Tau functions. KdV hierarchy

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- Definition
- Mixing of times and spectral variable
- Relation to isomonodromic deformations
- Pseudo-differential operators
- KdV hierarchy
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Last time

We introduced integrable hierarchies - systems of infinite number of pairwise commuting flows

$$\partial_{t_i} L = [M_i, L].$$

Commuting $\Rightarrow$

$$\partial_{t_i} M_i - \partial_{t_j} M_j - [M_i, M_j] = 0.$$

We defined a wawe function by

$$\partial_{t_i} \Psi = M_i \Psi.$$

Also we had

$$M_i = (g \xi_i g^{-1})_-, \qquad \xi_i = \frac{1}{\lambda^n} E_{\alpha\alpha}$$

$$\Psi(\lambda, \boldsymbol{t}) = g(\lambda, \boldsymbol{t}) e^{\xi(\lambda, \boldsymbol{t})}, \quad \xi(\lambda, \boldsymbol{t}) = \sum t_i \xi_i.$$

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Definition of τ-function

Let's work with a number of singular points λ_k. Consider a 1-form

$$\Omega = -\sum_k \operatorname{Tr}\operatorname{Res}_{\lambda=\lambda_k}\left((g^{(k)})^{-1}\partial_\lambda g^{(k)} d\xi\right), \quad d = \sum_i dt_i\ \partial_{t_i}$$

It is closed $d\Omega = 0$, therefore by Poincare lemma there is (locally) a function $\tau(t_1, t_2, \dots)$, such that $\Omega = d\log\tau$.

Proof.

Last time we had that in a certain gauge, the flows equations are equivalent to

$$\partial_{t_i} g^{(k)} = M_i^{(k)} g^{(k)} - g^{(k)}\partial_{t_i}\xi.$$

It can be written as

$$dg^{(k)} = \mathcal{M}^{(k)} g^{(k)} - g^{(k)} d\xi, \text{ for } \mathcal{M}^{(k)} = \sum M_i^{(k)} dt_i$$

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Definition of τ-function

Proof.

Therefore, we can write

$$d\Omega = \sum_k \operatorname{Tr}\operatorname{Res}_{\lambda=\lambda_k}\left(d\partial_\lambda \xi \wedge d\xi - \partial_\lambda \mathcal{M}^{(k)} \wedge g^{(k)} d\xi g^{-1}\right).$$

Note that ξ has a pole or the order at least 1, thus the first term has zero residue. Now note that $\mathcal{M}^{(k)}$ has the same polar part as $g^{(k)} d\xi (g^{(k)})^{-1}$ (as $M_i^{(k)} = (g^{(k)} d\xi_i (g^{(k)})^{-1})_-$), therefore we can write $g^{(k)} d\xi (g^{(k)})^{-1} = \mathcal{M}^{(k)} + \mathcal{N}^{(k)}$ for regular $\mathcal{N}^{(k)}$ and get

$$d\Omega = \sum_k \operatorname{Tr}\operatorname{Res}_{\lambda=\lambda_k}\left(\partial_\lambda \mathcal{M}^{(k)} \wedge (\mathcal{M}^{(k)} + \mathcal{N}^{(k)})\right)$$

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Definition of τ-function

Lemma

$$\operatorname{Tr}\operatorname{Res}_{\lambda=a}(\partial_\lambda \mathcal{M} \wedge \mathcal{N}) = -\frac{1}{2}\operatorname{Tr}\operatorname{Res}_{\lambda=a}(\partial_\lambda \mathcal{M} \wedge \mathcal{M})$$

Using the lemma we have

$$d\Omega = -\frac{1}{2}\sum_k \operatorname{Tr}\operatorname{Res}_{\lambda=\lambda_k}(\partial_\lambda \mathcal{M} \wedge \mathcal{M})$$

and it is zero since $\operatorname{Tr}(\partial_\lambda \mathcal{M} \wedge \mathcal{M})$ is a rational 1-form on a Riemann surface and therefore sum of its residues is 0.

Mixing times and spectral parameter

$$\Psi(\lambda, \boldsymbol{t}) = g(\lambda, \boldsymbol{t}) e^{\xi((\lambda, \boldsymbol{t}))} g^{-1}(\lambda, 0)$$
$$g(\lambda) = g_0 h(\lambda), \quad h(\lambda) = 1 + O(\lambda)$$

Theorem

$$h_{\alpha\alpha}(\boldsymbol{t}(\lambda, \boldsymbol{t}) = \frac{\tau(t - [\lambda]_\alpha)}{\tau(\boldsymbol{t})}$$

$$h_{\beta\alpha}(\boldsymbol{t}(\lambda, \boldsymbol{t}) = \lambda \frac{\tau_{\beta\alpha}(t - [\lambda]_\alpha)}{\tau(\boldsymbol{t})}$$

where

$$\boldsymbol{t} - [\lambda]_\alpha \Leftrightarrow t_{(n,\gamma)} - \delta_{\gamma\alpha} \frac{\lambda^n}{n}$$

i.e. $t_{(1,\alpha)} \to t_{(1,\alpha)} - \lambda; \;\; t_{(2,\alpha)} \to t_{(2,\alpha)} - \frac{\lambda^2}{2}; \;\; t_{(3,\alpha)} \Rightarrow t_{(3,\alpha)} - \frac{\lambda^3}{3}$

For a proof see [BBT03].

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Compairing to isomonodromic deformations

Recall the setup: we have a linear ODE:

$$\partial_x \Psi = M(x,t)\Psi,$$

it can be shown that there is also a well-defined system

$$d\Psi = \Theta(x,t)\Psi = \sum \Theta_j dt_j \Psi, \quad d = \sum dt_i \partial_i$$

The compatibility(integrability) condition of this system of PDEs gives the equations describing isomonodromic deformations. The integrable hierarchies equations also can be seen as integrability condition of the following sysytem:

$$\partial_i \Psi = M_i \Psi$$

$$L\Psi = \lambda \Psi$$

It justifies one more time the name "integrable hierarchy" and moreover it explains the name "spectral parameter".

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There is one more common feature: tau-functions for isomonodromic systems are defined in the exact same way as for the integrable hierachies!

But be careful, although the similarity between integrable systems and IMD is remarkable, it is just formal!

Fun fact: Painleve equations I and II appearing as isomonodromic equations can be extended to Painleve hierarchies! Google it.

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Pseudo-differential operators

Consider a function $f(x)$ and a differential operator $\partial := \frac{\partial}{\partial x}$. The composition of ∂ and miltiplication by f can be written as

$$\partial^n \circ f = \sum_{j \geqslant 0} \binom{n}{j} (\partial^j f) \circ \partial^{n-j}$$

Note that since $\binom{n}{j} = 0$ for $j > n$ the sum can go up to infinity. Now consider a more general series of this type:

$$L = \sum_{j=0}^{\infty} f_j \partial^{\alpha - j}$$

It is called a (formal) pseudo-differential operator of order $\leqslant \alpha$. The negative powers correspond to integration, which we do not define today.

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Pseudo-differential operators. Example

We can work with PDO pretty much in the same way as with usual power series.
Let's consier a differential operator $L = \partial^2 + u(x)$ and try to compute its square root $X = L^{1/2}$. Write

$$X = \partial + \sum_{n \geqslant 1} f_n \partial^{-n},$$

$$X^2 = \partial^2 + 2\sum_{n\geqslant 1} f_n \partial^{1-n} + \sum_{n\geqslant 1} (\partial f_n)\partial^{-n} + \sum_{m,n\geqslant 1,\, l\geqslant 0} \binom{-n}{l} f_n(\partial^l f_m)\partial^{-m-n-l},$$

where $\binom{-n}{l} = (-1)^l \binom{n+l-1}{l}$.

Pseudo-differential operators. Example

Compairing this to L we get

$$\begin{aligned} f_1 &= \frac{1}{2}u, \\ f_2 &= -\frac{1}{4}u_x, \\ f_3 &= -\frac{1}{8}(u^2 - u_{xx}), \\ f_4 &= -\frac{1}{16}u_{xxx} + \frac{3}{8}uu_x \\ &\dots \end{aligned}$$

See [Sed09] for a couple of further coefficients.

KdV hierarchy

Define a hierarchy by the following relations

$$\frac{\partial L}{\partial t_i} = [(L^{\frac{2i-1}{2}})_+, L],$$

where L is the quadratic operator from the previous example and $(A)_+$ means leaving only non-negative powers of ∂ in the opeartor A.
This hierarchy is called KdV hierarchy.

Integrability for KdV

Let's check the integrability condition:

$$\partial_i (L^{\frac{2j-1}{2}})_+ - \partial_j (L^{\frac{2i-1}{2}})_+ - [(L^{\frac{2i-1}{2}})_+, (L^{\frac{2j-1}{2}})_+] = 0.$$

Note that for any function $f(L)$ we have

$$\partial_i f(L) = [(L^{\frac{2i-1}{2}})_+, f(L)],$$

Thus

$$\partial_i (L^{\frac{2j-1}{2}})_+ = (\partial_i L^{\frac{2j-1}{2}})_+ = \left([(L^{\frac{2i-1}{2}})_+, L^{\frac{2j-1}{2}}]\right)_+.$$

Moreover we have identity

$$\left(\left[(L^{\frac{2i-1}{2}})_+, (L^{\frac{2j-1}{2}})\right]\right)_+ = \left[(L^{\frac{2i-1}{2}})_+, (L^{\frac{2j-1}{2}})_+\right] + \left(\left[(L^{\frac{2i-1}{2}})_+, (L^{\frac{2j-1}{2}})_-\right]\right)_+ =$$

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Note that $([A_-, B_-])_+ = 0$. Taking in consideration two last equalities we have integrability.

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Note that $([A_-, B_-])_+ = 0$. Taking in consideration two last equalities we have integrability.

KdV equations

If we spell out the flows equations explicitly we will get

$$\partial_1 u = u_x$$
$$\partial_2 u = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x$$
$$\partial_3 u = \frac{1}{16}u^{(5)} + \frac{5}{4}(u_x u_{xx} + \frac{1}{2}uu_{xxx}) + \frac{15}{8}u^2 u_x$$
$$\dots$$

We see that the first equation allows us to identify t_1 and x. The second is the classical KdV equation and others are the so called higher KdV equations.

Remarks on KdV

KP

If instead of $L^{1/2}$ we consider some general PDO of the form $X = \partial + \sum_{n \geqslant 1} f_n \partial^{-n}$ we will get the Kadomtsev–Petviashvili (KP) hierarchy and KdV is just a particular case of it.

q-version?

Is q-KdV obtained by doing similar things, but changing ∂ for $\partial_q = \frac{f(qx) - f(x)}{(q-1)x}$?

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τ-function and KdV

Let's look at the τ-function. KdV hierarchy is considered with singularity at $\lambda = \infty$, thus the wave function can be writen as

$$\Psi(\lambda, \boldsymbol{t}) = g(\lambda, \boldsymbol{t}) e^{\xi(\lambda, \boldsymbol{t})} = (1 + \frac{g_1}{\lambda} + \frac{g_2}{\lambda^2} + \dots) \exp(\lambda t_1 + \lambda^2 t_2 + \dots.)$$

Note that sometimes people write $\sum \lambda^{2i-1} t_i$ in the exponent since KdV is obtained by erasing even times of KP. I don't really understand this moment, but it is not important for how I am going to use Ψ. Recall that we have

$$\partial_1 \log \tau = -\operatorname{Res}_{\lambda=\infty} \left(g^{-1} \partial_\lambda g \partial_1(\xi) dt_1 \right).$$

Substituting the form of g and ξ we get $\partial_1 \log \tau(\boldsymbol{t}) = -g_1$.

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τ-function and KdV

The wave function is an eigenfunction of L, i.e.$L\Psi = \lambda^2\Psi$ or equivalently

$$L^{1/2}\Psi = \lambda\Psi.$$

Using this we can obtain that $g_1 = -u_x/2$, thus we have a notable relation

$$u(\boldsymbol{t}) = 2\frac{\partial^2}{\partial x^2}\log\tau.$$

Although in the case of KdV it is not that exciting, since we basically exchange one function for another, but in the case of KP (which have infinitely many unkwnon functions u_i) we would incorporate all of them in a single τ-function in a similar way. That is the beauty of it.

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Hirota bilinear relations

Theorem

[Dub09][BDY15] KdV τ-function satisfies

$$\mathrm{Res}_{\lambda=\infty}\, \tau(\boldsymbol{t} - [\lambda^{-1}])\tau(\tilde{\boldsymbol{t}} + [\lambda^{-1}]) \exp((t_j - \tilde{t}_j)\lambda^{2j-1})\lambda^{2p} dz = 0$$

for all $t_i, \tilde{t}_i$ and $p \geqslant 0$.

This relations are called Hirota bilinear relations. Writing out the expression in the theorem leads to an infinite set of differential equations (see Dubrovin's notes, p.140).

Before writing it down we introduce the following: consider f(x+y)g(x-y) and Taylor expand it around $y = 0$:

$$f(x+y)g(x-y) = \sum_{j\geqslant 0} \frac{1}{j!}(D_x^j f \cdot g) y^j,$$

the operator $(f, g) \to D_x^j f \cdot g$ is called Hirota derivative.

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Hirota bilinear relations

For example,

$$D_x f \cdot g = \frac{\partial f}{\partial x} g - f \frac{\partial f}{\partial x}, \quad D_x^2 f \cdot g = \frac{\partial^2 f}{\partial x^2} g - 2 \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2}$$

Using this, the Hirota bilinear relations becomes:

$$\left(4D_1 D_3 - D_1^4\right) \tau \cdot \tau = 0$$

$$\left(D_1^6 - 20 D_1^3 D_3 - 80 D_3^2 + 144 D_1 D_5\right) \tau \cdot \tau = 0$$

and so on. Thus we can derive τ-function by solving HBR.

Remark

The HBR are inherent to integrable systems. We can deal with integrable systems using some infinite-dimensional Grassmanians and then the HBR can be seen as a consequence of bilinear identities between Plücker coordinates of the embedding $\mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\Lambda^k V)$ *(but of course an infinite-dimensional version).*

Hirota bilinear relations

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$$D_x f \cdot g = \frac{\partial f}{\partial x} g - f \frac{\partial f}{\partial x}, \quad D_x^2 f \cdot g = \frac{\partial^2 f}{\partial x^2} g - 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2}$$

Using this, the Hirota bilinear relations becomes:

$$\left(4D_1D_3 - D_1^4\right)\tau \cdot \tau = 0$$

$$\left(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5\right)\tau \cdot \tau = 0$$

and so on. Thus we can derive τ-function by solving HBR.

Remark

The HBR are inherent to integrable systems. We can deal with integrable systems using some infinite-dimensional Grassmanians and then the HBR can be seen as a consequence of bilinear identities between Plücker coordinates of the embedding $\mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\Lambda^k V)$ *(but of course an infinite-dimensional version).*

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KdV and enumerative geometry

Note that a KdV τ-function depends on a solution of the KdV equation. Let's consider a τ-function specified by a solution with initial data

$$u(t_1 = x, t_2 = 0, t_3 = 0 \dots) = x$$

Partition function

$$\log\tau = \frac{1}{\epsilon^2}\left(\frac{t_0^3}{6} + \frac{t_0^3\, t_1}{6} + \frac{t_0^3\, t_1^2}{6} + \frac{t_0^3\, t_1^3}{6} + \frac{t_0^3\, t_1^4}{6} + \frac{t_0^4\, t_2}{24} + \frac{t_0^4\, t_1\, t_2}{8} + \frac{t_0^4\, t_1^2\, t_2}{4} + \frac{t_0^5\, t_2^2}{40} + \frac{t_0^5\, t_3}{120} + \frac{t_0^5\, t_1\, t_3}{30} + \frac{t_0^6\, t_4}{720} + \ldots\right)$$

$$+\left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0\, t_2}{24} + \frac{t_0\, t_1\, t_2}{12} + \frac{t_0\, t_1^2\, t_2}{8} + \frac{t_0^2\, t_2^2}{24} + \frac{t_0^2\, t_3}{48} + \frac{t_0^2\, t_1\, t_3}{16} + \frac{t_0^3\, t_4}{144} + \ldots\right)$$

$$+\epsilon^2\left(\frac{7\, t_2^3}{1440} + \frac{7\, t_1\, t_2^3}{288} + \frac{29\, t_2\, t_3}{5760} + \frac{29\, t_1\, t_2\, t_3}{1440} + \frac{29\, t_1^2\, t_2\, t_3}{576} + \frac{5\, t_0\, t_2^2\, t_3}{144} + \frac{29\, t_0\, t_3^2}{5760} + \frac{29\, t_0\, t_1\, t_3^2}{1152} + \frac{t_4}{1152} + \frac{t_1\, t_4}{384} + \frac{t_1^2\, t_4}{192} + \frac{t_1^3\, t_4}{96} + \frac{11\, t_0\, t_2\, t_4}{1440} + \frac{11\, t_0\, t_1\, t_2\, t_4}{288} + \frac{17\, t_0^2\, t_3\, t_4}{1920} + \ldots\right) + O(\epsilon^4)$$

GW

where ϵ is introduced via $\partial_t u = \frac{\epsilon^2}{4} u_{xxx} + \frac{3}{2} u u_x$.

This series coincides with the generating function of the intersection numbers of the Mumford - Morita - Miller classes in $H^*(\bar{\mathcal{M}}_{g,n})$. Here $\bar{\mathcal{M}}_{g,n}$ is the moduli space of stable algebraic curves of genus g with n punctures. Equivalently, this series is a Gromow-Witten potential of a point. We have

$$\log \tau = \sum \epsilon^{2g-2} \mathcal{F}_g,$$

$$\mathcal{F}_g = \sum_n \frac{1}{n!} t_{p_1} \dots t_{p_n} \int_{\bar{\mathcal{M}}_{g,n}} c_1^{p_1}(\mathcal{L}_1) \wedge \dots \wedge c_1^{p_n}(\mathcal{L}_n)$$

where $\mathcal{L}_i$ is the tautological line bundle over the moduli space corresponding to the i-th puncture.

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Conjecturally, this connection between GW theory and integrable systems is a general phenomenon. In [BCR12] we find:

Conjecture

Let $\mathcal{F}^X(\epsilon, \boldsymbol{t})$ be the all genus full descendant Gromow-Witten potential of X, or more generally a full potential of some CohFT. Then there exists a Hamiltonian integrable hierarchy, such that $\mathcal{F}^X(\epsilon, \boldsymbol{t})$ is the logarithm of a τ-function associated to one of solutions.

There has been progress on this conjecture:

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Integrable systems and GW theory

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Virasoro conjecture

The Kontsevich-Witten theorem can be reformulated to the following statement: The partition function $Z = \exp(\sum \epsilon^{2g-2}\mathcal{F}_g) = \tau$ satisfies

$$L_m Z = 0, \qquad m \geqslant -1.$$

And L_m satisfy commutation relations of the virasoro algebra. Conjecturaly this holds for every projective variety or even further.

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