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Topology Seminar, Winter 2020/21

Seminar: introduction to ℓ^2-invariants

Given an invariant for finite CW-complexes, one can get a more sophisticated version by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.

For example, the Betti numbers of a compact CW complex are defined as the dimensions of the homology vector spaces. When passing to the universal covering, we have infinitely generated objects, but using a normalised dimension one can define the so called *ℓ^2-Betti numbers*, which are positive real numbers containing interesting informations.

The key tool comes from functional analysis: the action of the deck transformation group can be normalised by means of the *von Neumann dimension.*

On one hand ℓ^2-invariants share many properties of their classical counterparts: for instance, the ℓ^2-Betti numbers are homotopy invariants. On the other hand, they are interesting for infinite coverings and provide new informations on the asymptotic properties of the space. In the setting of complete manifolds, one can deal with analytically defined ℓ^2-invariants.

The goal of the seminar is to get acquainted with the subject and learn some of the applications. We shall mainly follow the presentation of the recent lecture notes by Holger Kammeyer [Ka], which assume no prior knowledge of the functional analytic tools needed in the seminar.

Since ℓ^2-invariants pop up in different context (geometric group theory, Riemannian geometry, operator algebras) the last talks can be selected depending on the interest of the participants.

Prerequisites: a basic background in algebraic topology (fundamental group, covering spaces, homology and cohomology) and basics of functional analysis (Hilbert spaces, bounded operators).

Contact:

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Time and place:

Tuesday 14:15-15:45 via Zoom.
Please contact me by email if you are interested!

Preliminary plan of the talks:

1. **Introduction**
   Overview of the topic and the results, discussion on the organisation.

2. **Hilbert spaces, group von Neumann algebra** [Ka, Sections 2.1 and 2.2]
   Recall Hilbert spaces and their basic properties (include Example 2.14 and Theorem 2.15), bounded operators, adjoint, Riesz Lemma (Thm 2.18). The group von Neumann algebra $\mathcal{R}(G)$.

3. **von Neumann trace and dimension** [Ka, Section 2.3]
   von Neumann trace on $\mathcal{R}(G)$ and examples. Hilbert G-modules, properties of the trace, von Neumann dimension.

4. **G-CW-complexes and the cellular ℓ^2-chain complex** [Ka, Section 3.1, 3.2]
   G-CW-complexes, ℓ^2-completion, proper and finite type G-CW complexes.

5. **Cellular ℓ^2-Betti numbers and how to compute them** [Ka, Section 3.3]
   Reduced ℓ^2-homology, ℓ^2-Betti numbers, examples of computations. The ℓ^2-Euler–Poincaré formula, the Singer and Hopf conjectures.

6. **More properties ℓ^2-Betti numbers** [Ka, Sections 3.4 and 3.5]
Cohomological ℓ^2-Betti numnbers, ℓ^2-Hodge–de Rham decomposition. Atiyah's question on the values of ℓ^2-Betti numbers (cfr. also [Eck, page 209])

7. **L^2-Betti numbers as obstructions** [Ka, Section 3.6]
Obstructions for nontrivial self-coverings, mapping torus structures, circle actions.

8. **Extended von Neumann dimension** [Ka, Sections 4.1, 4.2]
Classifying spaces. Extended von Neumann dimension of a left $\mathcal{R}(G)$-module. Properties.

9. **ℓ^2-Betti numbers of groups** [Ka, Sections 4.3, 4.4]
ℓ^2-Betti numbers of G-spaces. ℓ^2-Betti numbers of groups, examples and ways to compute them.

10. **Euler characteristic of amenable groups** [Eck, Section 4.2 until Cor 4.3.4]
Definition of amenable group, characterisations via Følner condition. Vanishing of ℓ^2-Betti numbers for amenable groups (see also [Lu, 6.75]).

11. **Bounding the deficiency of finitely presented groups** [Ka, Section 4.5.2] and [Eck, Section 4.1]
Deficiency of a finitely presented group. Bound in terms of ℓ^2-Betti numbers. Examples.

12. **The analytic approach** [Lu, 1.3 until beginning of 1.3.2, Def. 1.71, Lemma 1.72]
On a complete Riemannian manifold, definition of L^2-harmonic forms, L^2-Hodge–de Rham decomposition and analytic definition of L^2-cohomology. Mention the applications to ℓ^2-Betti numbers of hyperbolic manifolds [Lu, Thm 1.62] using L^2-Hodge–de Rham theorem [Lu, Thm 1.59] (without proof).

13. possible subjects: a survey of Lück's approximation theorem [Ka, Ch. 5] ; Atiyah's question and Kaplansky's conjecture [Ka, Section 3.5];
or on the analytic side: survey of Atiyah's L^2-index theorem for coverings [At]; survey of the proof of L^2-Hodge–de Rham theorem [Lu, Thm 1.59], [Do]; other ℓ^2-invariants [Lu].

References

[Ka] Holger Kammeyer, *Introduction to ℓ^2-invariants*, Lecture notes in mathematics 2247, Springer (2019)
download from the campus network: `https://link-1springer-1com-10039ffhv0043.emedien3.sub.uni-hamburg.de/book/10.1007%2F978-3-030-28297-4`
(main source)

— Other references:

[At] M.F. Atiyah, Elliptic operators discrete groups and von Neumann algebras, *Astérisque* **32-33** (1976), 43–72.

[Do] J. Dodziuk, De Rham–Hodge theory for L^2-cohomology of infinite coverings, *Topology* **16** (1977), 157–165.

[Do2] J. Dodziuk, L^2-harmonic forms on rotationally symmetric Riemannian manifolds, *Proc. AMS* **77** (3) (1979), 395–400.

[Eck] B. Eckmann, Notes *Introduction to L^2-methods in topology: Reduced L^2-homology, harmonic chains, L^2-Betti numbers*, Israel J. of Math. 117 (2000), 183–219. The article can be downloaded at `https://link.springer.com/article/10.1007/BF02773570`

[Lu] Wolfgang Lück, *L^2-Invariants, Theory and Applications to Geometry and K-Theory*, Springer Verlag (2002)

[Pa] Pierre Pansu, *Introduction to L^2-Invariants*, notes available at
`https://www.imo.universite-paris-saclay.fr/~pansu/pansu_IntroToL2BettiNumbers.pdf`