The Ooguri-Vafa space as a moduli space of framed wild harmonic bundles

Ivan Tulli

Universität Hamburg

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Answer: Yes! The Ooguri-Vafa space can be interpreted as a certain class of (framed) wild harmonic bundles.

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- Furthermore, on the total space X we can define the three real symplectic forms:

$$\omega_j = (\frac{i}{2\pi}\Theta) \wedge \pi^* dx^j + \pi^*(V \star dx^j) \quad (5)$$

- Finally, from the ω_i we can obtain the I_i's and

$$g = V^{-1}(\frac{i}{2\pi}\Theta)\otimes(\frac{i}{2\pi}\Theta) + V\pi^*(dx^1\otimes dx^1 + dx^2\otimes dx^2 + dx^3\otimes dx^3) \tag{6}$$

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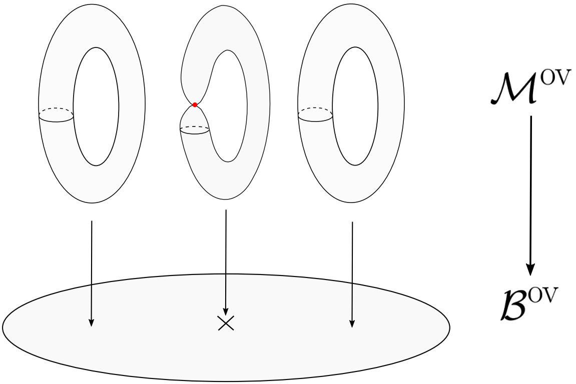

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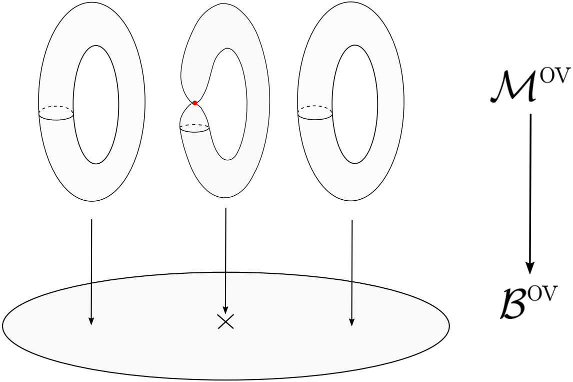

$\rightsquigarrow$ should think of $\mathcal{M}^{\mathrm{ov}}(\Lambda)$ as a "model HK space".

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- The reason for including framings in not obvious at this point, but it will become clear in the future.

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We call g a **compatible frame**; and the **equivalence classes** of $\mathcal{H}^{\mathrm{fr}}$ we denote by $\mathfrak{X}^{\mathrm{fr}}$.

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$$E = \mathbb{C}P^1 \times \mathbb{C}^2, \quad \overline{\partial}_E = \overline{\partial}, \quad \theta = zHdz, \quad h(e_i, e_j) = \delta_{ij}, \quad g = (e_1, e_2)|_\infty$$

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- Manifestation of the fact that (M, g, I_1, I_2, I_3) can be encoded holomorphically in the associated **twistor space of** M $\rightsquigarrow (\mathcal{Z}(M), \mathcal{I}, \Omega, \tau)$.

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  - **Remark:** we need framings so that Stokes data can be used as coordinates.

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Theorem [I.T.]: If $\Lambda = 4i$, then $\mathfrak{X}^{\text{fr}}(4i)$ can be identified with $\mathcal{M}^{\text{ov}}(4i)$. Under this identification $\mathfrak{X}^{\text{fr}}(4i)$ gets an induced hyperkähler structure, whose twistor family of holomorphic symplectic forms $\Omega(\xi)$ is described by

$$\Omega(\xi) = -\frac{1}{4\pi^2}\frac{d\mathcal{X}_e(\xi)}{\mathcal{X}_e(\xi)} \wedge \frac{d\mathcal{X}_m(\xi)}{\mathcal{X}_m(\xi)} \quad \text{for} \quad \xi \in \mathbb{C}^* \tag{15}$$

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- The tuple $(\mathcal{E}, \nabla, \tau)$ will be called a **framed meromorphic connection**.

Let $(\mathcal{E}, \nabla, \tau) \to (\mathbb{C}P^1, \infty)$ be as before.

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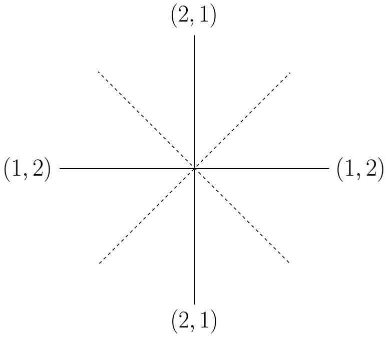
- The corresponding frames of flat sections exist on sectors determined by two consecutive Stokes rays. These have opening $\pi/2 + \pi/(k-1)$.

- We illustrate an example below, where

$$Q = \frac{1}{w^2} H = \mathrm{diag}(1/w^2, -1/w^2). \tag{19}$$

In this case $k = 3$, so we have 4 **Stokes rays** (the dotted rays bellow) and 4 sectors (determined by two Stokes rays with opening π).



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- $(S_1, ..., S_{2k-2}, \Lambda)$ completely characterizes the equivalence classes $[\mathcal{E}, \nabla, \tau]$ with fixed formal type (Q, Λ).

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- Because the Hitchin equation is satisfied, the connections

$$\nabla^{\xi} := D(\overline{\partial}_E, h) + \xi^{-1}\theta + \xi\theta^{\dagger_h} \text{ for } \xi \in \mathbb{C}^* \tag{21}$$

  define flat bundles $(E, \nabla^{\xi}) \to \mathbb{C}P^1 - \{\infty\}$.

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- Issue: there is no unique way to achieve this. The following filtered structure will allow us to consider all such possible extensions "at the same time".

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- We call $(\mathcal{P}^h_* \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*) \to (\mathbb{C}P^1, \infty)$ for $\xi \in \mathbb{C}^*$ the associated **framed filtered flat bundles**.

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For each $\xi \in \mathbb{C}^*$, let $(\mathcal{P}^h_* \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*) \to (\mathbb{C}P^1, \infty)$ be the framed filtered flat bundle associated to $(E, \overline{\partial}_E, \theta, h, g) \in \mathcal{H}^{\mathrm{fr}}$.

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- $S_i's$ and M_0 only depends $[\mathcal{P}^h_* \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*] \implies$ can associate $S_i(\xi)$ and $M_0(\xi)$ for $\xi \in \mathbb{C}^*$ to $[E, \overline{\partial}_E, \theta, h, g] \in \mathfrak{X}^{\mathrm{fr}}$.

Stokes data of a framed filtered flat bundle

For each $\xi \in \mathbb{C}^*$, let $(\mathcal{P}^h_* \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*) \to (\mathbb{C}P^1, \infty)$ be the framed filtered flat bundle associated to $(E, \overline{\partial}_E, \theta, h, g) \in \mathcal{H}^{\mathrm{fr}}$.

- For each $a \in \mathbb{R}$, we have the Stokes data associated to $(\mathcal{P}^h_a \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_a) \to (\mathbb{C}P^1, \infty)$.
- The S_i's **do not** depend on $a \in \mathbb{R}$, while Λ **does** depend on $a \in \mathbb{R}$. However, $M_0 = e^{-2\pi i \Lambda}$ **does not** depend on $a \in \mathbb{R}$.
- We associate S_i's and $M_0 = e^{-2\pi i \Lambda}$ to $(\mathcal{P}^h_* \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*) \to (\mathbb{C}P^1, \infty)$.
- $S_i' s$ and M_0 only depends $[\mathcal{P}^h_* \mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*] \implies$ can associate $S_i(\xi)$ and $M_0(\xi)$ for $\xi \in \mathbb{C}^*$ to $[E, \overline{\partial}_E, \theta, h, g] \in \mathfrak{X}^{\mathrm{fr}}$.
- The twistor family of Stokes data $(S_1(\xi), S_2(\xi), S_3(\xi), S_4(\xi), M_0(\xi))$, satisfies:
$$S_1(\xi) S_2(\xi) S_3(\xi) S_4(\xi) M_0^{-1}(\xi) = 1 \tag{23}$$

The electric twistor coordinate in $\mathfrak{X}^{\mathrm{fr}}$

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- On $\mathcal{M}^{\mathrm{ov}}(\Lambda)$:

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- On $\mathcal{M}^{\mathrm{ov}}(\Lambda)$:

$$\mathcal{X}_e^{\mathrm{ov}}(\xi) := \exp\left(\frac{\pi}{\xi} z + i\theta_e + \pi\xi\bar{z}\right) \quad z \in \mathcal{B}^{\mathrm{ov}},\ \theta_e = 2\pi x^3 \qquad (24)$$

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- For $[\mathcal{P}^h_*\mathcal{E}^\xi, \nabla^\xi, \tau^\xi_*]$ corresponding to $[E, \overline{\partial}_E, \theta, h, g] \in \mathfrak{X}^{\mathrm{fr}}$, M_0 equals:

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$$\exp\begin{bmatrix} -2\pi i(-\xi^{-1}m + m^{(3)} + \xi\overline{m}) & 0 \\ 0 & -2\pi i(\xi^{-1}m - m^{(3)} - \xi\overline{m}) \end{bmatrix} \tag{25}$$

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- We then define

$$\mathcal{X}_e([E, \overline{\partial}_E, \theta, h, g], \xi) := \exp(-2\pi i(\xi^{-1}m - m^{(3)} - \xi\overline{m})) \tag{26}$$

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- Get correspondence:

$$z \iff -2im, \qquad \theta_e \iff 2\pi m^{(3)} \tag{27}$$

The magnetic twistor coordinate of $\mathcal{M}^{\mathrm{ov}}(\Lambda)$

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$$\mathcal{X}_m^{\mathrm{sf}}(\xi) = \exp\Big(\frac{1}{\xi}\frac{(z\mathrm{Log}(z/\Lambda) - z)}{2i} + i\theta_m - \xi\frac{(\bar{z}\mathrm{Log}(\bar{z}/\bar{\Lambda}) - \bar{z})}{2i}\Big) \qquad (28)$$

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We write $\mathcal{X}^{\text{ov}}_m(\xi) = \mathcal{X}^{\text{sf}}_m(\xi)\mathcal{X}^{\text{inst}}_m(\xi)$ with

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and

$$\begin{aligned}\mathcal{X}^{\text{inst}}_m(\xi) =& \exp\Big(\frac{i}{4\pi}\int_{l_+(z)}\frac{d\xi'}{\xi'}\frac{\xi+\xi'}{\xi'-\xi}\text{Log}(1-\mathcal{X}^{\text{ov}}_e(\xi')) \\ & - \frac{i}{4\pi}\int_{l_-(z)}\frac{d\xi'}{\xi'}\frac{\xi+\xi'}{\xi'-\xi}\text{Log}(1-(\mathcal{X}^{\text{ov}}_e(\xi'))^{-1})\Big)\end{aligned} \qquad (29)$$

where

$$l_\pm(z) = \{\xi \in \mathbb{C}^* \mid \pm z/\xi < 0\} \qquad (30)$$

Key properties of the magnetic twistor coordinate

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- **Jumps**:

$$\mathcal{X}_m^{\mathrm{ov}}(\xi)^+ = \mathcal{X}_m^{\mathrm{ov}}(\xi)^-(1 - \mathcal{X}_e^{\mathrm{ov}}(\xi))^{-1} \quad \text{along} \quad \xi \in l_+(z)$$
$$\mathcal{X}_m^{\mathrm{ov}}(\xi)^+ = \mathcal{X}_m^{\mathrm{ov}}(\xi)^-(1 - \mathcal{X}_e^{\mathrm{ov}}(\xi)^{-1}) \quad \text{along} \quad \xi \in l_-(z) \tag{31}$$

where the $+$ or $-$ on the coordinate denotes the clockwise or counterclockwise limit to the ray, respectively.

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- **Asymptotics**:

$$\mathcal{X}_m^{\text{ov}}(\xi) \sim \begin{cases} \exp(-\frac{i}{2\xi}(z\text{Log}(z/\Lambda) - z) + i\theta_m + r(z,\theta_e)) & \text{as } \xi \to 0 \\ \exp(\frac{i\xi}{2}(\bar{z}\text{Log}(\bar{z}/\bar{\Lambda}) - \bar{z}) + i\theta_m - r(z,\theta_e)) & \text{as } \xi \to \infty \end{cases} \tag{32}$$

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$$\mathcal{X}_m^{\text{ov}}(\xi) = \overline{\mathcal{X}_m^{\text{ov}}(-1/\bar{\xi})}^{-1} \tag{33}$$

Key properties of the magnetic twistor coordinate

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This properties **uniquely** determine $\mathcal{X}_m^{\rm ov}(\xi)$! They are used to determine the analogous magnetic coordinate for $\mathfrak{X}^{\rm fr}$.

Magnetic twistor coordinate on $\mathfrak{X}^{\mathrm{fr}}$

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We consider $(S_1(\xi), S_2(\xi), S_3(\xi), S_4(\xi), M_0(\xi))$ for $\xi \in \mathbb{C}^*$, corresponding to $[E, \overline{\partial}_E, \theta, h, g] \in \mathfrak{X}^{\mathrm{fr}}$.

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- Let $a(\xi)$ and $b(\xi)$ be the non-trivial off-diagonal elements of $S_1(\xi)$ and $S_2(\xi)$.

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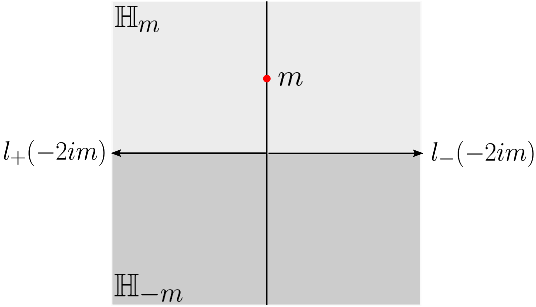
- Let $a(\xi)$ and $b(\xi)$ be the non-trivial off-diagonal elements of $S_1(\xi)$ and $S_2(\xi)$. Away from the locus where $m = 0$:

$$\mathcal{X}_m([E, \overline{\partial_E}, \theta, h, g], \xi) := \begin{cases} a(\xi) & \text{for} \quad \xi \in \mathbb{H}_m \\ -1/b(\xi) & \text{for} \quad \xi \in \mathbb{H}_{-m} \end{cases} \tag{34}$$



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- The **asymptotics** in ξ of $\mathcal{X}_m(\xi)$ are computed.

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- From these results, one is able to identify $\mathcal{X}_m(\xi)$ with $\mathcal{X}_m^{\text{ov}}(\xi)$ (under $z \iff -2im, \; 2\pi m^{(3)} \iff \theta_e$ and $\Lambda = 4i$).

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- The **reality** condition also holds: $\mathcal{X}_m(\xi) = \overline{\mathcal{X}_m(-1/\overline{\xi})}^{-1}$.
- From these results, one is able to identify $\mathcal{X}_m(\xi)$ with $\mathcal{X}_m^{\text{ov}}(\xi)$ (under $z \iff -2im$, $2\pi m^{(3)} \iff \theta_e$ and $\Lambda = 4i$).
- From the previous results, one can identify the subset $\mathfrak{X}^{\text{fr}}(4i) \subset \mathfrak{X}^{\text{fr}}$ with $\mathcal{M}^{\text{ov}}(4i)$.

Thanks!