On constructing integrable systems and elementary flows

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String Math Seminar (Summer 2020)

Lax pairs and r-matrices

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Today, we will address 1. and 2. by using Lax pairs and the Zakharov–Shabat construction.

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Here $L_1 = L \otimes 1$, $L_2 = 1 \otimes L$.

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satisfies the Lax equation $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$, equivalent to (1).

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Note: the parameters for L are the dynamical parameters whereas $F(L, \lambda)$ determines the dynamical flow.

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Nevertheless, the Zakharov–Shabat construction gives a systematic way to produce symplectic manifolds with many Poisson-commuting constants of motion.

Elementary flows and wave functions

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But: Can the $\mathbf{t}$-dependence be made more explicit?

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Solving (9) for small enough time t_i is equivalent to solving the factorization problem

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Thank you!