Exponential smallness and resurgence

What perturbative expansions secretly know

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Outline

1. Paradise lost
   - Divergent series
   - Mode splitting
   - Stokes phenomenon
2. Paradise resummed
   - Borel summation
   - Trans-series

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Perturbative expansions

- In physics and engineering, exact solutions are rare.
- Examples of what I mean by exact: harmonic oscillator, hydrogenic atom, shallow wave equation (Korteweg–de Vries), 2D Ising model, etc.
- However, problems of interest are often "close" to ones that we can exactly solve.
- Strategy: Add order-by-order corrections in some perturbative parameter λ.
- Hope: Resulting series converges when λ is small.

Real quartic potential in zero dimensions

- A real-valued field configuration $u : \{p\} \to \mathbb{R}$ on a point p is just a real number $u(p)$, so space of field configurations $\mathcal{F}$ is just $\mathbb{R}$.
- The action $S_\lambda : \mathcal{F} \to \mathbb{R}$ is an ordinary function and the path integral $Z(\lambda) = (2\pi)^{-1/2} \int_{\mathcal{F}} \mathrm{d}u\, e^{-S_\lambda(u)}$ is an ordinary integral.
- Consider the action with quartic potential $S_\lambda = \frac{1}{2}u^2 + \lambda u^4$ where $\lambda > 0$.
- $Z(0)$ is just a Gaussian integral that we can exactly solve, so we perturb around it as a series in λ.

Feynman's strategy

An illegal move

$$
\begin{aligned}
Z(\lambda) &= \int_{\mathcal{F}} \frac{\mathrm{d}u}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}} \sum_{n=0}^{\infty} \frac{u^{4n}}{n!} (-\lambda)^n \\
&\stackrel{?}{=} \sum_{n=0}^{\infty} \frac{1}{n!} (-\lambda)^n \int_{\mathcal{F}} \frac{\mathrm{d}u}{\sqrt{2\pi}}\, u^{4n} e^{-\frac{u^2}{2}} .
\end{aligned}
\tag{1}
$$

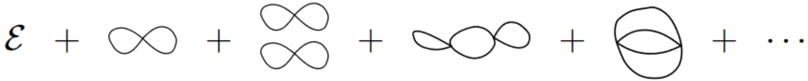

Figure: Feynman diagrammatics, from Kleiss, *Pictures, paths, particles, processes*

Dyson's argument

- Coefficient of $(-\lambda)^n/n! = (4!)^n \times$ number of Feynman diagrams with n vertices $\times$ symmetry factor.
- Number of Feynman diagrams with n vertices grows factorially with n.
- A typical Feynman diagram has symmetry factor 1.
- Conclusion: Perturbative expansions in QFT generically have zero radius of convergence.
- Case in point: $Z(\lambda) = \sum_{n=0}^{\infty} c_n \, (-\lambda)^n$, where $c_n = \frac{(4n)!}{n!(2n)!4^n} \sim 16^n n!$.

Optimal truncc

- Note that $|c_n(-\lambda)^n|$ usually decreases with n at first, say until $n = N$ before increasing.
- The optimal truncation heuristic prescribes truncating the series at $n = N$.
- Not only does this work in practice, it in fact works better than actual convergent series!

Carrier's rule

Divergent series converge faster than convergent series because they don't have to converge.

Question 1

Why does optimal truncation work?

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Asymptotic series

- $Z(\lambda) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!(2n)!4^n} (-\lambda)^n$ is an *asymptotic series*.

Asymptotic series

A series $\sum_{n=0}^{\infty} F_n \lambda^n$ is said be an *asymptotic series* for $F(\lambda)$ around $\lambda = 0$ if given a positive integer N, we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda^N} \left| F(\lambda) - \sum_{n=0}^{N} F_n \lambda^n \right| = 0.$$

Convergent series

Convergent series

In contrast, a series $\sum_{n=0}^{\infty} F_n \lambda^n$ is said be an *convergent series* for $F(\lambda)$ in an open set $U \ni 0$ if given a $\lambda \in U$, we have

$$\lim_{N\to\infty} \left| F(\lambda) - \sum_{n=0}^{N} F_n \lambda^n \right| = 0.$$

Asymptotic series in other contexts

- Asymptotic series arise whenever we try to use the method of Frobenius to solve ODEs around an irregular singularity.
- In our case, $Z(\lambda)$ can be obtained as a solution to the following ODE with irregular singularity at $\lambda = 0$.

Via Ward identities

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\lambda^2} + P(\lambda)\frac{\mathrm{d}Z}{\mathrm{d}\lambda} + Q(\lambda)Z = 0 \text{ where}$$
$$P(\lambda) = \frac{2}{\lambda} + \frac{1}{16\lambda^2}, \quad Q(\lambda) = \frac{3}{16\lambda^2}.$$

Method of Frobenius

- Substitute $Z(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^{n-\epsilon}$ into the ODE.

Indicial equation and recurrence

$$\epsilon = 0, \quad (16n(n-3)+3)c_n - (n+1)c_{n+1} = 0$$

- The ODE is second order, so it should have two linearly independent solutions.
- But the method of Frobenius gives only one asymptotic series.

Question 1

Why does optimal truncation work?

Question 2

What is the origin of the ambiguity in asymptotic series?

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Airy integral

- Let $T_z(u) = \frac{1}{3}u^3 + zu$ and define the Airy integral as $\mathrm{Ai}(z) = (2\pi)^{-1} \int_{\mathcal{F}} \mathrm{d}u\, e^{\mathrm{i}T_z(u)}$.
- The integral was introduced by Airy to study the propagation of S-shaped wavefronts that arise in the formation of rainbows by raindrops.
- The Airy function (of the first kind) $\mathrm{Ai}(z)$ solves the ODE $\mathrm{Ai}''(z) - z\,\mathrm{Ai}(z) = 0$, which has an irregular singularity at infinity.
- The Airy function of the second kind $\mathrm{Bi}(z)$ is another linear independent solution of the Airy ODE and is related to $\mathrm{Ai}(z)$ by a phase difference.

Asymptotic behaviour of the Airy functions

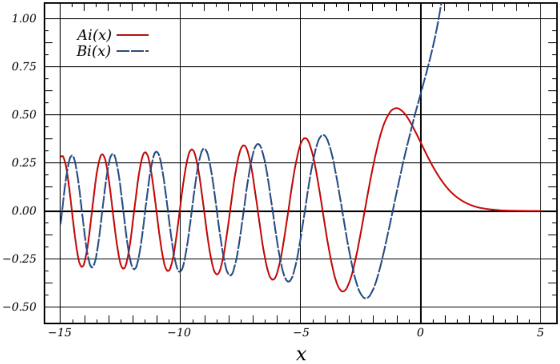

Figure: Airy functions, from Wikimedia Commons

Stokes phenomenon

Asymptotic behaviour as $z \to +\infty$

$$\mathrm{Ai}(+z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}, \quad \mathrm{Ai}(-z) \sim \frac{\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}z^{1/4}}$$

- A clearer picture emerges on allowing z to be complex; the asymptotic behaviour jumps across certain "Stokes rays."
- A given analytic function can have different asymptotic behaviour in different sectors.
- Given asymptotic behaviour may correspond to different analytic functions in different sectors.

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Why does optimal truncation work?

Question 2

What is the origin of the ambiguity in asymptotic series?

Question 3

Why does Stokes phenomenon occur?

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Borel transform

- Set $z = 1/\lambda$ and remove the classical term in $Z(\lambda)$.

Borel transform

The *Borel transform* $\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]$ is given by

$$\sum_{n=0}^{\infty} F_{n+1} z^{-n-1} \mapsto \sum_{n=0}^{\infty} F_{n+1} \frac{\zeta^n}{n!}.$$

Proposition

If $F = \sum_{n=0}^{\infty} F_{n+1} z^{-n-1}$ is such that $F_{n+1} \sim O(C^n n!)$ (Gevrey type 1), then $\mathcal{B}(F)$ has a nonzero radius of convergence around $\zeta = 0$.

Undoing the Borel transform

- Option A: Multiply the coefficients F_{n+1} by $n!$ order by order.
- Option B: Use the fact that $\int_0^\infty \mathrm{d}\zeta\, \zeta^n e^{-\zeta} = n!$, or more generally $\int_0^{e^{i\theta}\infty} \mathrm{d}\zeta\, \zeta^n e^{-\zeta} = n!$.

Directional Laplace transform

$$\mathcal{L}^\theta[\hat{F}](z) = \int_0^{e^{i\theta}\infty} \mathrm{d}\zeta\, e^{-z\zeta} \hat{F}(\zeta)$$

Proposition

If $\mathcal{U}$ is the set of analytical functions on the half-plane $\Re(ze^{i\theta}) > r$ and $\mathcal{V}$ the set of $O(e^{r|\zeta|})$ analytical functions on $e^{i\theta}\mathbb{R}_+$, then $\mathcal{L}^\theta \circ \mathcal{B}|_\mathcal{U} = \mathrm{id}_\mathcal{U}$ and $\mathcal{B} \circ \mathcal{L}^\theta|_\mathcal{V} = \mathrm{id}_\mathcal{V}$.

Two wrongs make a right

Another illegal move

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n+1} z^{-n-1} &= \sum_{n=0}^{\infty} F_{n+1} \int_0^{e^{i\theta}\infty} d\zeta\, e^{-z\zeta} \frac{\zeta^n}{n!} \\ &\stackrel{?}{=} \int_0^{e^{i\theta}\infty} d\zeta\, e^{-z\zeta} \sum_{n=0}^{\infty} F_{n+1} \frac{\zeta^n}{n!}\end{aligned} \tag{2}$$

Borel resummation

The *Borel resummation* $\mathcal{S}_\theta F$ of a formal series $F \in z^{-1}\mathbb{C}[[z^{-1}]]$ is defined to be $\mathcal{S}_\theta[F] = \mathcal{L}^\theta \circ \mathcal{B}[F]$.

Borel resummation with constant terms

- The Borel trasform turns multiplication into convolution i.e. $\mathcal{B}[F \cdot G] = \mathcal{B}[F] * \mathcal{B}[G]$.
- So $\mathcal{B}[1]$ must be the identity for convolution, which doesn't exist.
- We therefore introduce a formal identity $\delta = \mathcal{B}[1]$ (Dirac delta) and let the range of $\mathcal{B}$ be $\delta\,\mathbb{C} \oplus \mathbb{C}[[\zeta]]$.
- Now, the definition of the Borel sum can be extended to formal series with constant terms as well.

Quartic potential revisited

Quartic potential

$$Z(z^{-1}) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!(2n)!4^n} (-z)^{-n}$$

$$\begin{aligned}\mathcal{B}[Z \circ (\cdot)^{-1}](\zeta) &= \delta - \sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^2(2n)!4^n} (-\zeta)^{n-1} \\ &= \delta + \frac{1}{\zeta} \left[\frac{2\pi^{-1}}{(1+16\zeta)^{1/4}} K\left(\frac{\sqrt{1+16\zeta}-1}{2\sqrt{1+16\zeta}} \right) - 1 \right],\end{aligned}$$

$$\text{where } K(s) = \frac{1}{2} \int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)(1-st)}}$$

A first look at exponential smallness

Error

Let $F = \sum_{n=0}^{\infty} F_n z^{-n}$, where $F_{n+1} = \alpha(-C)^n n!$, with $\alpha, C > 0$.

$$
\begin{aligned}
\mathcal{S}_0 F - \sum_{n=0}^{N} F_n z^{-n} &= \int_0^\infty \mathrm{d}\zeta \, e^{-z\zeta} \sum_{n=N}^{\infty} F_{n+1} \frac{\zeta^n}{n!} \\
&= \int_0^\infty \mathrm{d}\zeta \, e^{-z\zeta} \sum_{n=N}^{\infty} \alpha(-C\zeta)^n \\
&= \alpha \int_0^\infty \mathrm{d}\zeta \, \frac{e^{-z\zeta}(-C\zeta)^N}{1 + C\zeta} \\
&= \frac{\alpha}{C} e^{-z/C} N! \, \Gamma\left(-N, \frac{z}{C}\right)
\end{aligned}
$$

A first look at exponential smallness

- As $z \to \infty$, the error is $\sim \frac{\alpha}{C} N! \left(\frac{C}{z}\right)^{N+1} = C^{-1} F_{N+1} z^{-N-1}$.
- Optimal truncation thus minimises this error given a large z.
- The minimum error is at $N+1 \approx z/C$ and is $e^{-z/C} = e^{-1/C\lambda}$.

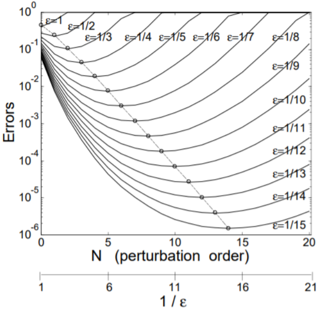
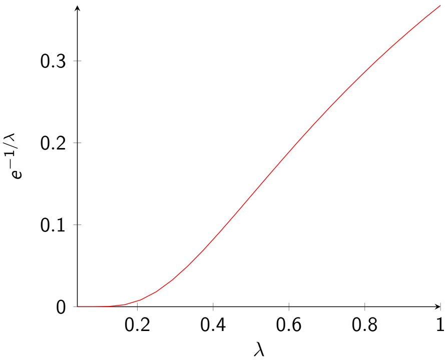


Figure: Exponential smallness in optimal truncation, from John Boyd, "The Devil's Invention"



Derivatives of all orders, and hence the power series expansion, of $e^{-1/\lambda}$ at $\lambda = 0$ vanishes.

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θ-dependence of Borel resummation

- Multiplying the coefficients F_{n+1} by $n!$ order by order doesn't depend on θ but S_θ does, so what gives?
- Contours can be deformed as long as as they don't pass through singularities, but formal series whose Borel transforms have no singularities are convergent to begin with.
- Consider $\mathcal{B}[Z]$ for the quartic potential, which has a branch point at $\zeta = -1/16$, so we should expect a jump when θ crosses π.
- This jump is essentially the reason for Stokes phenomenon.

A second look at exponential smallness

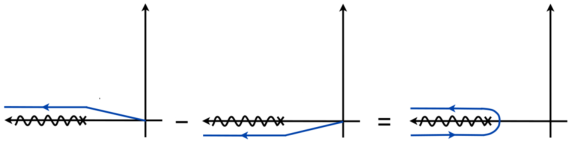

Figure: Hankel contour, from Dorigoni, "Introduction to trans-series, resurgence, and alien calculus"

$(\mathcal{S}_{\pi^+} - \mathcal{S}_{\pi^-})Z(z^{-1})$ is given by an integral along the branch cut at $\zeta = -1/16$ which gives terms of order $e^{z/16} = e^{1/16\lambda}$ and higher.

Complexifying the path integral

- In order to understand what is physically going on as the phase of λ varies, we have to first make sense of the complexification of the path integral Z.
- Note that for $\Re(\lambda) < 0$, the path integral along the real line diverges.
- So to give it meaning, we either have to smoothly deform the contour as we vary λ, or equivalently, make the change of variables $u = \lambda^{-1/4} v =: \mu^{1/2} v$ while keeping the contour fixed.
- The complexified path integral therefore can be written as $Z(\mu^{-2}) = \sqrt{\frac{\mu}{2\pi}} \int_{\mathbb{R}} \mathrm{d}v \, e^{-\tilde{S}_\mu(v)}$ where $\tilde{S}_\mu(v) = \frac{1}{2}\mu v^2 + v^4$.

Instantons and Lefshetz thimbles

- $v = 0$ is not the only solution of $\tilde{S}'_{\mu}(v) = 0$ i.e. it is not the only classical solution.
- There are in fact two other classical solutions given by $v = v_{\pm} = \pm\frac{\mathrm{i}}{2}\sqrt{\mu}$, and referred to as "instantons."
- Through each of the classical solutions, we have contours of steepest descent aka Lefshetz thimbles.
- The thimbles may be "good" (the path integral along them is well-defined) or "bad" (otherwise), and the real line can generically be uniquely decomposed into good thimbles.
- Applying Feynman's strategy around these points yields terms that can be organised into $e^{-\tilde{S}_{\lambda}(v_{\pm})} Z_{\pm} = e^{\mu^2/16} Z_{\pm} = e^{1/16\lambda} Z_{\pm}$ where $Z_{\pm}$ is again an asymptotic series.

Instantons and Lefschetz thimbles, visualised

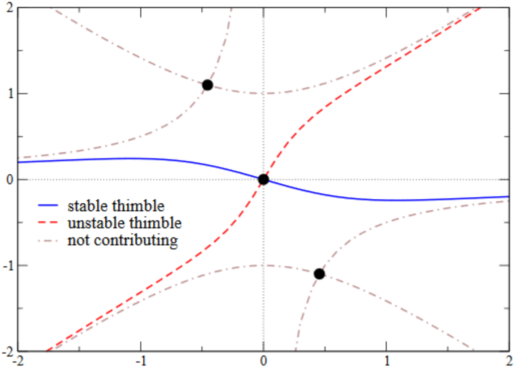

Figure: Instantons and thimbles for $\mu = 2(1+\mathrm{i})$, from Gert Aarts, "Langevin and Laguerre"

Stokes phenomenon via Lefschetz thimbles

- The coefficients in the decomposition of the real line into good thimbles can jump upon crossing a Stokes ray.

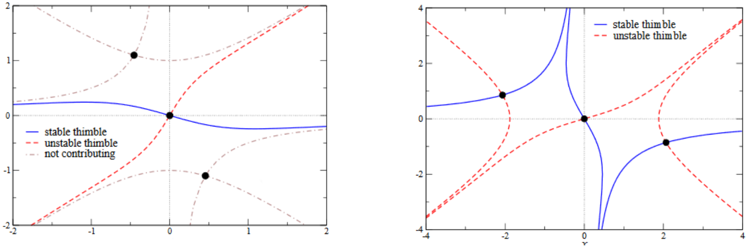

Figure: Stokes jump across $\arg(\mu) = \pi/2$, from Gerald Dunne, “A Beginners Guide to Resurgence and Trans-series in Quantum Theories”

Trans-series matter

- Asymptotic series should thus be regarded as “incomplete” formal representations of analytic functions.
- Instead asymptotic series should be augmented to include exponentially small terms.

Trans-series of height 1

$$F = \sum_{m,n} F_n^{(m)} \lambda^n e^{\sum_I m_I S_I} \prod_I \sigma_I^{m_I} =: \sum_m F^{(m)} e^{\sum_I m_I S_I} \sigma^m$$

- This is an example of a *trans-series* where $\sigma = (\sigma_I)$ is a formal parameter keeping track of the instantons (labelled I) with $m = (m_I)$ telling us how many (anti-)instantons of what type are simultaneously contributing.
- $\mathcal{S}_\theta F^{(m)}$ and the σ simultaneously jump across Stokes rays in a way such that F is continuous.

Summary

- Optimal truncation of an asymptotic series approximates the true value so well because it differs from it by an exponentially small quantity.
- Stokes phenomenon occurs because of an exponentially small ambiguity in asymptotic series corresponding to different integration contours in the convolutive model.
- This ambiguity can be repaired by passing to trans-series, which capture instanton contributions.
- Thus, nonperturbative information which naïvely appears to be invisible to perturbative expansions does resurge in the behavior of late terms.

A Zen koan

Before you study physics, perturbative expansions capture all information about a system.

While you're studying physics, perturbative expansions barely capture any information about a system.

After you've studied physics, perturbative expansions again capture all information about a system.

For Further Reading I

Jean Écalle.
Les fonctions résurgentes. Tome I, II et III.
Publications Mathématiques d'Orsay 81, Vol. 5 and 6.
Université de Paris-Sud, Département de Mathématique, Orsay (1980).

Daniele Dorigoni.
An introduction to resurgence, trans-series and alien calculus.
arXiv: hep-th/1411.3585.

Gerald V. Dunne, Mithat Ünsal.
What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles.
Proceedings, LATTICE 2015.

For Further Reading II

Brent Pym.
Lecture notes, *Resurgence in geometry and physics*.
University of Oxford, Trinity Term 2016.

John P. Boyd.
The devil's invention: asymptotic, superasymptotic and hyperasymptotic series.
Acta Applicandae Mathematicae 56(1): 1-98. Kluwer Academic Publishers (1999).

Back-up slides

Ward identities

Let f be a function of u and $\langle f\rangle_\lambda = (2\pi)^{-} \int_{\mathcal{F}} \mathrm{d}u\, f(u) e^{S_\lambda(u)}$. For any such f, we have a Ward identity $\langle fS'\rangle = \langle f'\rangle$. In particular:

$$\langle u(u+4\lambda u^3)\rangle_\lambda = \langle 1\rangle_\lambda$$
$$\langle u^3(u+4\lambda u^3)\rangle_\lambda = \langle 3u^2\rangle_\lambda$$
$$\langle u^5(u+4\lambda u^3)\rangle_\lambda = \langle 5u^4\rangle_\lambda, \text{ resulting in}$$
$$\langle u^8\rangle_\lambda - P(\lambda)\langle u^4\rangle_\lambda + Q(\lambda)\langle 1\rangle_\lambda = 0$$

Now use the fact $Z'(\lambda) = -\langle u^4\rangle_\lambda$ and $Z''(\lambda) = \langle u^8\rangle_\lambda$ to obtain an ODE for Z.

Integration along the Hankel contour

The Hankel contour integral takes the form

$$\oint \mathrm{d}\zeta \, \frac{e^{-z\zeta}\hat{g}(\zeta)}{\zeta + \frac{1}{16}} + \int_{-1/16}^{-\infty} \mathrm{d}\zeta \, e^{-z\zeta}\hat{h}(\zeta),$$

where $\hat{g}, \hat{h}$ are holomorphic functions of ζ with at most polynomial growth. Using the residue formula for the first term and carrying out a change of variables $\zeta = \xi - 1/16$ for the second, we get $e^{z/16}\hat{g}(-1/16) + e^{z/16}\mathcal{L}^{\pi}[\hat{h}](z)$.

Stokes phenomenon for Airy integral

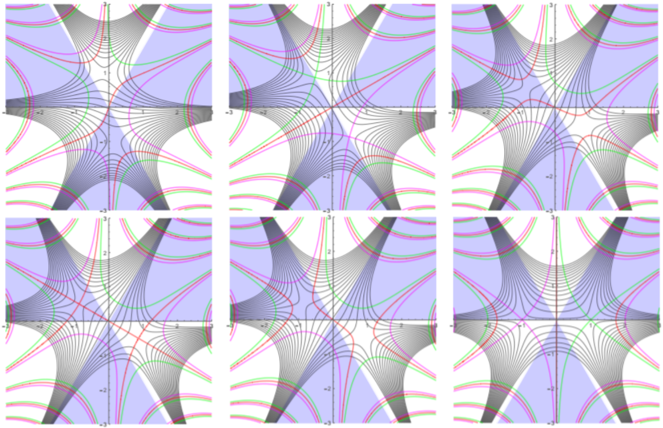

Figure: Thimble decompositions for Airy integral, from Falk Bruckmann, "Towards resurgence and trans-series"