Multiple Eisenstein series

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Workshop on Periods and Motives - YRS
Madrid 4th June 2012

Motivation

"classical" "multiple"

$$\zeta(s) \longrightarrow \zeta(s_1, \ldots, s_l)$$

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Numbers

$\zeta(s)$ → $\zeta(s_1, \dots, s_l)$

↑ const. term in Fourierexp.

Hol. Functions

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

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Hol. Functions

$G_k(\tau) \longrightarrow G_{s_1,\ldots,s_l}(\tau)$ = ???

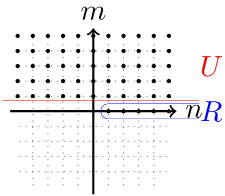
$$= \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

A particular order on lattices

Given $\tau \in \mathbb{H}$ we consider the lattice $\mathbb{Z}\tau + \mathbb{Z}$, then for lattice points $a_1 = m_1\tau + n_1$ and $a_2 = m_2\tau + n_2$ we write

$$a_1 \succ a_2$$

if $(m_1 - m_2, n_2 - n_1) \in \mathcal{P} := R \cup U$ with
$R = \{(0,n) \in \mathbb{Z}^2 \mid n > 0\}$ and $U = \{(m,n) \in \mathbb{Z}^2 \mid m > 0\}$.



Classical Eisenstein series are ordered sums

With this order on $\mathbb{Z}\tau + \mathbb{Z}$ one gets for even $k > 2$:

$$G_k(\tau) := \sum_{\substack{a \succ 0 \\ a \in \mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{a^k} = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}.$$

Using this modified definition for G_k we get in fact for all $k > 2$:

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

Multiple Eisenstein series

Definition

For $\tau \in \mathbb{H}$ and natural numbers $s_1 \geq 3$, $s_2, ..., s_l \geq 2$ we define the *multiple Eisenstein series* of weight $s_1 + ... + s_l$ and length l by

$$G_{s_1,...,s_l}(\tau) := \sum_{\substack{a_1 \succ ... \succ a_l \succ 0 \\ a_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{a_1^{s_1} \cdot ... \cdot a_l^{s_l}} .$$

Remark

- The Stuffle relations are fulfilled because the formal sum manipulations are the same as for MZV.
- Shuffle relations hold as long as they are consequences of the formal partial fraction decomposition.

(For $l = 2$ these functions were studied by Gangl, Kaneko & Zagier (2006))

Questions

- We have
$$G_{s_1,\dots,s_l}(\tau) = G_{s_1,\dots,s_l}(\tau+1)\,.$$
How does the Fourier expansion look like?
- Modularity (for even k)?
- modified definitions of $G_{s_1,\dots,s_l}$ in the cases where MZV are defined, e.g. for all $s_1 \geq 2$, s_2, ..., $s_l \geq 1$? Stuffle? Shuffle?

Fourier expansion of multiple Eisenstein series

Theorem

For $s_1 \geq 3$ and $s_2, ..., s_l \geq 2$ one has:

$$G_{s_1,...,s_l} = \zeta(s_1, ..., s_l) + \sum_{n>0} a_n q^n \,,$$

where

$$a_n = \sum_{\substack{1 \leq j \leq l \\ a_1+...+a_l = s_1+...+s_l}} \alpha_j^{a_1,...,a_l} \cdot (\pi i)^{a_1+...+a_j} \cdot \zeta(a_{j+1}, ..., a_l) \cdot \sigma_{a_1-1,...,a_j-1}(n)$$

with generalised divisor functions

$$\sigma_{s_1,...,s_l}(n) := \sum_{\substack{u_1 v_1 + ... + u_l v_l = n \\ u_1 > ... > u_l > 0}} v_1^{s_1} \cdot ... \cdot v_l^{s_l}$$

and the numbers $\alpha_j^{a_1,...,a_l} \in \mathbb{Q}$ can be computed algorithmically.

Examples

$$G_{4,4}(\tau) = \zeta(4,4)$$
$$+\sum_{n>0}\left(\frac{(-2\pi i)^{4+4}}{3!\cdot 3!}\sigma_{3,3}(n) + 20(-2\pi i)^2\zeta(6)\sigma_1(n) + \frac{(-2\pi i)^4}{2}\zeta(4)\sigma_3(n)\right)q^n.$$

Notation:

$$\tilde{\zeta}(s_1,...,s_l) := (-2\pi i)^{-s_1+...+s_l}\zeta(s_1,...,s_l)$$
$$\tilde{G}_{s_1,..,s_l}(\tau) := (-2\pi i)^{-s_1+...+s_l}G_{s_1,..,s_l}(\tau)$$

e.g. with further simplifications by explicit known MZV's

$$\tilde{G}_{4,4}(\tau) = \frac{1}{29030400} + \frac{1}{36}\sum_{n>0}\left(\sigma_{3,3}(n) - \frac{1}{84}\sigma_1(n) + \frac{1}{80}\sigma_3(n)\right)q^n$$

Examples

$$
\begin{aligned}
\tilde{G}_{4,5,6}(\tau) &= (-2\pi i)^{-15} G_{4,5,6}(\tau) = \tilde{\zeta}(4,5,6) \\
&+ \frac{(-1)^{4+5+6}}{c} \sum_{n>0} \left(\sigma_{3,4,5}(n) - \frac{281}{2882880}\sigma_0(n) + \frac{130399}{605404800}\sigma_2(n) - \frac{37}{1330560}\sigma_4(n) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(3600\sigma_4(n)\tilde{\zeta}(6,4) + 293760\sigma_2(n)\tilde{\zeta}(7,5) + 302400\sigma_2(n)\tilde{\zeta}(8,4) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(1814400\sigma_0(n)\tilde{\zeta}(8,6) + 2903040\sigma_0(n)\tilde{\zeta}(9,5) + 2177280\sigma_0(n)\tilde{\zeta}(10,4) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(-\frac{1}{168}\sigma_{2,5}(n) - \frac{1}{120}\sigma_{3,2}(n) + \frac{1}{168}\sigma_{3,4}(n) + \frac{1}{240}\sigma_{4,5}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(-\frac{\zeta(5)}{20\pi^5}\sigma_1(n) + \frac{\zeta(5)}{14\pi^5}\sigma_3(n) - \frac{\zeta(5)}{80\pi^5}\sigma_5(n) - \frac{3\zeta(5)}{\pi^5}\sigma_{3,5}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{45\zeta(5)^2}{32\pi^{10}}\sigma_4(n) + \frac{25\zeta(7)}{64\pi^7}\sigma_1(n) + \frac{21\zeta(7)}{32\pi^7}\sigma_3(n) - \frac{105\zeta(7)}{64\pi^7}\sigma_5(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{315\zeta(7)}{8\pi^7}\sigma_{1,5}(n) - \frac{315\zeta(7)}{4\pi^7}\sigma_{3,3}(n) - \frac{2835\zeta(5)\zeta(7)}{16\pi^{12}}\sigma_2(n) + \frac{42525\zeta(7)^2}{128\pi^{14}}\sigma_0(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{189\zeta(9)}{16\pi^9}\sigma_1(n) - \frac{945\zeta(9)}{16\pi^9}\sigma_3(n) + \frac{1125\zeta(9)}{64\pi^9}\sigma_5(n) + \frac{2835\zeta(9)}{4\pi^9}\sigma_{3,1}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{8505\zeta(5)\zeta(9)}{16\pi^{14}}\sigma_0(n) + \frac{28755\zeta(11)}{64\pi^{11}}\sigma_3(n) - \frac{135135\zeta(13)}{128\pi^{13}}\sigma_1(n) \right) q^n, \quad (c = 3!\cdot 4!\cdot 5!)
\end{aligned}
$$

Fourier expansion - Sketch of the proof

In the classical case $(l = 1)$ one splits the sum of G_k into two parts:

$$G_k(\tau) = \sum_{\substack{a \succ 0 \\ a \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{a^k} = \underbrace{\sum_{a \in R} \frac{1}{a^k}}_{G_k^R :=} + \underbrace{\sum_{a \in U} \frac{1}{a^k}}_{G_k^U :=}$$

$$= \zeta(k) + \sum_{m > 0} \underbrace{\sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}}_{\Psi_k(m\tau) :=}$$

and then uses the Lipschitz summation formula for the second part:

$$\Psi_k(\tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d=1}^{\infty} d^{k-1} q^d \,.$$

to get

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

Fourier expansion - Sketch of the proof

For the case $l = 2$ one has to consider $4 = 2^2$ sums:

$$\begin{aligned} G_{s_1,s_2}(\tau) &= G^{R^2}_{s_1,s_2}(\tau) + G^{UR}_{s_1,s_2}(\tau) + G^{U^2}_{s_1,s_2}(\tau) + G^{RU}_{s_1,s_2}(\tau) \\ &= \zeta(s_1, s_2) + \sum_{m>0} \Psi_{s_1}(m\tau)\zeta(s_2) \\ &+ \sum_{m_1>m_2>0} \Psi_{s_1}(m_1\tau)\Psi_{s_2}(m_2\tau) + \sum_{m>0} \Psi_{s_1,s_2}(m\tau)\,, \end{aligned}$$

where

$$\Psi_{s_1,\dots,s_l}(\tau) := \sum_{\substack{n_1>\dots>n_l \\ n_i\in\mathbb{Z}}} \prod_{i=1}^{l} \frac{1}{(\tau+n_i)^{s_i}}.$$

Fourier expansion - Sketch of the proof

For the Fourier expansion of
$G^{U^2}_{s_1,s_2}(\tau) = \sum_{m_1>m_2>0} \Psi_{s_1}(m_1\tau)\Psi_{s_2}(m_2\tau)$ we use the lemma:

Lemma

For $s_1, ..., s_l > 1$ we have

$$\sum_{m_1>...>m_l>0} \Psi_{s_1}(m_1\tau) \cdot ... \cdot \Psi_{s_l}(m_l\tau) = \frac{(-2\pi i)^{s_1+...+s_l}}{(s_1-1)! \cdot ... \cdot (s_l-1)!} \sum_{n>0} \sigma_{s_1-1,...,s_l-1}(n) q^n \, .$$

Proof: Use the Lipschitz summation formula l-times.

Fourier expansion - Sketch of the proof

For the term $G^{RU}_{s_1,s_2}(\tau) = \sum_{m>0} \Psi_{s_1,s_2}(m\tau)$ we use the following theorem:

Theorem

We have

$$\Psi_{s_1,\dots,s_l}(x) = \sum_{h=2}^{k} \lambda_h^{s_1,\dots,s_l} \Psi_h(x)\,,$$

where $k = s_1 + \dots + s_l$ and the coefficients $\lambda_h^{s_1,\dots,s_l}$ are linear combinations of MZV's of weight $k - h$.

Proof: Use the partial fraction decomposition. (Compare to the Result in the talk of O. Bouillot)

Combining the Lemma and Theorem also works for the general case $l \geq 2$. $\square$

Modularity, easy case

Because of of the stuffle relation we have for example

$$G_4^2 = 2G_{4,4} + G_8$$

so $G_{4,4}$ is a modular form of weight 8. In general we have

Theorem

If all $s_1, ..., s_l$ are even and all $s_j > 2$, then we have

$$\sum_{\sigma \in \Sigma_l} G_{s_{\sigma(1)},\ldots,s_{\sigma(l)}} \in M_k(\mathrm{SL}_2(\mathbb{Z}))\,,$$

where the weight k is given by $k = s_1 + ... + s_l$.

Proof: Easy induction using stuffle relation.

Modularity, cusp forms

By the double shuffle relations and Eulers formula $\zeta(2k) = \lambda \cdot \pi^{2k}$ for $\lambda \in \mathbb{Q}$ one can show:

$$-\frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17}\zeta(12) - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17}\zeta(6,3,3)$$

$$-\frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17}\zeta(4,5,3) + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17}\zeta(7,5)$$

$=$ some sum of products of even Zeta values by double shuffle relations, sorry the coeficients are at home...

$= 0$ because of Euler's formula .

Modularity, cusp forms

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\begin{aligned}
&- \frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17} \zeta(12) - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17} \zeta(6,3,3) \\
&\quad - \frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17} \zeta(4,5,3) + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17} \zeta(7,5) \\
&= \text{some sum of products of even Zeta values} \quad \text{by double shuffle relations, sorry the coeficients are at home...} \\
&= 0 \quad \text{because of Euler's formula} \, .
\end{aligned}
$$

But in the context of multiple Eisenstein series we get:

Theorem

$$
\begin{aligned}
&- \frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17} \tilde{G}_{12} - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17} \tilde{G}_{6,3,3} \\
&- \frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17} \tilde{G}_{4,5,3} + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17} \tilde{G}_{7,5} = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z})) \, .
\end{aligned}
$$

Modularity & cusp forms

Proof: The first identity of the above MZV Relation hold also for multiple Eisensten series. It follows from the Stuffle relation and partial fraction decompositions which replaces Shuffle.
But in the second identity, i.e., the place when Eulers formula is needed, one gets the "error term" , because in general whenever $s_1 + s_2 \geq 12$ we have

$$0 \neq G_{s_1} \cdot G_{s_2} - \frac{\zeta(s_1)\zeta(s_2)}{\zeta(s_1+s_2)} G_{s_1+s_2} \in S_{s_1+s_2}(\mathrm{SL}_2(\mathbb{Z})).$$

So the failure of Euler's relation give us the cusp forms □

Remark

- There are many more such linear relations which give cusp forms
- The Algebra spanned by the multiple Eisenstein series will be studied in another PhD-project in Hamburg.

Modularity, cusp forms

From such identities we get new relations between Fourier coefficients of modular forms and generalized divisor sums, e.g.:

Corollary - Formula for the Ramanujan τ-function

For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\tau(n) = &\frac{2\cdot 7\cdot 691}{3^2\cdot 11\cdot 17}\sigma_1(n) - \frac{43\cdot 691}{2^3\cdot 3^2\cdot 5\cdot 17}\sigma_3(n) + \frac{691}{2\cdot 3^3\cdot 7\cdot 17}\sigma_5(n)\\ &- \frac{757}{2^3\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 17}\sigma_{11}(n) - \frac{2^3\cdot 5\cdot 691}{3\cdot 17}\sigma_{2,2}(n) + \frac{2^2\cdot 5\cdot 691}{3\cdot 17}\sigma_{3,1}(n)\\ &- \frac{2^2\cdot 7\cdot 691}{3\cdot 17}\sigma_{3,3}(n) + \frac{2\cdot 7\cdot 691}{17}\sigma_{4,2}(n) + \frac{2^2\cdot 7\cdot 691}{3\cdot 17}\sigma_{5,1}(n)\\ &+ \frac{2\cdot 5\cdot 691}{3\cdot 17}\sigma_{6,4}(n) + \frac{2^4\cdot 5\cdot 7\cdot 691}{17}\sigma_{3,4,2}(n) - \frac{2^4\cdot 5\cdot 7\cdot 691}{17}\sigma_{5,2,2}(n)\,.\end{aligned}$$

$$\Delta(\tau) = q\prod_{i=1}^{\infty}(1-q^n)^{24} = \sum_{n>0}\tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

"Non convergent" multiple Eisenstein series

For $l = 2$ Kaneko, Gangl & Zagier give an extended definition for the Double Eisenstein series $G^*_{3,1}$, $G^*_{2,2}$,... .

Natural question: What should $G^*_{2,...,2}$ in general be?
We want our multiple Eisenstein series to fulfill the same linear relations as the corresponding MZV (modulo cusp forms), therefore we have to imitate the following well known result in the context of multiple Eisenstein series:

Theorem

For $\lambda_n := (-1)^{n-1} \cdot 2^{2n-1} \cdot (2n+1) \cdot B_{2n}$ we have

$$\zeta(2n) - \lambda_n \cdot \zeta(\underbrace{2, ..., 2}_{n}) = 0\,.$$

"Non convergent" multiple Eisenstein series

Ansatz: Define $G_{2,\dots,2}$ to be the function obtained by setting all s_i to 2 in the formula of the Fourier Expansion. e.g.

$$G_2(\tau) = \zeta(2) + (-2\pi i)^2 \sum_{n>0} \sigma_1(n) q^n$$

Will this give the "right" definition of $G^*_{2,\dots,2}$?

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Will this give the "right" definition of $G^*_{2,\dots,2}$?

No, because with this definition the function

$$G_{2n} - \lambda_n \cdot G_{\underbrace{2\dots,2}_{n}} \notin S_{2n}(\mathrm{SL}_2(\mathbb{Z}))$$

is not a cusp form. It is not modular but quasi-modular, this property is used for the following modified definition:

The multiple Eisenstein series $G_{2,\ldots,2}$

Theorem

Let $X_n(\tau) := (2\pi i)^{-2l} G_{\underbrace{2,\,\ldots,\,2}_{n}}(\tau)$, $D := q\frac{d}{dq}$ and

$$\tilde{G}^{*}_{\underbrace{2,\,\ldots,\,2}_{n}}(\tau) := X_n(\tau) + \sum_{j=1}^{n-1} \frac{(2n-2-j)!}{2^j \cdot j! \cdot (2n-2)!} D^j X_{n-j}(\tau)\,.$$

then we have with $\lambda_n \in \mathbb{Q}$ as above

$$\tilde{G}_{2n} - \lambda_n \cdot \tilde{G}^{*}_{2,\ldots,2} \in S_{2n}(\mathrm{SL}_2\,\mathbb{Z})\,.$$

Proof: Following a suggestion by Zagier we use the generating function of $G_{2,\ldots,2}$ and the theory of Jacobi-like forms. □

The multiple Eisenstein series $G^*_{2,\dots,2}$

Examples:

$$G^*_{2,2} = \tilde{G}_{2,2} + \frac{1}{4} D^1 \tilde{G}_2 \,,$$

$$G^*_{2,2,2} = \tilde{G}_{2,2,2} + \frac{1}{8} D^1 \tilde{G}_{2,2} + \frac{1}{96} D^2 \tilde{G}_2 \,,$$

$$G^*_{2,2,2,2} = \tilde{G}_{2,2,2,2} + \frac{1}{12} D^1 \tilde{G}_{2,2,2} + \frac{1}{240} D^2 \tilde{G}_{2,2} + \frac{1}{5760} D^3 \tilde{G}_2 \,.$$

More details on this talk are in my master thesis. It is available on my webpage http://www.math.uni-hamburg.de/home/bachmann/