.

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Multiple zeta-values

Definition

For natural numbers $s_1 \geq 2, s_2, ..., s_l \geq 1$ the multiple zeta-value (mzv) of weight $s_1 + ... + s_l$ and length l ist defined by

$$\zeta(s_1, ..., s_l) = \sum_{n_1 > ... > n_l > 0} \frac{1}{n_1^{s_1} \cdot ... \cdot n_2^{s_2}} .$$

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle relation). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s) .$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle relation) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.

Multiple zeta-values

Example:

$$\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(2,1)=\zeta(3).$$

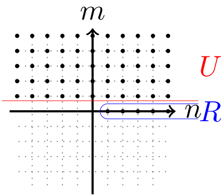
These follow from the "extended double shuffle relations" where one use the same combinatorics as above for "$\zeta(1)\cdot\zeta(2)$" in a formal setting. The extended double shuffle relations are conjectured to give all relations between MZV.

A particular order on lattices

Given $\tau \in \mathbb{H}$ we consider the lattice $\mathbb{Z}\tau + \mathbb{Z}$, then for lattice points $a_1 = m_1\tau + n_1$ and $a_2 = m_2\tau + n_2$ we write

$$a_1 \succ a_2$$

if $(m_1 - m_2, n_2 - n_1) \in \mathcal{P} := R \cup U$ with
$R = \left\{(0,n) \in \mathbb{Z}^2 \mid n > 0\right\}$ and $U = \left\{(m,n) \in \mathbb{Z}^2 \mid m > 0\right\}$.



Classical Eisenstein series are ordered sums

With this order on $\mathbb{Z}\tau + \mathbb{Z}$ one gets for even $k > 2$:

$$G_k(\tau) := \sum_{\substack{a \succ 0 \\ a \in \mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{a^k} = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}.$$

Using this modified definition for G_k we get in fact for all $k > 2$:

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

Multiple Eisenstein series

Definition

For $\tau \in \mathbb{H}$ and natural numbers $s_1 \geq 3, s_2, ..., s_l \geq 2$ we define the *multiple Eisenstein series* of weight $s_1 + ... + s_l$ and length l by

$$G_{s_1,...,s_l}(\tau) := \sum_{\substack{a_1 \succ ... \succ a_l \succ 0 \\ a_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{a_1^{s_1} \cdot ... \cdot a_l^{s_l}} .$$

Remark

- The Stuffle relations are fulfilled because the formal sum manipulations are the same as for MZV.
- Shuffle relations hold as long as they are consequences of the formal partial fraction decomposition.

(For $l = 2$ these functions were studied by Gangl, Kaneko & Zagier (2006))

Questions

- We have
$$G_{s_1,\dots,s_l}(\tau) = G_{s_1,\dots,s_l}(\tau+1)\,.$$
How does the Fourier expansion look like?
- Modularity (for even k)?
- modified definitions of $G_{s_1,\dots,s_l}$ in the cases where MZV are defined, e.g. for all $s_1 \geq 2$, s_2, ..., $s_l \geq 1$? Stuffle? Shuffle?

Fourier expansion of multiple Eisenstein series

Theorem

For $s_1 \geq 3$ and $s_2, \ldots, s_l \geq 2$ one has:

$$G_{s_1,\ldots,s_l} = \zeta(s_1,\ldots,s_l) + \sum_{n>0} a_n q^n \,,$$

where

$$a_n = \sum_{\substack{1 \leq j \leq l \\ a_1+\ldots+a_l = s_1+\ldots+s_l}} \alpha_j^{a_1,\ldots,a_l} \cdot (\pi i)^{a_1+\ldots+a_j} \cdot \zeta(a_{j+1},\ldots,a_l) \cdot \sigma_{a_1-1,\ldots,a_j-1}(n)$$

with generalised divisor functions

$$\sigma_{s_1,\ldots,s_l}(n) := \sum_{\substack{u_1 v_1+\ldots+u_l v_l = n \\ u_1 > \ldots > u_l > 0}} v_1^{s_1} \cdot \ldots \cdot v_l^{s_l}$$

and the numbers $\alpha_j^{a_1,\ldots,a_l} \in \mathbb{Q}$ can be computed algorithmically.

Examples

$$G_{4,4}(\tau) = \zeta(4,4)$$
$$+ \sum_{n>0} \left(\frac{(-2\pi i)^{4+4}}{3! \cdot 3!} \sigma_{3,3}(n) + 20(-2\pi i)^2 \zeta(6) \sigma_1(n) + \frac{(-2\pi i)^4}{2} \zeta(4) \sigma_3(n) \right) q^n .$$

Notation:

$$\tilde{\zeta}(s_1, ..., s_l) := (-2\pi i)^{-s_1+...+s_l} \zeta(s_1, ..., s_l)$$
$$\tilde{G}_{s_1,..,s_l}(\tau) := (-2\pi i)^{-s_1+...+s_l} G_{s_1,..,s_l}(\tau)$$

e.g. with further simplifications by explicit known MZV's

$$\tilde{G}_{4,4}(\tau) = \frac{1}{29030400} + \frac{1}{36} \sum_{n>0} \left(\sigma_{3,3}(n) - \frac{1}{84} \sigma_1(n) + \frac{1}{80} \sigma_3(n) \right) q^n$$

Examples

$$
\begin{aligned}
\tilde{G}_{4,5,6}(\tau) &= (-2\pi i)^{-15} G_{4,5,6}(\tau) = \tilde{\zeta}(4,5,6) \\
&+ \frac{(-1)^{4+5+6}}{c} \sum_{n>0} \left(\sigma_{3,4,5}(n) - \frac{281}{2882880}\sigma_0(n) + \frac{130399}{605404800}\sigma_2(n) - \frac{37}{1330560}\sigma_4(n) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(3600\sigma_4(n)\tilde{\zeta}(6,4) + 293760\sigma_2(n)\tilde{\zeta}(7,5) + 302400\sigma_2(n)\tilde{\zeta}(8,4) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(1814400\sigma_0(n)\tilde{\zeta}(8,6) + 2903040\sigma_0(n)\tilde{\zeta}(9,5) + 2177280\sigma_0(n)\tilde{\zeta}(10,4) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(-\frac{1}{168}\sigma_{2,5}(n) - \frac{1}{120}\sigma_{3,2}(n) + \frac{1}{168}\sigma_{3,4}(n) + \frac{1}{240}\sigma_{4,5}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(-\frac{\zeta(5)}{20\pi^5}\sigma_1(n) + \frac{\zeta(5)}{14\pi^5}\sigma_3(n) - \frac{\zeta(5)}{80\pi^5}\sigma_5(n) - \frac{3\zeta(5)}{\pi^5}\sigma_{3,5}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{45\zeta(5)^2}{32\pi^{10}}\sigma_4(n) + \frac{25\zeta(7)}{64\pi^7}\sigma_1(n) + \frac{21\zeta(7)}{32\pi^7}\sigma_3(n) - \frac{105\zeta(7)}{64\pi^7}\sigma_5(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{315\zeta(7)}{8\pi^7}\sigma_{1,5}(n) - \frac{315\zeta(7)}{4\pi^7}\sigma_{3,3}(n) - \frac{2835\zeta(5)\zeta(7)}{16\pi^{12}}\sigma_2(n) + \frac{42525\zeta(7)^2}{128\pi^{14}}\sigma_0(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{189\zeta(9)}{16\pi^9}\sigma_1(n) - \frac{945\zeta(9)}{16\pi^9}\sigma_3(n) + \frac{1125\zeta(9)}{64\pi^9}\sigma_5(n) + \frac{2835\zeta(9)}{4\pi^9}\sigma_{3,1}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{8505\zeta(5)\zeta(9)}{16\pi^{14}}\sigma_0(n) + \frac{28755\zeta(11)}{64\pi^{11}}\sigma_3(n) - \frac{135135\zeta(13)}{128\pi^{13}}\sigma_1(n) \right) q^n, \quad (c = 3! \cdot 4! \cdot 5!)
\end{aligned}
$$

Fourier expansion - Glimpse of the proof

- Nice combinatorics but lengthy.
- Unfortunately I have no time to explain it now. But you can ask me later.

Modularity, easy case

Because of of the stuffle relation we have for example

$$G_4^2 = 2G_{4,4} + G_8$$

so $G_{4,4}$ is a modular form of weight 8. In general we have

Theorem

If all $s_1, ..., s_l$ are even and all $s_j > 2$, then we have

$$\sum_{\sigma \in \Sigma_l} G_{s_{\sigma(1)},...,s_{\sigma(l)}} \in M_k(\mathrm{SL}_2(\mathbb{Z}))\,,$$

where the weight k is given by $k = s_1 + ... + s_l$.

Proof: Easy induction using stuffle relation.

Modularity, cusp forms

By the double shuffle relations and Eulers formula $\zeta(2k) = \lambda \cdot \pi^{2k}$ for $\lambda \in \mathbb{Q}$ one can show:

$$
\begin{aligned}
&-\frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17}\zeta(12) - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17}\zeta(6,3,3) \\
&\quad - \frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17}\zeta(4,5,3) + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17}\zeta(7,5) \\
&= -40\zeta(4)^3 + 49\zeta(6)^2 \\
&= 0 \text{ because of Euler's formula} .
\end{aligned}
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&= -40\zeta(4)^3 + 49\zeta(6)^2 \\
&= 0 \text{ because of Euler's formula}\,.
\end{aligned}
$$

But in the context of multiple Eisenstein series we get:

Theorem

$$
\begin{aligned}
&- \frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17}\tilde{G}_{12} - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17}\tilde{G}_{6,3,3} \\
&- \frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17}\tilde{G}_{4,5,3} + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17}\tilde{G}_{7,5} = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))\,.
\end{aligned}
$$

Modularity & cusp forms

Proof: The first identity of the above MZV Relation hold also for multiple Eisensten series. It follows from the Stuffle relation and partial fraction decompositions which replaces Shuffle.

But in the second identity, i.e., the place when Eulers formula is needed, one gets the "error term" , because in general whenever $s_1 + s_2 \geq 12$ the following function doesn't vanish

$$G_{s_1} \cdot G_{s_2} - \frac{\zeta(s_1)\zeta(s_2)}{\zeta(s_1 + s_2)} G_{s_1+s_2} \in S_{s_1+s_2}(\mathrm{SL}_2(\mathbb{Z})).$$

So the failure of Euler's relation give us the cusp forms □

Remark

- There are many more such linear relations which give cusp forms

Modularity, cusp forms

From such identities we get new relations between Fourier coefficients of modular forms and generalized divisor sums, e.g.:

Corollary - Formula for the Ramanujan τ-function

For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\tau(n) = {} & \frac{2\cdot 7\cdot 691}{3^2\cdot 11\cdot 17}\sigma_1(n) - \frac{43\cdot 691}{2^3\cdot 3^2\cdot 5\cdot 17}\sigma_3(n) + \frac{691}{2\cdot 3^3\cdot 7\cdot 17}\sigma_5(n) \\ & - \frac{757}{2^3\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 17}\sigma_{11}(n) - \frac{2^3\cdot 5\cdot 691}{3\cdot 17}\sigma_{2,2}(n) + \frac{2^2\cdot 5\cdot 691}{3\cdot 17}\sigma_{3,1}(n) \\ & - \frac{2^2\cdot 7\cdot 691}{3\cdot 17}\sigma_{3,3}(n) + \frac{2\cdot 7\cdot 691}{17}\sigma_{4,2}(n) + \frac{2^2\cdot 7\cdot 691}{3\cdot 17}\sigma_{5,1}(n) \\ & + \frac{2\cdot 5\cdot 691}{3\cdot 17}\sigma_{6,4}(n) + \frac{2^4\cdot 5\cdot 7\cdot 691}{17}\sigma_{3,4,2}(n) - \frac{2^4\cdot 5\cdot 7\cdot 691}{17}\sigma_{5,2,2}(n)\,.\end{aligned}$$

$$\Delta(\tau) = q\prod_{i=1}^{\infty}(1-q^n)^{24} = \sum_{n>0}\tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

"Non convergent" multiple Eisenstein series

For $l = 2$ Kaneko, Gangl & Zagier give an extended definition for the Double Eisenstein series $G^*_{3,1}$, $G^*_{2,2}$,... .

Natural question: What should $G^*_{2,...,2}$ in general be?
We want our multiple Eisenstein series to fulfill the same linear relations as the corresponding MZV (modulo cusp forms), therefore we have to imitate the following well known result in the context of multiple Eisenstein series:

Theorem

For $\lambda_n := (-1)^{n-1} \cdot 2^{2n-1} \cdot (2n+1) \cdot B_{2n}$ we have

$$\zeta(2n) - \lambda_n \cdot \zeta(\underbrace{2, ..., 2}_{n}) = 0\,.$$

"Non convergent" multiple Eisenstein series

Ansatz: Define $G_{2,\ldots,2}$ to be the function obtained by setting all s_i to 2 in the formula of the Fourier Expansion. e.g.

$$G_2(\tau) = \zeta(2) + (-2\pi i)^2 \sum_{n>0} \sigma_1(n) q^n \,,$$

$$G_{2,2}(\tau) = \zeta(2,2) + (-2\pi i)^4 \sum_{n>0} \left(\sigma_{1,1}(n) - \frac{1}{8}\sigma_1(n) \right) q^n \,.$$

Will this give the "right" definition of $G^*_{2,\ldots,2}$?

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Will this give the "right" definition of $G^*_{2,\dots,2}$?
No, because with this definition the function

$$G_{2n} - \lambda_n \cdot G_{\underbrace{2\dots,\,2}_{n}} \notin S_{2n}(\mathrm{SL}_2(\mathbb{Z}))$$

is not a cusp form. It is not modular but quasi-modular, this property is used for the following modified definition:

Theorem

Let $X_n(\tau) := (2\pi i)^{-2l} G_{\underbrace{2,\dots,2}_{n}}(\tau)$, $D := q\frac{d}{dq}$ and

$$\tilde{G}^*_{\underbrace{2,\dots,2}_{n}}(\tau) := X_n(\tau) + \sum_{j=1}^{n-1} \frac{(2n-2-j)!}{2^j \cdot j! \cdot (2n-2)!} D^j X_{n-j}(\tau)\,.$$

then we have with $\lambda_n \in \mathbb{Q}$ as above

$$\tilde{G}_{2n} - \lambda_n \cdot \tilde{G}^*_{2,\dots,2} \in S_{2n}(\mathrm{SL}_2\,\mathbb{Z})\,.$$

Examples:

$$G^*_{2,2} = \tilde{G}_{2,2} + \frac{1}{4} D^1 \tilde{G}_2 \,,$$
$$G^*_{2,2,2} = \tilde{G}_{2,2,2} + \frac{1}{8} D^1 \tilde{G}_{2,2} + \frac{1}{96} D^2 \tilde{G}_2 \,,$$
$$G^*_{2,2,2,2} = \tilde{G}_{2,2,2,2} + \frac{1}{12} D^1 \tilde{G}_{2,2,2} + \frac{1}{240} D^2 \tilde{G}_{2,2} + \frac{1}{5760} D^3 \tilde{G}_2 \,.$$
$$\ldots$$

In weight 12 we get

$$\tilde{G}_{12} - \lambda_6 \cdot \tilde{G}^*_{2,2,2,2,2,2} = \frac{17}{3^6 \cdot 5^4 \cdot 7^2} \Delta$$

which gives another expression for $\tau(n)$ in $\sigma_{11}(n)$ and $\sigma_{1,\ldots,1}(n)$ (see last slide).

Sketch of the proof

- First show following identity for the generating function of X_n

$$\Phi(\tau, T) := \sum_{n \geq 0} X_n(\tau)(-4\pi i T)^n = \exp\left(-2 \sum_{l \geq 1} \frac{(-1)^l}{(2l)!} E_{2l}(\tau)(-4\pi i T)^l\right)$$

where $E_k(\tau) = -\frac{B_k}{2k} + \sum_{n>0} \sigma_{k-1}(n) q^n$.

- Using the modularity of the E_k for $k > 2$ one sees easily that Φ is a Jacobi-like form of weight 0, i.e.:

$$\Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{T}{(c\tau + d)^2}\right) = \exp\left(\frac{cT}{(c\tau + d)}\right) \Phi(\tau, T) \quad , \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

- One can show that coefficients of such functions always give rise to modular forms as above.

Application

New formular for fourier coefficients of cusp forms, e.g.:

$$
\begin{aligned}
-\frac{17}{9794925}\tau(n) = &\left(\frac{1}{7560}n^5 - \frac{1}{1008}n^4 + \frac{13}{4320}n^3 - \frac{41}{9072}n^2 + \frac{671}{201600}n - \frac{73}{76032}\right)\sigma_1(n) \\
&+ \left(\frac{1}{126}n^4 - \frac{5}{54}n^3 + \frac{23}{54}n^2 - \frac{227}{252}n + \frac{631}{864}\right)\sigma_{1,1}(n) \\
&+ \left(\frac{4}{9}n^3 - \frac{56}{9}n^2 + \frac{154}{5}n - \frac{479}{9}\right)\sigma_{1,1,1}(n) \\
&+ \left(\frac{64}{3}n^2 - 288n + 1032\right)\sigma_{1,1,1,1}(n) \\
&+ (768n - 7040)\,\sigma_{1,1,1,1,1}(n) \\
&+ 15360\sigma_{1,1,1,1,1,1}(n) - \frac{1}{17512704}\sigma_{11}(n)
\end{aligned}
$$

Reminder:

$$
\sigma_{\underbrace{1,\dots,1}_{l}}(n) = \sum_{\substack{u_1 v_1 + \dots + u_l v_l = n \\ u_1 > \dots > u_l > 0}} v_1 \cdot \dots \cdot v_l \,.
$$