Generating series of multiple divisor sums and other interesting q-series

Henrik Bachmann - University of Hamburg

1th July 2014

Content of this talk

- We are interested in a family of q-series which arises in a theory which combines multiple zeta values, partitions and modular forms.
- There are a lot of linear relations between these q-series. For example:

$$\sum_{n_1>n_2>0} \frac{q^{n_1} n_2 q^{n_2}}{(1-q^{n_1})(1-q^{n_2})} = \frac{1}{2}\sum_{n>0} \frac{n^2 q^n}{1-q^n} + \frac{1}{2}\sum_{n>0} \frac{n q^n}{1-q^n} - \sum_{n>0} \frac{n q^n}{(1-q^n)^2} .$$

- First we start by an elementary definition of these q-series using partitions...

Partitions

By a partititon of a natural number n with l different parts we denote a representation of n as a sum of l different numbers.

For example

$$15 = 4 + 4 + 3 + 2 + 1 + 1$$

is a partition of 15 with 4 different parts.

Partitions

By a partititon of a natural number n with l different parts we denote a representation of n as a sum of l different numbers.

For example

$$15 = 4 + 4 + 3 + 2 + 1 + 1$$

is a partition of 15 with 4 different parts.

We identify a partition of n with l different parts with a tupel $\binom{u}{v}$, with $u, v \in \mathbb{N}^l$.

- The u_j are the l different summands.
- The v_j count their appearence in the sum.

The above partition is therefore given by $\binom{u}{v} = \binom{4,3,2,1}{2,1,1,2}$.

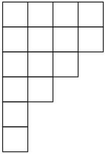
Partitions

We denote the set of all partition of n with l different parts by $P_l(n)$, i.e. we set

$$P_l(n) := \left\{ \binom{u}{v} \in \mathbb{N}^l \times \mathbb{N}^l \mid n = u_1 v_1 + \dots + u_l v_l \,, \;\; u_1 > \dots > u_l > 0 \right\} .$$

An element in $P_l(n)$ can be represented by a Young tableau. For example

$$\begin{pmatrix} 4,3,2,1 \\ 2,1,1,2 \end{pmatrix} =$$



$$\in P_4(15)$$

With this it is easy to see that this element gives rise to a different element $P_4(15)$:

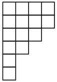
Partitions

We denote the set of all partition of n with l different parts by $P_l(n)$, i.e. we set

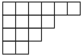
$$P_l(n) := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{N}^l \times \mathbb{N}^l \;\middle|\; n = u_1 v_1 + \dots + u_l v_l \,,\;\; u_1 > \dots > u_l > 0 \right\} .$$

On the set $P_l(n)$ we have an involution ρ given by the conjugation of partitionen

$$\begin{pmatrix} 4,3,2,1 \\ 2,1,1,2 \end{pmatrix} =$$



$$\xrightarrow{\qquad\qquad\rho\qquad\qquad}$$



$$= \begin{pmatrix} 6,4,3,2 \\ 1,1,1,1 \end{pmatrix}$$

On $P_l(n)$ this ρ is explicitly given by $\rho\left(\binom{u}{v}\right) = \binom{u'}{v'}$, where
$u'_j = v_1 + \dots + v_{l-j+1}$ and $v'_j = u_{l-j+1} - u_{l-j+2}$ with $u_{l+1} := 0$, i.e.

$$\rho : \begin{pmatrix} u_1, \dots, u_l \\ v_1, \dots, v_l \end{pmatrix} \longmapsto \begin{pmatrix} v_1 + \dots + v_l, \dots, v_1 + v_2, v_1 \\ u_l, u_{l-1} - u_l, \dots, u_1 - u_2 \end{pmatrix} .$$

We now want to construct q-series $\sum_{n>0} a_n q^n$, where the coefficients a_n are given by sums over $P_l(n)$. The map ρ will then give linear relations between these series.

Example Consider the following series

$$\sum_{n>0} \left(\sum_{\binom{u}{v} \in P_2(n)} v_1 \cdot v_2 \right) q^n$$

We now want to construct q-series $\sum_{n>0} a_n q^n$, where the coefficients a_n are given by sums over $P_l(n)$. The map ρ will then give linear relations between these series.

Example Consider the following series

$$\sum_{n>0}\left(\sum_{\binom{u}{v}\in P_2(n)} v_1\cdot v_2\right)q^n=\sum_{n>0}\left(\sum_{\rho\left(\binom{u}{v}\right)=\binom{u'}{v'}\in P_2(n)} u_2'\cdot(u_1'-u_2')\right)q^n$$

$$=\sum_{n>0}\left(\sum_{\binom{u}{v}\in P_2(n)} u_2\cdot u_1\right)q^n-\sum_{n>0}\left(\sum_{\binom{u}{v}\in P_2(n)} u_2^2\right)q^n\,.$$

bi-brackets

Definition

For $r_1,\dots,r_l \geq 0$, $s_1,\dots,s_l > 0$ and $c := (r_1!(s_1-1)!\dots r_l!(s_l-1)!)^{-1}$ we define th following q-series

$$\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix} := c \cdot \sum_{n>0} \left(\sum_{\binom{u}{v} \in P_l(n)} u_1^{r_1} v_1^{s_1-1} \dots u_l^{r_l} v_l^{s_l-1} \right) q^n$$

$$= \sum_{\substack{u_1>\dots>u_l>0 \\ v_1,\dots,v_l>0}} \frac{u_1^{r_1}}{r_1!} \dots \frac{u_l^{r_l}}{r_1!} \cdot \frac{v_1^{s_1-1} \dots v_l^{s_l-1}}{(s_1-1)! \dots (s_l-1)!} \cdot q^{u_1 v_1 + \dots + u_l v_l} \in \mathbb{Q}[[q]]$$

which we call **bi-brackets** of weight $r_1+\dots+r_k+s_1+\dots+s_l$, upper weight $s_1+\dots+s_l$, lower weight $r_1+\dots+r_l$ and length l. By $\mathcal{BD}$ we denote the $\mathbb{Q}$-vector space spanned by all bi-brackets and 1.

bi-brackets

The bi-brackets can also be written as

$$\begin{bmatrix} s_1, \ldots, s_l \\ r_1, \ldots, r_l \end{bmatrix} = c \cdot \sum_{n_1 > \cdots > n_l > 0} \frac{n_1^{r_1} P_{s_1-1}(q^{n_1}) \ldots n_l^{r_l} P_{s_l-1}(q^{n_l})}{(1-q^{n_1})^{s_1} \ldots (1-q^{n_l})^{s_l}} \,,$$

where the $P_{k-1}(t)$ are the Eulerian polynomials defined by

$$\frac{P_{k-1}(t)}{(1-t)^k} = \sum_{d>0} d^{k-1} t^d \,.$$

Examples:

$$P_0(t) = P_1(t) = t \,, P_2(t) = t^2 + t \,, P_3(t) = t^3 + 4t^2 + t \,,$$

$$\begin{bmatrix} 1, 1 \\ 0, 1 \end{bmatrix} = \sum_{n_1 > n_2 > 0} \frac{q^{n_1} n_2 q^{n_2}}{(1-q^{n_1})(1-q^{n_2})} \,.$$

For $r_1 = \cdots = r_l = 0$ we also write

$$\begin{bmatrix} s_1, \ldots, s_l \\ 0, \ldots, 0 \end{bmatrix} = [s_1, \ldots, s_l] =: \frac{1}{(s_1 - 1)! \ldots (s_l - 1)!} \sum_{n>0} \sigma_{s_1-1,\ldots,s_l-1}(n) q^n .$$

and denote the space spanned by all $[s_1, \ldots, s_l]$ and 1 by $\mathcal{MD}$. We call the coefficients $\sigma_{s_1-1,\ldots,s_l-1}(n)$ **multiple divisor sums**. In the case $l = 1$ these are the classical divisor sums $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and

$$[k] = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n .$$

multiple divisor sums and modular forms

For $r_1 = \cdots = r_l = 0$ we also write

$$\begin{bmatrix} s_1, \dots, s_l \\ 0, \dots, 0 \end{bmatrix} = [s_1, \dots, s_l] =: \frac{1}{(s_1-1)! \dots (s_l-1)!} \sum_{n>0} \sigma_{s_1-1,\dots,s_l-1}(n) q^n .$$

and denote the space spanned by all $[s_1, \dots, s_l]$ and 1 by $\mathcal{MD}$. We call the coefficients $\sigma_{s_1-1,\dots,s_l-1}(n)$ **multiple divisor sums**. In the case $l = 1$ these are the classical divisor sums $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and

$$[k] = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n .$$

These function appear in the Fourier expansion of classical Eisenstein series which are modular forms for $SL_2(\mathbb{Z})$, for example

$$G_2 = -\frac{1}{24} + [2] , \quad G_4 = \frac{1}{1440} + [4] , \quad G_6 = -\frac{1}{60480} + [6] .$$

We will see that we have an inclusion of algebras

$$M_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \mathcal{MD} \subset \mathcal{BD} ,$$

where $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_4, G_6]$ and $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_2, G_4, G_6]$ are the algebras of modular forms and quasi-modular forms.

bi-brackets - examples

$$[2] = \sum_{n>0} \sigma_1(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \dots,$$

$$\begin{bmatrix} 2,2 \\ 1,0 \end{bmatrix} = 2q^3 + 7q^4 + 23q^5 + 42q^6 + 89q^7 + 142q^8 + 221q^9 + 342q^{10} + \dots,$$

$$\begin{bmatrix} 2,2 \\ 0,1 \end{bmatrix} = q^3 + 3q^4 + 10q^5 + 16q^6 + 35q^7 + 52q^8 + 78q^9 + 120q^{10} + \dots,$$

$$\begin{bmatrix} 1,1,1 \\ 1,2,3 \end{bmatrix} = \frac{1}{12}\left(12q^6 + 28q^7 + 96q^8 + 481q^9 + 747q^{10} + 2042q^{11} + \dots\right),$$

$$[4,4,4] = \frac{1}{216}\left(q^6 + 9q^7 + 45q^8 + 190q^9 + 642q^{10} + 1899q^{11} + \dots\right),$$

$$[3,1,3,1] = \frac{1}{4}\left(q^{10} + 2q^{11} + 8q^{12} + 16q^{13} + 43q^{14} + 70q^{15} + \dots\right).$$

bi-brackets

- There are a lot of relations between bi-brackets. For example the discussion from the beginning gives
$$\begin{bmatrix} 2,2 \\ 0,0 \end{bmatrix} = \begin{bmatrix} 1,1 \\ 1,1 \end{bmatrix} - 2 \begin{bmatrix} 1,1 \\ 0,2 \end{bmatrix}.$$
- To obtain more relations we have to study the algebraic structure of the space $\mathcal{BD}$ which is "similar" to the algebraic structure of the space of multiple zeta values.
- The brackets $[s_1, \dots, s_l]$ have a direct connection to multiple zeta values and the Fourier expansion of multiple Eisenstein series.

Multiple zeta values

Definition

For natural numbers $s_1 \geq 2, s_2, ..., s_l \geq 1$ the multiple zeta value (MZV) of weight $s_1 + ... + s_l$ and length l is defined by

$$\zeta(s_1, ..., s_l) = \sum_{n_1 > ... > n_l > 0} \frac{1}{n_1^{s_1} \dots n_l^{s_l}} \,.$$

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle product). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r,s) + \zeta(s,r) + \zeta(r+s) \,.$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV. Conjecturally these are all relations between MZV.

Multiple zeta values

Example:

$$\zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2) + 6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV, which can be proven by using (extended) double shuffle relations. e.g.:

$$\begin{aligned}
\zeta(4) &= \zeta(2,1,1)\,,\\
\zeta(5) &= \zeta(4,1) + \zeta(3,2) + \zeta(2,3)\,,\\
16\zeta(3,2,2) &= 18\zeta(5,2) + 21\zeta(4,3) - 2\zeta(7)\,,\\
\frac{5197}{691}\zeta(12) &= 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)\,.
\end{aligned}$$

Filtrations

On $\mathcal{BD}$ we have the increasing filtrations $\mathrm{Fil}^{\mathrm{W}}_{\bullet}$ given by the upper weight,$\mathrm{Fil}^{\mathrm{D}}_{\bullet}$ given by the lower weight and $\mathrm{Fil}^{\mathrm{L}}_{\bullet}$ given by the length, i.e., we have for $A \subseteq \mathcal{BD}$

$$\mathrm{Fil}^{\mathrm{W}}_{k}(A) := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \in A \,\middle|\, 0 \leq l \leq k\,,\, s_1 + \dots + s_l \leq k \right\rangle_{\mathbb{Q}}$$

$$\mathrm{Fil}^{\mathrm{D}}_{k}(A) := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \in A \,\middle|\, 0 \leq l \leq k\,,\, r_1 + \dots + r_l \leq k \right\rangle_{\mathbb{Q}}$$

$$\mathrm{Fil}^{\mathrm{L}}_{l}(A) := \left\langle \begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} \in A \,\middle|\, r \leq l \right\rangle_{\mathbb{Q}}.$$

If we consider the length and weight filtration at the same time we use the short notation $\mathrm{Fil}^{\mathrm{W,L}}_{k,l} := \mathrm{Fil}^{\mathrm{W}}_{k} \mathrm{Fil}^{\mathrm{L}}_{l}$ and simlar for the other filtrations.

bi-brackets - algebra structure

Theorem

The space $\mathcal{BD}$ is a filtered differential $\mathbb{Q}$-algebra with the differential given by $\mathrm{d} = q\frac{d}{dq}$ and

$$\mathrm{Fil}^{\mathrm{W,D,L}}_{k_1,d_1,l_1}(\mathcal{BD}) \cdot \mathrm{Fil}^{\mathrm{W,D,L}}_{k_2,d_2,l_2}(\mathcal{BD}) \subset \mathrm{Fil}^{\mathrm{W,D,L}}_{k_1+k_2,d_1+d_2,l_1+l_2}(\mathcal{BD})\,.$$

As in the case of multiple zeta values we also have two different ways, also called stuffle and shuffle, of writing the product of two bi-brackets.

Examples:

$$[1]\cdot[1] = 2[1,1] + [2] - [1]$$

$$[1]\cdot\begin{bmatrix}1\\1\end{bmatrix} \overset{st}{=} \begin{bmatrix}1,1\\0,1\end{bmatrix} + \begin{bmatrix}1,1\\1,0\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix}$$

$$[1]\cdot\begin{bmatrix}1\\1\end{bmatrix} \overset{sh}{=} \begin{bmatrix}1,1\\1,0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

$$\mathrm{d}\begin{bmatrix}1,2\\3,4\end{bmatrix} = 4\begin{bmatrix}2,2\\4,4\end{bmatrix} + 10\begin{bmatrix}1,3\\3,5\end{bmatrix}.$$

bi-brackets - algebra structure

That the space $\mathcal{BD}$ is closed under $\mathrm{d} = q\frac{d}{dq}$ is easy to see, since $\mathrm{d}\sum_{n>0} a_n q^n = \sum_{n>0} n a_n q^n$ one obtains:

Proposition

The operater d on $\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix}$ is given by

$$\mathrm{d}\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix} = \sum_{j=1}^{l}\left(s_j(r_j+1)\begin{bmatrix} s_1,\dots,s_{j-1},s_j+1,s_{j+1},\dots,s_l \\ r_1,\dots,r_{j-1},r_j+1,r_{j+1},\dots,r_l \end{bmatrix}\right).$$

Example:

$$\mathrm{d}[k] = k\begin{bmatrix} k+1 \\ 1 \end{bmatrix}, \qquad \mathrm{d}[s_1,s_2] = s_1\begin{bmatrix} s_1+1, s_2 \\ 1,0 \end{bmatrix} + s_2\begin{bmatrix} s_1, s_2+1 \\ 0,1 \end{bmatrix}.$$

bi-brackets - generating series

To prove that $\mathcal{BD}$ is closed under multiplication we will consider the generating series of bi-brackets

Definition

For the generating function of the bi-brackets we write

$$\left|\begin{matrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{matrix}\right| := \sum_{\substack{s_1,\dots,s_l>0 \\ r_1,\dots,r_l>0}} \begin{bmatrix} s_1, \dots, s_l \\ r_1-1, \dots, r_l-1 \end{bmatrix} X_1^{s_1-1} \dots X_l^{s_l-1} \cdot Y_1^{r_1-1} \dots Y_l^{r_l-1}$$

bi-brackets - generating series

For $n \in \mathbb{N}$ set $L_n(X) := \frac{e^X q^n}{1-e^X q^n} \in \mathbb{Q}[[q,X]]$.

Theorem

For all $l \geq 1$ we have the following two expressions for the generating series of bi-brackets

$$\left|\begin{matrix} X_1,\dots,X_l \\ Y_1,\dots,Y_l \end{matrix}\right| = \sum_{u_1>\dots>u_l>0} \prod_{j=1}^{l} e^{u_j Y_j} L_{u_j}(X_j)$$

$$= \sum_{u_1>\dots>u_l>0} \prod_{j=1}^{l} e^{u_j (X_{l+1-j} - X_{l+2-j})} L_{u_j}(Y_1 + \dots + Y_{l-j+1})$$

(where $X_{l+1} := 0$)

Corollary (partition relation)

For all $l \geq 1$ we have

$$\left|\begin{matrix} X_1,\dots,X_l \\ Y_1,\dots,Y_l \end{matrix}\right| = \left|\begin{matrix} Y_1+\dots+Y_l,\dots,Y_1+Y_2,Y_1 \\ X_l, X_{l-1}-X_l,\dots,X_1-X_2 \end{matrix}\right|$$

The proof of the theorem uses the conjugation ρ which was given by

$$\rho : \begin{pmatrix} u_1, \dots, u_l \\ v_1, \dots, v_l \end{pmatrix} \longmapsto \begin{pmatrix} v_1 + \dots + v_l, \dots, v_1 + v_2, v_1 \\ u_l, u_{l-1} - u_l, \dots, u_1 - u_2 \end{pmatrix}$$

on the set of partitions $P_l(n)$.
It is therefore not surprising that we have the similar looking equality

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \dots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}.$$

bi-brackets - partition relation

Corollary (partition relation in length 1 and 2)

For $r, r_1, r_2 \geq 0$ and $s, s_1, s_2 > 0$ we obtain for the coefficients in the corollary before

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} r+1 \\ s-1 \end{bmatrix},$$

$$\begin{bmatrix} s_1, s_2 \\ r_1, r_2 \end{bmatrix} = \sum_{\substack{0 \leq j \leq r_1 \\ 0 \leq k \leq s_2 - 1}} (-1)^k \binom{s_1 - 1 + k}{k} \binom{r_2 + j}{j} \begin{bmatrix} r_2 + j + 1\,, r_1 - j + 1 \\ s_2 - 1 - k\,, s_1 - 1 + k \end{bmatrix}.$$

Examples:

$$\begin{bmatrix} 1,1 \\ 1,1 \end{bmatrix} = \begin{bmatrix} 2,2 \\ 0,0 \end{bmatrix} + 2 \begin{bmatrix} 3,1 \\ 0,0 \end{bmatrix},$$

$$\begin{bmatrix} 3,3 \\ 0,0 \end{bmatrix} = 6 \begin{bmatrix} 1,1 \\ 0,4 \end{bmatrix} - 3 \begin{bmatrix} 1,1 \\ 1,3 \end{bmatrix} + \begin{bmatrix} 1,1 \\ 2,2 \end{bmatrix},$$

$$\begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} = -2 \begin{bmatrix} 2,2 \\ 0,2 \end{bmatrix} + \begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} - 4 \begin{bmatrix} 3,1 \\ 0,2 \end{bmatrix} + 2 \begin{bmatrix} 3,1 \\ 1,1 \end{bmatrix}.$$

Lemma

For $L_n(X) = \frac{e^X q^n}{1-e^X q^n}$ we have

$$L_n(X) \cdot L_n(Y) = \coth\left(\frac{X-Y}{2}\right) \cdot \frac{L_n(X) - L_n(Y)}{2} - \frac{L_n(X) + L_n(Y)}{2}$$

$$= \sum_{k>0} \frac{B_k}{k!} (X-Y)^{k-1} \left(L_n(X) + (-1)^{k-1} L_n(Y)\right) + \frac{L_n(X) - L_n(Y)}{X-Y}$$

Proof: By definition it is

$$\coth(X) = \frac{e^X + e^{-X}}{e^X - e^{-X}} = 1 + \frac{2}{e^{2X} - 1}$$

and by direct calculation

$$L_n(X) \cdot L_n(Y) = \frac{1}{e^{X-Y} - 1} L_n(X) + \frac{1}{e^{Y-X} - 1} L_n(Y) .$$

This gives the first equation and the second one follows by the generating series of the Bernoulli numbers $\frac{X}{e^X - 1} = \sum_{n \geq 0} \frac{B_n}{n!} X^n$.

Proposition

The product of the generating series in length one can be written as:

("stuffle product of the generating series in length one")

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} = \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \frac{1}{X_1 - X_2} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} - \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right)$$
$$+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (X_1 - X_2)^{k-1} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right).$$

("shuffle product of the generating series in length one")

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} = \begin{vmatrix} X_1 + X_2, X_1 \\ Y_2 , Y_1 - Y_2 \end{vmatrix} + \begin{vmatrix} X_1 + X_2, X_2 \\ Y_1, Y_2 - Y_1 \end{vmatrix}$$
$$+ \frac{1}{Y_1 - Y_2} \left(\begin{vmatrix} X_1 + X_2 \\ Y_1 \end{vmatrix} - \begin{vmatrix} X_1 + X_2 \\ Y_2 \end{vmatrix} \right)$$
$$+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (Y_1 - Y_2)^{k-1} \left(\begin{vmatrix} X_1 + X_2 \\ Y_1 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_1 + X_2 \\ Y_2 \end{vmatrix} \right).$$

Sketch of the proof:

- For the stuffle product consider

$$\begin{aligned}\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} &= \sum_{n_1>0} e^{n_1 Y_1} L_n(X_1) \cdot \sum_{n_2>0} e^{n_2 Y_2} L_n(X_2) \\ &= \sum_{n_1>n_2>0} \dots + \sum_{n_2>n_1>0} \dots + \sum_{n_1=n_2>0} \dots \\ &= \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \sum_{n>0} e^{n(Y_1+Y_2)} L_n(X_1) L_n(X_2)\end{aligned}$$

and then use the lemma for the term $L_n(X_1)L_n(X_2)$.

- For the shuffle product first use the partition relation on the left hand side, i.e. use $\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} = \begin{vmatrix} Y_1 \\ X_1 \end{vmatrix}$ and then use the partition relation again on the right hand side.

Idea of proof for the algebra structure:

In order to proof that an arbitrary product of bi-brackets is again an element in $\mathcal{BD}$ one considers the product of the generating series in general

$$\left|\begin{matrix} X_1,\ldots,X_m \\ Y_1,\ldots,Y_m \end{matrix}\right| \cdot \left|\begin{matrix} X_{m+1},\ldots,X_l \\ Y_{m+1},\ldots,Y_l \end{matrix}\right| = \sum_{n_1>\cdots>n_m>0} \cdots \cdot \sum_{n_{m+1}>\cdots>n_l>0} \cdots$$

where we again consider all possible sums over shuffles of $n_1,\ldots,n_m$ and $n_{m+1},\ldots,n_l$ plus the sums with equalities $n_j = n_i$ with $1 \leq j \leq m$ and $m+1 \leq i \leq l$.

$$\sum_{n_1>n_2>0} \cdot \sum_{n_3>0} = \sum_{n_1>n_2>n_3>0} + \sum_{n_1>n_3>n_2>0} + \sum_{n_3>n_1>n_2>0} + \sum_{n_1=n_3>n_2>0} + \sum_{n_1>n_3=n_2>0} .$$

For each equality $n_j = n_i$ one uses the lemma to rewrite $L_{n_j}(X_j) \cdot L_{n_i}(X_i)$.

Corollary (stuffle product)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{st}{=} \begin{bmatrix} s_1, s_2 \\ r_1, r_2 \end{bmatrix} + \begin{bmatrix} s_2, s_1 \\ r_2, r_1 \end{bmatrix} + \binom{r_1 + r_2}{r_1} \begin{bmatrix} s_1 + s_2 \\ r_1 + r_2 \end{bmatrix}$$
$$+ \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_1} \frac{(-1)^{s_2 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_1 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}$$
$$+ \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_2} \frac{(-1)^{s_1 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_2 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}$$

Notice: If $r_1 = r_2 = 0$, i.e. when the two brackets are elements in $\mathcal{MD}$, all elements on the right hand side are also elements in $\mathcal{MD}$.

bi-brackets - shuffle product

Corollary (shuffle product)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{aligned}
\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{sh}{=} & \sum_{\substack{1 \leq j \leq s_1 \\ 0 \leq k \leq r_2}} \binom{s_1+s_2-j-1}{s_1-j} \binom{r_1+r_2-k}{r_1} (-1)^{r_2-k} \begin{bmatrix} s_1+s_2-j, j \\ k, r_1+r_2-k \end{bmatrix} \\
& + \sum_{\substack{1 \leq j \leq s_2 \\ 0 \leq k \leq r_1}} \binom{s_1+s_2-j-1}{s_1-1} \binom{r_1+r_2-k}{r_1-k} (-1)^{r_1-k} \begin{bmatrix} s_1+s_2-j, j \\ k, r_1+r_2-k \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \begin{bmatrix} s_1+s_2-1 \\ r_1+r_2+1 \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \sum_{j=0}^{r_1} \frac{(-1)^{r_2} B_{r_1+r_2-j+1}}{(r_1+r_2-j+1)!} \binom{r_1+r_2-j}{r_1-j} \begin{bmatrix} s_1+s_2-1 \\ j \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \sum_{j=0}^{r_2} \frac{(-1)^{r_1} B_{r_1+r_2-j+1}}{(r_1+r_2-j+1)!} \binom{r_1+r_2-j}{r_2-j} \begin{bmatrix} s_1+s_2-1 \\ j \end{bmatrix}
\end{aligned}$$

bi-brackets - stuffle & shuffle product

Using the shuffle and stuffle product we obtain linear relations in $\mathcal{BD}$.

Example:

$$\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix}\overset{st}{=}\begin{bmatrix}1,2\\3,4\end{bmatrix}+\begin{bmatrix}2,1\\4,3\end{bmatrix}-\frac{35}{2}\begin{bmatrix}2\\7\end{bmatrix}+35\begin{bmatrix}3\\7\end{bmatrix},$$

$$\begin{aligned}\begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}2\\4\end{bmatrix}\overset{sh}{=}&-35\begin{bmatrix}1,2\\0,7\end{bmatrix}+15\begin{bmatrix}1,2\\1,6\end{bmatrix}-5\begin{bmatrix}1,2\\2,5\end{bmatrix}+\begin{bmatrix}1,2\\3,4\end{bmatrix}-5\begin{bmatrix}2,1\\1,6\end{bmatrix}\\&+5\begin{bmatrix}2,1\\2,5\end{bmatrix}-3\begin{bmatrix}2,1\\3,4\end{bmatrix}+\begin{bmatrix}2,1\\4,3\end{bmatrix}-\frac{1}{6048}\begin{bmatrix}2\\2\end{bmatrix}+\frac{1}{720}\begin{bmatrix}2\\4\end{bmatrix}+\begin{bmatrix}2\\8\end{bmatrix}\end{aligned}$$

bi-brackets

This procedure works in general. For example the stuffle product for the generating series of length one and length two is given by

$$\begin{aligned}
\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2, X_3 \\ Y_2, Y_3 \end{vmatrix} &= \begin{vmatrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{vmatrix} + \begin{vmatrix} X_2, X_1, X_3 \\ Y_2, Y_1, Y_3 \end{vmatrix} + \begin{vmatrix} X_2, X_3, X_1 \\ Y_2, Y_3, Y_1 \end{vmatrix} \\
&+ \frac{1}{X_1 - X_2} \left(\begin{vmatrix} X_1, X_3 \\ Y_1 + Y_2, Y_3 \end{vmatrix} - \begin{vmatrix} X_2, X_3 \\ Y_1 + Y_2, Y_3 \end{vmatrix} \right) \\
&+ \frac{1}{X_1 - X_3} \left(\begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 + Y_3 \end{vmatrix} - \begin{vmatrix} X_2, X_3 \\ Y_2, Y_1 + Y_3 \end{vmatrix} \right) \\
&+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (X_1 - X_2)^{k-1} \left(\begin{vmatrix} X_1, X_3 \\ Y_1 + Y_2, Y_3 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_2, X_3 \\ Y_1 + Y_2, Y_3 \end{vmatrix} \right) \\
&+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (X_1 - X_3)^{k-1} \left(\begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 + Y_3 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_2, X_3 \\ Y_2, Y_1 + Y_3 \end{vmatrix} \right)
\end{aligned}$$

The partition relation and the two ways of writing the product give a large family of linear relations in $\mathcal{BD}$ and conjecturally these are all relations.
Indeed there are so many relations that numerical experiments suggest the following conjecture:

Conjecture

The algebra $\mathcal{BD}$ of bi-brackets is a subalgebra of $\mathcal{MD}$ and in particular it is

$$\mathrm{Fil}^{\mathrm{W,D,L}}_{k,d,l}(\mathcal{BD}) \subset \mathrm{Fil}^{\mathrm{W,L}}_{k+d,l+d}(\mathcal{MD})\,.$$

This conjecture is interesting, because the elements in $\mathcal{MD}$ have a connection to multiple zeta values.

bi-brackets - connections to mzv

Denote the space of all admissible brackets by

$$\mathrm{q}\mathcal{MZ} := \left\langle [s_1, \dots, s_l] \in \mathcal{MD} \;\middle|\; s_1 > 1 \right\rangle_{\mathbb{Q}}.$$

Proposition

For $[s_1, \dots, s_l] \in \mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ define the map Z_k by

$$Z_k\left([s_1, \dots, s_l]\right) = \lim_{q \to 1} (1-q)^k [s_1, \dots, s_l].$$

then it is

$$Z_k\left([s_1, \dots, s_l]\right) = \begin{cases} \zeta(s_1, \dots, s_l)\,, & s_1 + \dots + s_l = k, \\ 0\,, & s_1 + \dots + s_l < k\,. \end{cases}$$

The map Z_k is linear on $\mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$, i.e. relations in $\mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ give rise to relations between MZV.

Example:

$$[4] = 2[2,2] - 2[3,1] + [3] - \frac{1}{3}[2] \quad \overset{Z_4}{\Longrightarrow} \quad \zeta(4) = 2\zeta(2,2) - 2\zeta(3,1)\,.$$

bi-brackets - connections to mzv

All relations between MZV are in the kernel of Z_k and therefore we are interested in the elements of it.

Theorem

For the kernel of Z_k we have

- For $s_1 + \cdots + s_l < k$ it is $Z_k([s_1, \ldots, s_l]) = 0$.
- If $f \in \mathrm{Fil}^{\mathrm{W}}_{k-2}(\mathcal{MD})$ then $Z_k(\mathrm{d}(f)) = 0$.
- Every cusp form $f \in \mathrm{Fil}^{\mathrm{W}}_{k}(\mathcal{MD})$ is in the kernel of Z_k.

But these are not all elements in the kernel of Z_k.

There are elements in the kernel of Z_k which can't be "described" by just using elements of $\mathcal{MD}$ in the list above.

bi-brackets - connections to mzv

In weight 4 one has the following relation of MZV

$$\zeta(4) = \zeta(2,1,1)\,,$$

i.e. it is $[4] - [2,1,1] \in \ker Z_4$. But this element can't be written as a linear combination of cusp forms, lower weight brackets or derivatives. But one can show that

$$[4] - [2,1,1] = \frac{1}{2}\left(\mathrm{d}[1] + \mathrm{d}[2]\right) - \frac{1}{3}[2] - [3] + \begin{bmatrix} 2,1 \\ 1,0 \end{bmatrix}$$

and $\begin{bmatrix} 2,1 \\ 1,0 \end{bmatrix} \in \ker Z_4$.

In general one can show that most of the bi-brackets $\begin{bmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{bmatrix}$ where at least one $r_j \neq 0$ is in the kernel of Z_k.

Conjecture (rough version)

Every element in the kernel of Z_k can be desribed by using bi-brackets.

Summary

- bi-brackets are q-series whose coefficients are rational numbers given by sums over partitions.
- The space $\mathcal{BD}$ spanned by all bi-brackets form a differential $\mathbb{Q}$-algebra and there are two different ways to express a product of bi-brackets.
- This give rise to a lot of linear relations between bi-brackets and conjecturally every element in $\mathcal{BD}$ can be written as a linear combination of elements in $\mathcal{MD}$.
- The elements in $\mathcal{MD}$ have a connection to multiple zeta values and elements in the kernel of Z_k give rise to relations between them.
- Conjecturally the elements in the kernel of Z_k can be described by using bi-brackets.
- In a recent joint work with K. Tasaka it turns out that bi-brackets are also necessery to give a definition of "shuffle regularized multiple Eisenstein series".