Generating series of multiple divisor sums and other interesting q-series

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content of this talk

- We are interested in a family of q-series which arises in a theory which combines multiple zeta values, partitions and modular forms.
- We will see that the space spanned by these q-series form an algebra where the product can be written in two different ways which then yields linear relations.
- For example:

$$\sum_{n_1>n_2>0} \frac{q^{n_1} n_2 q^{n_2}}{(1-q^{n_1})(1-q^{n_2})} = \frac{1}{2}\sum_{n>0} \frac{n^2 q^n}{1-q^n} + \frac{1}{2}\sum_{n>0} \frac{n q^n}{1-q^n} - \sum_{n>0} \frac{n q^n}{(1-q^n)^2} \,.$$

- Linear relations between these series induce linear relations (conjecturally all) between multiple zeta values.

bi-brackets

Definition

For $r_1, \dots, r_l \geq 0$, $s_1, \dots, s_l > 0$ and $c := (r_1!(s_1-1)! \dots r_l!(s_l-1)!)^{-1}$ we define the following q-series

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} := c \cdot \sum_{\substack{u_1 > \dots > u_l > 0 \\ v_1, \dots, v_l > 0}} u_1^{r_1} v_1^{s_1 - 1} \dots u_l^{r_l} v_l^{s_l - 1} q^{u_1 v_1 + \dots + u_l v_l} ,$$

which we call **bi-brackets** of weight $s_1 + \dots + s_l + r_1 + \dots + r_l$, upper weight $s_1 + \dots + s_l$, lower weight $r_1 + \dots + r_l$ and length l.

By $\mathcal{BD}$ we denote the $\mathbb{Q}$-vector space spanned by all bi-brackets and 1.

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sum_{n>0} \sigma_1(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \dots ,$$

$$\begin{bmatrix} 1,1,1 \\ 1,2,3 \end{bmatrix} = \frac{1}{12} \left(12q^6 + 28q^7 + 96q^8 + 481q^9 + 747q^{10} + 2042q^{11} + \dots \right) .$$

bi-brackets

The bi-brackets can also be written as

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} = c \cdot \sum_{n_1 > \dots > n_l > 0} \frac{n_1^{r_1} P_{s_1 - 1}(q^{n_1}) \dots n_l^{r_l} P_{s_l - 1}(q^{n_l})}{(1 - q^{n_1})^{s_1} \dots (1 - q^{n_l})^{s_l}} \,,$$

where the $P_{k-1}(t)$ are the Eulerian polynomials defined by

$$\frac{P_{k-1}(t)}{(1-t)^k} = \mathrm{Li}_{1-k}(t) = \sum_{d>0} d^{k-1} t^d \,.$$

Examples:

$$P_0(t) = P_1(t) = t \,, \quad P_2(t) = t^2 + t \,, \quad P_3(t) = t^3 + 4t^2 + t \,,$$

$$\begin{bmatrix} 1, 1 \\ 0, 1 \end{bmatrix} = \sum_{n_1 > n_2 > 0} \frac{q^{n_1} n_2 q^{n_2}}{(1 - q^{n_1})(1 - q^{n_2})} \,,$$

$$\begin{bmatrix} 4, 2, 1 \\ 2, 0, 5 \end{bmatrix} = \frac{1}{3! \cdot 2! \cdot 5!} \sum_{n_1 > n_2 > n_3 > 0} \frac{n_1^2 (q^{3n_1} + 4q^{2n_1} + q^{n_1}) \cdot q^{n_2} \cdot n_3^5 q^{n_3}}{(1 - q^{n_1})^4 \cdot (1 - q^{n_1})^2 \cdot (1 - q^{n_1})^1} \,.$$

bi-brackets - filtrations

Filtrations

On $\mathcal{BD}$ we have the increasing filtrations $\mathrm{Fil}^{\mathrm{W}}_{\bullet}$ given by the upper weight, $\mathrm{Fil}^{\mathrm{D}}_{\bullet}$ given by the lower weight and $\mathrm{Fil}^{\mathrm{L}}_{\bullet}$ given by the length, i.e., we have for $A \subseteq \mathcal{BD}$

$$\mathrm{Fil}^{\mathrm{W}}_{k}(A) := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \in A \,\middle|\, 0 \leq l \leq k\,,\ s_1 + \dots + s_l \leq k \right\rangle_{\mathbb{Q}}$$

$$\mathrm{Fil}^{\mathrm{D}}_{k}(A) := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \in A \,\middle|\, 0 \leq l \leq k\,,\ r_1 + \dots + r_l \leq k \right\rangle_{\mathbb{Q}}$$

$$\mathrm{Fil}^{\mathrm{L}}_{l}(A) := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \in A \,\middle|\, r \leq l \right\rangle_{\mathbb{Q}}.$$

If we consider the length and weight filtration at the same time we use the short notation $\mathrm{Fil}^{\mathrm{W,L}}_{k,l} := \mathrm{Fil}^{\mathrm{W}}_{k}\,\mathrm{Fil}^{\mathrm{L}}_{l}$ and similar for the other filtrations.

For $r_1 = \cdots = r_l = 0$ we also write

$$\begin{bmatrix} s_1, \ldots, s_l \\ 0, \ldots, 0 \end{bmatrix} = [s_1, \ldots, s_l] =: \frac{1}{(s_1 - 1)! \ldots (s_l - 1)!} \sum_{n>0} \sigma_{s_1-1,\ldots,s_l-1}(n) q^n .$$

and denote the space spanned by all $[s_1, \ldots, s_l]$ and 1 by $\mathcal{MD} = \mathrm{Fil}_0^{\mathrm{D}}(\mathcal{BD})$.

We call the coefficients $\sigma_{s_1-1,\ldots,s_l-1}(n)$ **multiple divisor sums** and their generating series $[s_1, \ldots, s_l]$ will be called **brackets**.

These brackets have a direct connection to multiple zeta values and to the Fourier expansion of multiple Eisenstein series.

multiple divisor sums and modular forms

In the case $l = 1$ we get the classical divisor sums $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and

$$[k] = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \,.$$

These function appear in the Fourier expansion of classical Eisenstein series which are modular forms for $SL_2(\mathbb{Z})$, for example

$$G_2 = -\frac{1}{24} + [2] \,, \quad G_4 = \frac{1}{1440} + [4] \,, \quad G_6 = -\frac{1}{60480} + [6] \,.$$

We will see that we have an inclusion of algebras

$$M_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \mathcal{MD} \subset \mathcal{BD} \,,$$

where $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_4, G_6]$ and $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_2, G_4, G_6]$ are the algebras of modular forms and quasi-modular forms.

bi-brackets - differential

It is a well-known fact that the space of quasi-modular forms is closed under the operator $\mathrm{d} = q\frac{d}{dq}$. This is also true for the space $\mathcal{BD}$.
Since $\mathrm{d}\sum_{n>0} a_n q^n = \sum_{n>0} n a_n q^n$ one obtains:

Proposition

The operator d on $\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix}$ is given by

$$\mathrm{d}\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix} = \sum_{j=1}^{l} \left(s_j(r_j+1) \begin{bmatrix} s_1,\dots,s_{j-1},s_j+1,s_{j+1},\dots,s_l \\ r_1,\dots,r_{j-1},r_j+1,r_{j+1},\dots,r_l \end{bmatrix} \right).$$

Example:

$$\mathrm{d}[k] = k\begin{bmatrix} k+1 \\ 1 \end{bmatrix}, \qquad \mathrm{d}[s_1,s_2] = s_1\begin{bmatrix} s_1+1,s_2 \\ 1,0 \end{bmatrix} + s_2\begin{bmatrix} s_1,s_2+1 \\ 0,1 \end{bmatrix}.$$

Remark: It is more difficult to show that $\mathcal{MD}$ is also closed under d.

bi-brackets - generating series

Many statements on bi-brackets are obtained by using their generating function.

Definition

For the generating function of the bi-brackets we write

$$\left| \begin{matrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{matrix} \right| := \sum_{\substack{s_1,\dots,s_l>0 \\ r_1,\dots,r_l>0}} \begin{bmatrix} s_1, \dots, s_l \\ r_1 - 1, \dots, r_l - 1 \end{bmatrix} X_1^{s_1-1} \dots X_l^{s_l-1} \cdot Y_1^{r_1-1} \dots Y_l^{r_l-1}$$

bi-brackets - partition relation

Theorem (partition relation)

For all $l \geq 1$ we have

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \dots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}$$

This theorem gives linear relations between bi-brackts in a fixed length, for example

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} r+1 \\ s-1 \end{bmatrix} \quad \text{for all } r, s \in \mathbb{N},$$

$$\begin{bmatrix} 3,3 \\ 0,0 \end{bmatrix} = 6\begin{bmatrix} 1,1 \\ 0,4 \end{bmatrix} - 3\begin{bmatrix} 1,1 \\ 1,3 \end{bmatrix} + \begin{bmatrix} 1,1 \\ 2,2 \end{bmatrix},$$

$$\begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} = -2\begin{bmatrix} 2,2 \\ 0,2 \end{bmatrix} + \begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} - 4\begin{bmatrix} 3,1 \\ 0,2 \end{bmatrix} + 2\begin{bmatrix} 3,1 \\ 1,1 \end{bmatrix}.$$

Idea of proof: Interpret the sum as a sum over partitions and then use the conjugation of partitions. For this we will now introduce some notation.

bi-brackets - partition relation - idea of proof

By a partititon of a natural number n with l different parts we denote a representation of n as a sum of l different numbers, which are allowed to appear with some multiplicities.

For example

$$\begin{aligned} 15 &= 4+4+3+2+1+1 \\ &= 4\cdot 2+3\cdot 1+2\cdot 1+1\cdot 2 \end{aligned}$$

is a partition of 15 with the 4 different parts $4,3,2,1$ and multiplicities $2,1,1,2$.

We identify a partition of n with l different parts with a tupel $\binom{u}{v}$, with $u,v\in\mathbb{N}^l$.

- The u_j are the l different summands.
- The v_j count their appearence in the sum.

The above partition is therefore given by $\binom{u}{v}=\binom{4,3,2,1}{2,1,1,2}$.

bi-brackets - partition relation - idea of proof

We denote the set of all partition of n with l different parts by $P_l(n)$, i.e. we set

$$P_l(n) := \left\{ \binom{u}{v} \in \mathbb{N}^l \times \mathbb{N}^l \;\middle|\; n = u_1 v_1 + \dots + u_l v_l \,,\;\; u_1 > \dots > u_l > 0 \right\} .$$

With this the bi-brackets can be written as

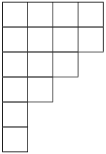
$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} := c \cdot \sum_{\substack{u_1 > \dots > u_l > 0 \\ v_1, \dots, v_l > 0}} u_1^{r_1} v_1^{s_1 - 1} \dots u_l^{r_l} v_l^{s_l - 1} q^{u_1 v_1 + \dots + u_l v_l}$$

$$= c \cdot \sum_{n > 0} \left(\sum_{\binom{u}{v} \in P_l(n)} u_1^{r_1} v_1^{s_1 - 1} \dots u_l^{r_l} v_l^{s_l - 1} \right) q^n .$$

On the set $P_l(n)$ we have an involution ρ given by the conjugation of partitions.

To see this one represents an element in $P_l(n)$ by a Young tableau.
In $P_4(15)$ we have for example

$$\begin{pmatrix} 4,3,2,1 \\ 2,1,1,2 \end{pmatrix} =$$



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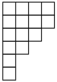
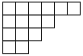
The conjugation ρ of this partition is given by

$$\begin{pmatrix} 4,3,2,1 \\ 2,1,1,2 \end{pmatrix} =$$



$$\xrightarrow{\quad\rho\quad}$$



$$= \begin{pmatrix} 6,4,3,2 \\ 1,1,1,1 \end{pmatrix}$$

bi-brackets - partition relation - idea of proof

We now can apply the conjugation ρ to the set $P_l(n)$ in the summation as in the following example

$$\begin{bmatrix} 2,2 \\ 0,0 \end{bmatrix} = \sum_{n>0} \left(\sum_{\binom{u}{v} \in P_2(n)} v_1 \cdot v_2 \right) q^n = \sum_{n>0} \left(\sum_{\binom{u'}{v'} = \rho\left(\binom{u}{v}\right) \in P_2(n)} v_1' \cdot v_2' \right) q^n$$

$$= \sum_{n>0} \left(\sum_{\binom{u'}{v'} = \rho\left(\binom{u}{v}\right) \in P_2(n)} u_2 \cdot (u_1 - u_2) \right) q^n$$

$$= \sum_{n>0} \left(\sum_{\binom{u}{v} \in P_2(n)} u_2 \cdot u_1 \right) q^n - \sum_{n>0} \left(\sum_{\binom{u}{v} \in P_2(n)} u_2^2 \right) q^n$$

$$= \begin{bmatrix} 1,1 \\ 1,1 \end{bmatrix} - 2 \begin{bmatrix} 1,1 \\ 0,2 \end{bmatrix}.$$

bi-brackets - partition relation - idea of proof

In general the conjugation ρ on the partitions $P_l(n)$ is explicitly given by

$$\rho : \begin{pmatrix} u_1, \dots, u_l \\ v_1, \dots, v_l \end{pmatrix} \longmapsto \begin{pmatrix} v_1 + \dots + v_l, \dots, v_1 + v_2, v_1 \\ u_l, u_{l-1} - u_l, \dots, u_1 - u_2 \end{pmatrix}.$$

The partition relation of bi-brackets follows by applying the conjugation ρ to the $P_l(n)$ in the summation of the generating function.

Now we have seen the main idea used in the proof of the partition relation

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \dots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}.$$

bi-brackets - algebra structure

Lemma

Set $L_n(X) = \frac{e^X q^n}{1-e^X q^n}$ then we have the following two statements

- The generating function of the bi-brackets can be written as

$$\left|\begin{matrix} X_1, \ldots, X_l \\ Y_1, \ldots, Y_l \end{matrix}\right| = \sum_{u_1 > \cdots > u_l > 0} \prod_{j=1}^{l} e^{u_j Y_j} L_{u_j}(X_j) \,.$$

- The product of the function L_n is given by

$$L_n(X) \cdot L_n(Y) = \sum_{k>0} \frac{B_k}{k!} (X-Y)^{k-1} \left(L_n(X) + (-1)^{k-1} L_n(Y) \right) + \frac{L_n(X) - L_n(Y)}{X - Y}$$

Proof: For the second statement one shows by direct calculation that

$$L_n(X) \cdot L_n(Y) = \frac{1}{e^{X-Y} - 1} L_n(X) + \frac{1}{e^{Y-X} - 1} L_n(Y)$$

and then uses the gen. series $\frac{X}{e^X - 1} = \sum_{n \geq 0} \frac{B_n}{n!} X^n$ of the Bernoulli numbers. □

bi-brackets - algebra structure - stuffle product

Proposition (stuffle product - special case of the algebra structure)

The product of the generating series in length one can be written as:

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \frac{1}{X_1 - X_2} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} - \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right)$$

$$+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (X_1 - X_2)^{k-1} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right).$$

Proof sketch: Do the following calculation and then use the second statement of the lemma to rewrite $L_n(X_1)L_n(X_2)$:

$$\begin{aligned}
\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} &= \sum_{n_1>0} e^{n_1 Y_1} L_n(X_1) \cdot \sum_{n_2>0} e^{n_2 Y_2} L_n(X_2) \\
&= \sum_{n_1>n_2>0} \dots + \sum_{n_2>n_1>0} \dots + \sum_{n_1=n_2>0} \dots \\
&= \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \sum_{n>0} e^{n(Y_1+Y_2)} L_n(X_1) L_n(X_2)
\end{aligned}$$

bi-brackets - algebra structure

Theorem

The space $\mathcal{BD}$ is a filtered $\mathbb{Q}$-algebra with a derivation given by d and

$$\mathrm{Fil}^{\mathrm{W,D,L}}_{k_1,d_1,l_1}(\mathcal{BD}) \cdot \mathrm{Fil}^{\mathrm{W,D,L}}_{k_2,d_2,l_2}(\mathcal{BD}) \subset \mathrm{Fil}^{\mathrm{W,D,L}}_{k_1+k_2,d_1+d_2,l_1+l_2}(\mathcal{BD})\,.$$

As in the case of multiple zeta values we also have two different ways, called - in analogy to multiple zeta values - stuffle ($\overset{st}{=}$) and shuffle ($\overset{sh}{=}$), of writing the product of two bi-brackets.

Examples:

$$[1]\cdot[1] = 2[1,1] + [2] - [1]$$

$$[1]\cdot\begin{bmatrix}1\\1\end{bmatrix} \overset{st}{=} \begin{bmatrix}1,1\\0,1\end{bmatrix} + \begin{bmatrix}1,1\\1,0\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}$$

$$[1]\cdot\begin{bmatrix}1\\1\end{bmatrix} \overset{sh}{=} \begin{bmatrix}1,1\\1,0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

Using the **stuffle product** and the **partition relation** we obtain a second representation for the product of the generating function which we call **shuffle product**:

Corollary (shuffle product)

The product of the generating series in length one can be written as:

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} = \begin{vmatrix} X_1 + X_2, X_1 \\ Y_2\,, Y_1 - Y_2 \end{vmatrix} + \begin{vmatrix} X_1 + X_2, X_2 \\ Y_1, Y_2 - Y_1 \end{vmatrix}$$
$$+ \frac{1}{Y_1 - Y_2} \left(\begin{vmatrix} X_1 + X_2 \\ Y_1 \end{vmatrix} - \begin{vmatrix} X_1 + X_2 \\ Y_2 \end{vmatrix} \right)$$
$$+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (Y_1 - Y_2)^{k-1} \left(\begin{vmatrix} X_1 + X_2 \\ Y_1 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_1 + X_2 \\ Y_2 \end{vmatrix} \right) .$$

Sketch of the proof: The partition relation in length one and two (P) and the stuffle product (st) states:

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \overset{P}{=} \begin{vmatrix} Y_1 \\ X_1 \end{vmatrix}, \quad \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} \overset{P}{=} \begin{vmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{vmatrix}, \quad \begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \dots$$

and therefore we get

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{P}{=} \begin{vmatrix} Y_1 \\ X_1 \end{vmatrix} \cdot \begin{vmatrix} Y_2 \\ X_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} Y_1, Y_2 \\ X_1, X_2 \end{vmatrix} + \dots \overset{P}{=} \begin{vmatrix} X_1 + X_2, X_1 \\ Y_2 \, , Y_1 - Y_2 \end{vmatrix} + \dots .$$

bi-brackets - stuffle product

Comparing the coefficients in the stuffle product of the generating function we obtain:

Proposition (explicit stuffle product)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{st}{=} \begin{bmatrix} s_1, s_2 \\ r_1, r_2 \end{bmatrix} + \begin{bmatrix} s_2, s_1 \\ r_2, r_1 \end{bmatrix} + \binom{r_1 + r_2}{r_1} \begin{bmatrix} s_1 + s_2 \\ r_1 + r_2 \end{bmatrix}$$
$$+ \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_1} \frac{(-1)^{s_2 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_1 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}$$
$$+ \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_2} \frac{(-1)^{s_1 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_2 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}$$

Notice: If $r_1 = r_2 = 0$, i.e. when the two brackets are elements in $\mathcal{MD}$, all elements on the right hand side are also elements in $\mathcal{MD}$.

Proposition (explicit shuffle product)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{aligned}
\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{sh}{=} & \sum_{\substack{1 \leq j \leq s_1 \\ 0 \leq k \leq r_2}} \binom{s_1+s_2-j-1}{s_1-j} \binom{r_1+r_2-k}{r_1} (-1)^{r_2-k} \begin{bmatrix} s_1+s_2-j, j \\ k, r_1+r_2-k \end{bmatrix} \\
& + \sum_{\substack{1 \leq j \leq s_2 \\ 0 \leq k \leq r_1}} \binom{s_1+s_2-j-1}{s_1-1} \binom{r_1+r_2-k}{r_1-k} (-1)^{r_1-k} \begin{bmatrix} s_1+s_2-j, j \\ k, r_1+r_2-k \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \begin{bmatrix} s_1+s_2-1 \\ r_1+r_2+1 \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \sum_{j=0}^{r_1} \frac{(-1)^{r_2} B_{r_1+r_2-j+1}}{(r_1+r_2-j+1)!} \binom{r_1+r_2-j}{r_1-j} \begin{bmatrix} s_1+s_2-1 \\ j \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \sum_{j=0}^{r_2} \frac{(-1)^{r_1} B_{r_1+r_2-j+1}}{(r_1+r_2-j+1)!} \binom{r_1+r_2-j}{r_2-j} \begin{bmatrix} s_1+s_2-1 \\ j \end{bmatrix}
\end{aligned}$$

bi-brackets - stuffle & shuffle product

Using the shuffle and stuffle product we obtain linear relations in $\mathcal{BD}$ which we call **double shuffle relations**.

Example:

$$\begin{aligned}
\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix} &\overset{st}{=} \begin{bmatrix}1,2\\3,4\end{bmatrix}+\begin{bmatrix}2,1\\4,3\end{bmatrix}-\frac{35}{2}\begin{bmatrix}2\\7\end{bmatrix}+35\begin{bmatrix}3\\7\end{bmatrix},\\
\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix} &\overset{sh}{=} -35\begin{bmatrix}1,2\\0,7\end{bmatrix}+15\begin{bmatrix}1,2\\1,6\end{bmatrix}-5\begin{bmatrix}1,2\\2,5\end{bmatrix}+\begin{bmatrix}1,2\\3,4\end{bmatrix}-5\begin{bmatrix}2,1\\1,6\end{bmatrix}\\
&\quad+5\begin{bmatrix}2,1\\2,5\end{bmatrix}-3\begin{bmatrix}2,1\\3,4\end{bmatrix}+\begin{bmatrix}2,1\\4,3\end{bmatrix}-\frac{1}{6048}\begin{bmatrix}2\\2\end{bmatrix}+\frac{1}{720}\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}2\\8\end{bmatrix}
\end{aligned}$$

bi-brackets - conjectures

The partition relation and the two ways of writing the product give a large family of linear relations in $\mathcal{BD}$ and we have the following conjecture:

Conjecture

- All linear relations between bi-brackets come from the partition relation and the double shuffle relations.
- Every bi-bracket can be written as a linear combination of brackets, i.e. the algebra $\mathcal{BD}$ is a subalgebra of $\mathcal{MD}$ and in particular it is

$$\mathrm{Fil}^{\mathrm{W,D,L}}_{k,d,l}(\mathcal{BD}) \subset \mathrm{Fil}^{\mathrm{W,L}}_{k+d,l+d}(\mathcal{MD}) .$$

The second part of the conjecture is interesting, because the elements in $\mathcal{MD}$ have a connection to multiple zeta values.

multiple zeta values

Definition

For natural numbers $s_1 \geq 2, s_2, ..., s_l \geq 1$ the multiple zeta value (MZV) of weight $s_1 + ... + s_l$ and length l is defined by

$$\zeta(s_1, ..., s_l) = \sum_{n_1 > ... > n_l > 0} \frac{1}{n_1^{s_1} \dots n_l^{s_l}} .$$

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle product). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s) .$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (extended double shuffle relations) between MZV. Conjecturally these are all relations between MZV.

Example:

$$\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

Compare this to the shuffle and stuffle product of bi-brackets:

$$[2,3]+3[3,2]+6[4,1]-3[4]+3\begin{bmatrix}4\\1\end{bmatrix} \overset{sh}{=} [2]\cdot[3] \overset{st}{=} [2,3]+[3,2]+[5]-\frac{1}{12}[3]\,.$$

bi-brackets - connections to mzv

Denote the space of all admissible brackets by

$$\mathrm{q}\mathcal{MZ} := \Big\langle [s_1, \ldots, s_l] \in \mathcal{MD} \;\Big|\; s_1 > 1 \Big\rangle_{\mathbb{Q}}.$$

It has a filtration given by the weight $k = s_1 + \cdots + s_l$.

Proposition

For $[s_1, \ldots, s_l] \in \mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ define the map Z_k by

$$Z_k\left([s_1, \ldots, s_l]\right) = \lim_{q \to 1} (1-q)^k [s_1, \ldots, s_l].$$

then it is

$$Z_k\left([s_1, \ldots, s_l]\right) = \begin{cases} \zeta(s_1, \ldots, s_l)\,, & s_1 + \cdots + s_l = k, \\ 0\,, & s_1 + \cdots + s_l < k\,. \end{cases}$$

The map Z_k is linear on $\mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$, i.e. relations in $\mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ give rise to relations between MZV.

Example:

$$[4] = 2[2,2] - 2[3,1] + [3] - \frac{1}{3}[2] \quad \overset{Z_4}{\Longrightarrow} \quad \zeta(4) = 2\zeta(2,2) - 2\zeta(3,1)\,.$$

bi-brackets - connections to mzv

All relations between MZV are in the kernel of Z_k and therefore we are interested in the elements of it.

Theorem

For the kernel of Z_k we have

- For $s_1 + \dots + s_l < k$ it is $Z_k([s_1, \dots, s_l]) = 0$.
- If $f \in \mathrm{Fil}^{\mathrm{W}}_{k-2}(\mathcal{MD})$ then $Z_k(\mathrm{d}(f)) = 0$.
- Every cusp form $f \in \mathrm{Fil}^{\mathrm{W}}_{k}(\mathcal{MD})$ is in the kernel of Z_k.

Remark: $Z_k\left(\begin{bmatrix} k-1 \\ 1 \end{bmatrix}\right) = 0$, since $\mathrm{d}[k-2] = (k-2)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}$.

bi-brackets - connections to mzv - example I

To get the first relation $\zeta(2,1) = \zeta(3)$ between MZV by using bi-brackets one considers the double shuffle relation for $[1] \cdot [2]$. It is:

$$[1,2] + 2[2,1] - [2] + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \overset{sh}{=} [1] \cdot [2] \overset{st}{=} [1,2] - \frac{1}{2}[2] + [2,1] + [3]$$

and therefore

$$[2,1] + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [3] + \frac{1}{2}[2] \,.$$

Since $[2], \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] \in \ker Z_3$ one obtains this relation by applying Z_3.

bi-brackets - connections to mzv - example II

We also rediscover exotic relations related to cusp forms, e.g. the cusp form $\Delta = q\prod_{n>0}(1-q^n)^{24}$ can be written as

$$\frac{-1}{2^6\cdot 5\cdot 691}\Delta = 168[5,7]+150[7,5]+28[9,3] + \frac{1}{1408}[2] - \frac{83}{14400}[4] + \frac{187}{6048}[6] - \frac{7}{120}[8] - \frac{5197}{691}[12]\,.$$

Letting Z_{12} act on both sides one obtains the relation

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7)+150\zeta(7,5)+28\zeta(9,3)\,.$$

These type of relations can also be explained via the theory of period polynomials (Gangl-Kaneko-Zagier; Schneps; Baumard; Pollack) or via multiple modular values (Brown).

bi-brackets - connections to mzv

To summarize we have the following objects in the kernel of Z_k, i.e. ways of getting relations between multiple zeta values using brackets.

- Elements of lower weights, i.e. elements in $\mathrm{Fil}^{\mathrm{W}}_{k-1}(\mathcal{MD})$.
- Derivatives
- Modular forms, which are cusp forms
- Since $0 \in \ker Z_k$, any linear relation between brackets in $\mathrm{Fil}^{\mathrm{W}}_{k}(\mathcal{MD})$ gives an element in the kernel.

But these are not all elements in the kernel of Z_k.

There are elements in the kernel of Z_k which can't be "described" by just using elements of $\mathcal{MD}$ in the list above.

bi-brackets - connections to mzv

In weight 4 one has the following relation of MZV

$$\zeta(4) = \zeta(2,1,1)\,,$$

i.e. it is $[4] - [2,1,1] \in \ker Z_4$. But this element can't be written as a linear combination of cusp forms, lower weight brackets or derivatives. But one can show that

$$[4] - [2,1,1] = \frac{1}{2}\left(\mathrm{d}[1] + \mathrm{d}[2]\right) - \frac{1}{3}[2] - [3] + \begin{bmatrix} 2,1 \\ 1,0 \end{bmatrix}$$

and $\begin{bmatrix} 2,1 \\ 1,0 \end{bmatrix} \in \ker Z_4$.

Conjecture (rough version)

The kernel of Z_k is spanned by the elements of the above list and (essentially) the bi-brackets with at least one $r_j \neq 0$.

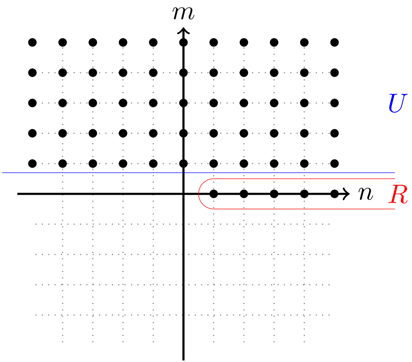
another application: multiple Eisenstein series

Let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$. We define an order $\succ$ on Λ_τ by setting

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \Lambda_\tau$ and the following set which we call the set of positive lattice points

$$P := \{m\tau + n \in \Lambda_\tau \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R$$



another application: multiple Eisenstein series

Definition

For $s_1 \geq 3, s_2, \dots, s_l \geq 2$ we define the *multiple Eisenstein series* of weight $k = s_1 + \cdots + s_l$ and length l by

$$G_{s_1,\dots,s_l}(\tau) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_l \succ 0 \\ \lambda_i \in \Lambda_\tau}} \frac{1}{\lambda_1^{s_1} \dots \lambda_l^{s_l}} \,.$$

- The multiple Eisenstein series fulfill the stuffle product, e.g.

$$G_r(\tau) \cdot G_s(\tau) = G_{r,s}(\tau) + G_{s,r}(\tau) + G_{r+s}(\tau) \,.$$

- We have a Fourier expansion of the form

$$G_{s_1,\dots,s_l}(\tau) = \zeta(s_1, \dots, s_l) + \sum_{n>0} a_n q^n, \qquad (q = e^{2\pi i \tau}) \,.$$

- In length $l = 1$ and weight k the multiple Eisenstein are the Eisenstein series

$$G_k(\tau) = \zeta(k) + (2\pi i)^k [k] \,.$$

another application: multiple Eisenstein series

Theorem

The Fourier expansion of multiple Eisenstein series equals a $\mathbb{Q}$-linear combination of multiple zeta values and brackets in $\mathcal{MD}$, i.e. there exist $a_{\underline{s}}(\underline{s}', \underline{s}'') \in \mathbb{Q}$ s.t.

$$G_{\underline{s}} = \zeta(\underline{s}) + \sum_{\underline{s}', \underline{s}''} a_{\underline{s}}(\underline{s}', \underline{s}'') \zeta(\underline{s}') \cdot (2\pi i)^{\mathrm{wt}(\underline{s}'')} [\underline{s}''] .$$

Examples:

$$G_{4,4}(\tau) = \zeta(4,4) + 20\zeta(6)(2\pi i)^2[2] + 3\zeta(4)(2\pi i)^4[4] + (2\pi i)^8[4,4] ,$$

$$\begin{aligned} G_{3,2,2}(\tau) = & \zeta(3,2,2) + \left(\frac{54}{5}\zeta(2,3) + \frac{51}{5}\zeta(3,2) \right) (2\pi i)^2[2] \\ & + \frac{16}{3}\zeta(2,2)(2\pi i)^3[3] + 3\zeta(3)(2\pi i)^4[2,2] + 4\zeta(2)(2\pi i)^5[3,2] \\ & + (2\pi i)^7[3,2,2] . \end{aligned}$$

another application: multiple Eisenstein series

Theorem (H.B., K. Tasaka - work in progress)

For all $s_1 \geq 2, s_2, \dots, s_l \geq 1$ there exist $a_{s_1,\dots,s_l}(\underline{s}', \underline{s}'') \in \mathbb{Q}$ and $f_{s_1,\dots,s_l}(\underline{s}'') \in \mathcal{BD}$ such that the holomorphic function on the upper half plane given by

$$E_{s_1,..,s_l} = \zeta(s_1,..,s_l) + \sum_{\underline{s}',\underline{s}''} a_{s_1,..,s_l}(\underline{s}', \underline{s}'')\zeta(\underline{s}') \cdot (2\pi i)^{\mathrm{wt}(\underline{s}'')} f_{s_1,\dots,s_l}(\underline{s}'') \,.$$

satisfies

- For $s_1 \geq 3, s_2, \dots, s_l \geq 2$ it is $E_{s_1,\dots,s_l} = G_{s_1,\dots,s_l}$, i.e. in this case they fulfill the stuffle product.
- The $E_{s_1,\dots,s_l}$ fulfill the shuffle product, i.e. it is for example

$$E_2 \cdot E_3 = E_{2,3} + 3E_{3,2} + 6E_{4,1} \,.$$

Sketch of proof: To prove this theorem we first identify the $a_{s_1,..,s_l}(\underline{s}', \underline{s}'')$ with the coefficients in the Goncharov coproduct. This observation together with the results mentioned in this talk on bi-brackets will then be used to define $E_{s_1,\dots,s_l}$.

summary

- bi-brackets are q-series whose coefficients are rational numbers given by sums over partitions.
- The space $\mathcal{BD}$ spanned by all bi-brackets form a differential $\mathbb{Q}$-algebra and there are two different ways to express a product of bi-brackets.
- This give rise to a lot of linear relations between bi-brackets and conjecturally every element in $\mathcal{BD}$ can be written as a linear combination of elements in $\mathcal{MD}$.
- This setup can also be seen as a combinatorial theory of modular forms. For example it follows directly by the double shuffle relations that G_4^2 is a multiple of G_8.
- The elements in $\mathcal{MD}$ have a connection to multiple zeta values and elements in the kernel of Z_k give rise to relations between them.
- Conjecturally the elements in the kernel of Z_k can be described by using bi-brackets.
- In a recent joint work with K. Tasaka it turns out that bi-brackets are also necessery to give a definition of "shuffle regularized multiple Eisenstein series".