Generating series of multiple divisor sums and other interesting q-series

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content of this talk

- We are interested in a family of q-series which arises in a theory which combines multiple zeta values, partitions and modular forms.
- There are a lot of linear relations between these q-series. For example:

$$\sum_{n_1>n_2>0} \frac{q^{n_1} n_2 q^{n_2}}{(1-q^{n_1})(1-q^{n_2})} = \frac{1}{2}\sum_{n>0} \frac{n^2 q^n}{1-q^n} + \frac{1}{2}\sum_{n>0} \frac{n q^n}{1-q^n} - \sum_{n>0} \frac{n q^n}{(1-q^n)^2}\,.$$

- We will see that the space spanned by these q-series form an algebra where the product can be written in two different ways which then yields linear relations.
- Linear relations between these series induce (conjecturally) all linear relations between multiple zeta values.

bi-brackets

Definition

For $r_1, \dots, r_l \geq 0$, $s_1, \dots, s_l > 0$ and $c := (r_1!(s_1-1)! \dots r_l!(s_l-1)!)^{-1}$ we define the following q-series

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} := c \cdot \sum_{\substack{u_1 > \dots > u_l > 0 \\ v_1, \dots, v_l > 0}} u_1^{r_1} v_1^{s_1 - 1} \dots u_l^{r_l} v_l^{s_l - 1} q^{u_1 v_1 + \dots + u_l v_l} ,$$

which we call **bi-brackets** of weight $s_1 + \dots + s_l + r_1 + \dots + r_l$, upper weight $s_1 + \dots + s_l$, lower weight $r_1 + \dots + r_l$ and length l.

By $\mathcal{BD}$ we denote the $\mathbb{Q}$-vector space spanned by all bi-brackets and 1.

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sum_{n>0} \sigma_1(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \dots ,$$

$$\begin{bmatrix} 1, 1, 1 \\ 1, 2, 3 \end{bmatrix} = \frac{1}{12} \left(12q^6 + 28q^7 + 96q^8 + 481q^9 + 747q^{10} + 2042q^{11} + \dots \right) .$$

bi-brackets

The bi-brackets can also be written as

$$\begin{bmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{bmatrix} = c \cdot \sum_{n_1 > \dots > n_l > 0} \frac{n_1^{r_1} P_{s_1-1}(q^{n_1}) \dots n_l^{r_l} P_{s_l-1}(q^{n_l})}{(1-q^{n_1})^{s_1} \dots (1-q^{n_l})^{s_l}} \,,$$

where the $P_{k-1}(t)$ are the Eulerian polynomials defined by

$$\frac{P_{k-1}(t)}{(1-t)^k} = \sum_{d>0} d^{k-1} t^d \,.$$

Examples:

$$P_0(t) = P_1(t) = t \,, \quad P_2(t) = t^2 + t \,, \quad P_3(t) = t^3 + 4t^2 + t \,,$$

$$\begin{bmatrix} 1, 1 \\ 0, 1 \end{bmatrix} = \sum_{n_1 > n_2 > 0} \frac{q^{n_1} n_2 q^{n_2}}{(1-q^{n_1})(1-q^{n_2})} \,,$$

$$\begin{bmatrix} 4, 2, 1 \\ 2, 0, 5 \end{bmatrix} = \frac{1}{3! \cdot 2! \cdot 5!} \sum_{n_1 > n_2 > n_3 > 0} \frac{n_1^2 (q^{3n_1} + 4q^{2n_1} + q^{n_1}) \cdot q^{n_2} \cdot n_3^5 q^{n_3}}{(1-q^{n_1})^4 \cdot (1-q^{n_1})^2 \cdot (1-q^{n_1})^1} \,.$$

For $r_1 = \cdots = r_l = 0$ we also write

$$\begin{bmatrix} s_1, \ldots, s_l \\ 0, \ldots, 0 \end{bmatrix} = [s_1, \ldots, s_l] =: \frac{1}{(s_1 - 1)! \ldots (s_l - 1)!} \sum_{n>0} \sigma_{s_1 - 1, \ldots, s_l - 1}(n) q^n \,.$$

and denote the space spanned by all $[s_1, \ldots, s_l]$ and 1 by $\mathcal{MD}$. We call the coefficients $\sigma_{s_1-1,\ldots,s_l-1}(n)$ **multiple divisor sums**. The brackets $[s_1, \ldots, s_l]$ have a direct connection to multiple zeta values and the Fourier expansion of multiple Eisenstein series.

In the case $l = 1$ these are the classical divisor sums $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and

$$[k] = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n .$$

These function appear in the Fourier expansion of classical Eisenstein series which are modular forms for $SL_2(\mathbb{Z})$, for example

$$G_2 = -\frac{1}{24} + [2] , \quad G_4 = \frac{1}{1440} + [4] , \quad G_6 = -\frac{1}{60480} + [6] .$$

We will see that we have an inclusion of algebras

$$M_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \mathcal{MD} \subset \mathcal{BD} ,$$

where $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_4, G_6]$ and $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_2, G_4, G_6]$ are the algebras of modular forms and quasi-modular forms.

bi-brackets - generating series

Many statements on bi-brackets are obtained by using their generating function.

Definition

For the generating function of the bi-brackets we write

$$\left|\begin{matrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{matrix}\right| := \sum_{\substack{s_1,\dots,s_l>0 \\ r_1,\dots,r_l>0}} \begin{bmatrix} s_1\,, \dots\,, s_l \\ r_1 - 1\,, \dots\,, r_l - 1 \end{bmatrix} X_1^{s_1-1} \dots X_l^{s_l-1} \cdot Y_1^{r_1-1} \dots Y_l^{r_l-1}$$

bi-brackets - partition relation

Theorem (partition relation)

For all $l \geq 1$ we have

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \dots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}$$

Idea of proof: Interpret the sum as a sum over partitions and then use the conjugation of partitions.

This theorem gives linear relations between bi-brackts in a fixed length, for example

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} r+1 \\ s-1 \end{bmatrix} \quad \text{for all } r, s \in \mathbb{N},$$

$$\begin{bmatrix} 3,3 \\ 0,0 \end{bmatrix} = 6 \begin{bmatrix} 1,1 \\ 0,4 \end{bmatrix} - 3 \begin{bmatrix} 1,1 \\ 1,3 \end{bmatrix} + \begin{bmatrix} 1,1 \\ 2,2 \end{bmatrix},$$

$$\begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} = -2 \begin{bmatrix} 2,2 \\ 0,2 \end{bmatrix} + \begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix} - 4 \begin{bmatrix} 3,1 \\ 0,2 \end{bmatrix} + 2 \begin{bmatrix} 3,1 \\ 1,1 \end{bmatrix}.$$

bi-brackets - algebra structure

Lemma

Set $L_n(X) = \frac{e^X q^n}{1-e^X q^n}$ then we have the following two statements

- The generating function of the bi-brackets can be written as

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \sum_{u_1 > \dots > u_l > 0} \prod_{j=1}^{l} e^{u_j Y_j} L_{u_j}(X_j) \,.$$

- The product of the function L_n is given by

$$L_n(X) \cdot L_n(Y) = \sum_{k>0} \frac{B_k}{k!} (X-Y)^{k-1} \left(L_n(X) + (-1)^{k-1} L_n(Y) \right) + \frac{L_n(X) - L_n(Y)}{X-Y}$$

Proof: For the second statement one shows by direct calculation that

$$L_n(X) \cdot L_n(Y) = \frac{1}{e^{X-Y}-1} L_n(X) + \frac{1}{e^{Y-X}-1} L_n(Y)$$

and then uses the gen. series $\frac{X}{e^X - 1} = \sum_{n \geq 0} \frac{B_n}{n!} X^n$ of the Bernoulli numbers. □

bi-brackets - algebra structure - stuffle product

Proposition (stuffle product - special case of the algebra structure)

The product of the generating series in length one can be written as:

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \frac{1}{X_1 - X_2} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} - \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right)$$
$$+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (X_1 - X_2)^{k-1} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right) .$$

Proof sketch: Do the following calculation and then use the second statement of the lemma to rewrite $L_n(X_1)L_n(X_2)$:

$$\begin{aligned} \begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} &= \sum_{n_1>0} e^{n_1 Y_1} L_n(X_1) \cdot \sum_{n_2>0} e^{n_2 Y_2} L_n(X_2) \\ &= \sum_{n_1>n_2>0} \dots + \sum_{n_2>n_1>0} \dots + \sum_{n_1=n_2>0} \dots \\ &= \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \sum_{n>0} e^{n(Y_1+Y_2)} L_n(X_1) L_n(X_2) \end{aligned}$$

bi-brackets - algebra structure

Theorem

The space $\mathcal{BD}$ is a filtered differential $\mathbb{Q}$-algebra (where the multiplication respects the filtration given by the weights and length) with the differential given by $\mathrm{d} = q\frac{d}{dq}$.

As in the case of multiple zeta values we also have two different ways, called - in analogy to multiple zeta values - stuffle ($\overset{st}{=}$) and shuffle ($\overset{sh}{=}$), of writing the product of two bi-brackets.

Examples:

$$[1] \cdot [1] = 2[1,1] + [2] - [1]$$

$$[1] \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \overset{st}{=} \begin{bmatrix} 1,1 \\ 0,1 \end{bmatrix} + \begin{bmatrix} 1,1 \\ 1,0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[1] \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \overset{sh}{=} \begin{bmatrix} 1,1 \\ 1,0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathrm{d} \begin{bmatrix} 1,2 \\ 3,4 \end{bmatrix} = 4 \begin{bmatrix} 2,2 \\ 4,4 \end{bmatrix} + 10 \begin{bmatrix} 1,3 \\ 3,5 \end{bmatrix}.$$

bi-brackets - algebra structure - shuffle product

Using the **stuffle product** and the **partition relation** we obtain a second representation for the product of the generating function which we call **shuffle product**:

Corollary (shuffle product)

The product of the generating series in length one can be written as:

$$\begin{aligned}\left|\begin{matrix}X_1\\Y_1\end{matrix}\right| \cdot \left|\begin{matrix}X_2\\Y_2\end{matrix}\right| &= \left|\begin{matrix}X_1+X_2, X_1\\Y_2\,, Y_1-Y_2\end{matrix}\right| + \left|\begin{matrix}X_1+X_2, X_2\\Y_1, Y_2-Y_1\end{matrix}\right| \\ &+ \frac{1}{Y_1-Y_2}\left(\left|\begin{matrix}X_1+X_2\\Y_1\end{matrix}\right| - \left|\begin{matrix}X_1+X_2\\Y_2\end{matrix}\right|\right) \\ &+ \sum_{k=1}^{\infty}\frac{B_k}{k!}(Y_1-Y_2)^{k-1}\left(\left|\begin{matrix}X_1+X_2\\Y_1\end{matrix}\right| + (-1)^{k-1}\left|\begin{matrix}X_1+X_2\\Y_2\end{matrix}\right|\right).\end{aligned}$$

bi-brackets - algebra structure - shuffle product

Sketch of the proof: The partition relation in length one and two (P) and the stuffle product (st) states:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} \overset{P}{=} \begin{bmatrix} Y_1 \\ X_1 \end{bmatrix}, \quad \begin{bmatrix} X_1, X_2 \\ Y_1, Y_2 \end{bmatrix} \overset{P}{=} \begin{bmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{bmatrix}, \quad \begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \dots$$

and therefore we get

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{P}{=} \begin{vmatrix} Y_1 \\ X_1 \end{vmatrix} \cdot \begin{vmatrix} Y_2 \\ X_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} Y_1, Y_2 \\ X_1, X_2 \end{vmatrix} + \dots \overset{P}{=} \begin{vmatrix} X_1 + X_2, X_1 \\ Y_2 , Y_1 - Y_2 \end{vmatrix} + \dots .$$

bi-brackets - stuffle product

Comparing the coefficients in the stuffle product of the generating function we obtain:

Proposition (explicit stuffle product)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{aligned}\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{st}{=} & \begin{bmatrix} s_1, s_2 \\ r_1, r_2 \end{bmatrix} + \begin{bmatrix} s_2, s_1 \\ r_2, r_1 \end{bmatrix} + \binom{r_1 + r_2}{r_1} \begin{bmatrix} s_1 + s_2 \\ r_1 + r_2 \end{bmatrix} \\ & + \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_1} \frac{(-1)^{s_2 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_1 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix} \\ & + \binom{r_1 + r_2}{r_1} \sum_{j=1}^{s_2} \frac{(-1)^{s_1 - 1} B_{s_1 + s_2 - j}}{(s_1 + s_2 - j)!} \binom{s_1 + s_2 - j - 1}{s_2 - j} \begin{bmatrix} j \\ r_1 + r_2 \end{bmatrix}\end{aligned}$$

Notice: If $r_1 = r_2 = 0$, i.e. when the two brackets are elements in $\mathcal{MD}$, all elements on the right hand side are also elements in $\mathcal{MD}$.

bi-brackets - shuffle product

Proposition (explicit shuffle product)

For $s_1, s_2 > 0$ and $r_1, r_2 \geq 0$ we have

$$\begin{aligned}
\begin{bmatrix} s_1 \\ r_1 \end{bmatrix} \cdot \begin{bmatrix} s_2 \\ r_2 \end{bmatrix} \overset{sh}{=} & \sum_{\substack{1 \leq j \leq s_1 \\ 0 \leq k \leq r_2}} \binom{s_1+s_2-j-1}{s_1-j} \binom{r_1+r_2-k}{r_1} (-1)^{r_2-k} \begin{bmatrix} s_1+s_2-j, j \\ k, r_1+r_2-k \end{bmatrix} \\
& + \sum_{\substack{1 \leq j \leq s_2 \\ 0 \leq k \leq r_1}} \binom{s_1+s_2-j-1}{s_1-1} \binom{r_1+r_2-k}{r_1-k} (-1)^{r_1-k} \begin{bmatrix} s_1+s_2-j, j \\ k, r_1+r_2-k \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \begin{bmatrix} s_1+s_2-1 \\ r_1+r_2+1 \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \sum_{j=0}^{r_1} \frac{(-1)^{r_2} B_{r_1+r_2-j+1}}{(r_1+r_2-j+1)!} \binom{r_1+r_2-j}{r_1-j} \begin{bmatrix} s_1+s_2-1 \\ j \end{bmatrix} \\
& + \binom{s_1+s_2-2}{s_1-1} \sum_{j=0}^{r_2} \frac{(-1)^{r_1} B_{r_1+r_2-j+1}}{(r_1+r_2-j+1)!} \binom{r_1+r_2-j}{r_2-j} \begin{bmatrix} s_1+s_2-1 \\ j \end{bmatrix}
\end{aligned}$$

bi-brackets - stuffle & shuffle product

Using the shuffle and stuffle product we obtain linear relations in $\mathcal{BD}$ which we call **double shuffle relations**.

Example:

$$\begin{bmatrix}1\\3\end{bmatrix} \cdot \begin{bmatrix}2\\4\end{bmatrix} \overset{st}{=} \begin{bmatrix}1,2\\3,4\end{bmatrix} + \begin{bmatrix}2,1\\4,3\end{bmatrix} - \frac{35}{2}\begin{bmatrix}2\\7\end{bmatrix} + 35\begin{bmatrix}3\\7\end{bmatrix},$$

$$\begin{bmatrix}1\\3\end{bmatrix} \cdot \begin{bmatrix}2\\4\end{bmatrix} \overset{sh}{=} -35\begin{bmatrix}1,2\\0,7\end{bmatrix} + 15\begin{bmatrix}1,2\\1,6\end{bmatrix} - 5\begin{bmatrix}1,2\\2,5\end{bmatrix} + \begin{bmatrix}1,2\\3,4\end{bmatrix} - 5\begin{bmatrix}2,1\\1,6\end{bmatrix}$$
$$+ 5\begin{bmatrix}2,1\\2,5\end{bmatrix} - 3\begin{bmatrix}2,1\\3,4\end{bmatrix} + \begin{bmatrix}2,1\\4,3\end{bmatrix} - \frac{1}{6048}\begin{bmatrix}2\\2\end{bmatrix} + \frac{1}{720}\begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}2\\8\end{bmatrix}$$

bi-brackets - conjectures

The partition relation and the two ways of writing the product give a large family of linear relations in $\mathcal{BD}$ and we have the following conjecture:

Conjecture

- All linear relations between bi-brackets come from the partition relation and the double shuffle relations.
- Every bi-bracket can be written as a linear combination of brackets, i.e. the algebra $\mathcal{BD}$ is a subalgebra of $\mathcal{MD}$.

The second part of the conjecture is interesting, because the elements in $\mathcal{MD}$ have a connection to multiple zeta values.

multiple zeta values

Definition

For natural numbers $s_1 \geq 2, s_2, ..., s_l \geq 1$ the multiple zeta value (MZV) of weight $s_1 + ... + s_l$ and length l is defined by

$$\zeta(s_1, ..., s_l) = \sum_{n_1 > ... > n_l > 0} \frac{1}{n_1^{s_1} \dots n_l^{s_l}} .$$

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle product). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s) .$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (extended double shuffle relations) between MZV. Conjecturally these are all relations between MZV.

Example:

$$\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV, which can be proven by using (extended) double shuffle relations. e.g.:

$$\begin{aligned}
\zeta(4) &= \zeta(2,1,1)\,,\\
\zeta(5) &= \zeta(4,1)+\zeta(3,2)+\zeta(2,3)\,,\\
16\zeta(3,2,2) &= 18\zeta(5,2)+21\zeta(4,3)-2\zeta(7)\,,\\
\frac{5197}{691}\zeta(12) &= 168\zeta(5,7)+150\zeta(7,5)+28\zeta(9,3)\,.
\end{aligned}$$

bi-brackets - connections to mzv

Denote the space of all admissible brackets by

$$\mathrm{q}\mathcal{MZ} := \left\langle [s_1,\dots,s_l] \in \mathcal{MD} \;\middle|\; s_1 > 1 \right\rangle_{\mathbb{Q}}.$$

It has a filtration given by the weight $k = s_1 + \dots + s_l$.

Proposition

For $[s_1,\dots,s_l] \in \mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ define the map Z_k by

$$Z_k\left([s_1,\dots,s_l]\right) = \lim_{q\to 1}(1-q)^k [s_1,\dots,s_l].$$

then it is

$$Z_k\left([s_1,\dots,s_l]\right) = \begin{cases} \zeta(s_1,\dots,s_l)\,, & s_1+\dots+s_l = k,\\ 0\,, & s_1+\dots+s_l < k\,. \end{cases}$$

The map Z_k is linear on $\mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$, i.e. relations in $\mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ give rise to relations between MZV.

Example:

$$[4] = 2[2,2] - 2[3,1] + [3] - \frac{1}{3}[2] \quad \stackrel{Z_4}{\Longrightarrow} \quad \zeta(4) = 2\zeta(2,2) - 2\zeta(3,1)\,.$$

All relations between MZV are in the kernel of Z_k and therefore we are interested in the elements of it.

Theorem

For the kernel of Z_k we have

- For $s_1 + \cdots + s_l < k$ it is $Z_k([s_1, \ldots, s_l]) = 0$.
- If $f \in \mathrm{Fil}^{\mathrm{W}}_{k-2}(\mathcal{MD})$ then $Z_k(\mathrm{d}(f)) = 0$.
- Every cusp form $f \in \mathrm{Fil}^{\mathrm{W}}_{k}(\mathcal{MD})$ is in the kernel of Z_k.

But these are not all elements in the kernel of Z_k.

There are elements in the kernel of Z_k which can't be "described" by just using elements of $\mathcal{MD}$ in the list above.

bi-brackets - connections to mzv

In weight 4 one has the following relation of MZV

$$\zeta(4) = \zeta(2,1,1)\,,$$

i.e. it is $[4] - [2,1,1] \in \ker Z_4$. But this element can't be written as a linear combination of cusp forms, lower weight brackets or derivatives. But one can show that

$$[4] - [2,1,1] = \frac{1}{2}\left(\mathrm{d}[1] + \mathrm{d}[2]\right) - \frac{1}{3}[2] - [3] + \begin{bmatrix} 2,1 \\ 1,0 \end{bmatrix}$$

and $\left[\begin{smallmatrix} 2,1 \\ 1,0 \end{smallmatrix}\right] \in \ker Z_4$.

Conjecture (rough version)

The kernel of Z_k is spanned by the elements of the above list and (essentially) the bi-brackets with at least one $r_j \neq 0$.

summary

- bi-brackets are q-series whose coefficients are rational numbers given by sums over partitions.
- The space $\mathcal{BD}$ spanned by all bi-brackets form a differential $\mathbb{Q}$-algebra and there are two different ways to express a product of bi-brackets.
- This give rise to a lot of linear relations between bi-brackets and conjecturally every element in $\mathcal{BD}$ can be written as a linear combination of elements in $\mathcal{MD}$.
- This setup can also be seen as a combinatorial theory of modular forms. For example it follows directly by the double shuffle relations that G_4^2 is a multiple of G_8.
- The elements in $\mathcal{MD}$ have a connection to multiple zeta values and elements in the kernel of Z_k give rise to relations between them.
- Conjecturally the elements in the kernel of Z_k can be described by using bi-brackets.
- In a recent joint work with K. Tasaka it turns out that bi-brackets are also necessery to give a definition of "shuffle regularized multiple Eisenstein series".