<u>Yesterday:</u> Self-duality equations in rank one

$$(SD_1) \quad \boxed{\begin{aligned} F^{\nabla^h} &= 0 \\ \bar\partial \Phi &= 0 \end{aligned}}$$

$L \to C$ hol. bdl.
w/ $\deg L = 0$
h hermitian metric on L
$\Phi \in A^{1,0}(C)$

$$\updownarrow$$

Higgs bdl.s (L, Φ) w/ $\deg L = 0$.

We have seen that the moduli space $\mathcal{M}_{Dol}(C, \mathbb{C}^*)$ of Higgs bdl.s as above is

$$\mathcal{M}_{Dol}(C, \mathbb{C}^*) \cong T^*\mathrm{Jac}(C) = V/\Gamma_C \oplus \overline{V}.$$

$V = \overline{H^0(C, K_C)} = H^{0,1}(C)$

In particular, it is a complex mf. in a natural way (in fact Kähler which we will discuss in the HK setting later).
As a cotangent bundle, it has a canonical hol. sympl. structure:

<u>Lem.:</u> Let M be a complex mf. Then T^*M carries a canonical hol. symplectic structure Ω_c.

proof: Let $p: T^*M \to M$ be the vbdl. projection.
(idea)
Define the tautological 1-form

$$\eta_\alpha(v) = \alpha \circ \underbrace{dp_\alpha(v)}_{\in T_{p(\alpha)}M}$$

$v \in T_\alpha T^*M$

Then $\boxed{\Omega_{\mathbb{C}} := d\eta}$ is a holo. symplectic form.
If $(z_j)_j$ are local coords & $(w_j)_j$ the fibre coordinates ξ, then

$$\Omega_{\mathbb{C}} = \sum_j dw_j \wedge dz_j .$$

(■)

Next we relate $\mathcal{M}_{Dol}(C, \mathbb{C}^*)$ to

$$\mathcal{M}_{dR}(C, \mathbb{C}^*) := \left\{ \begin{matrix} \nabla \text{ flat connection} \\ \text{on } \underline{\mathbb{C}} \to C \end{matrix} \right\} \Big/ \begin{matrix} \text{gauge} \\ g: C \to \mathbb{C}^* \end{matrix}$$

as in the general non-abelian Hodge story.
We begin by showing/indicating that it is a complex mf.

Thm.:

$$\mathcal{M}_{dR}(C, \mathbb{C}^*) \cong H^1(C, \mathbb{C}) / H^1(C, \mathbb{Z})$$

$$\cong (\mathbb{C}^*)^{2g},$$

so it is a complex mf. in a natural way.

rank one

proof (sketch) also of Jac(C): Every connection ∇ on $\underline{\mathbb{C}}$ is of the form

$$\nabla = d + \alpha, \qquad \alpha \in \mathcal{A}^1(C,\mathbb{C}).$$

The gauge action of $\mathcal{A}^0(C,\mathbb{C}^*)$ is then given by

$$g \cdot \nabla = d + \alpha - g^{-1} dg.$$
$$g \cdot \nabla = g \circ \nabla \circ g^{-1}, \qquad g^{-1}dg = d \log g$$

In particular, if $g = e^{-f}$, $f \in \mathcal{A}^0(C,\mathbb{C}^*)$, then

$$g \cdot \nabla = d + \alpha + df.$$

So it remains to understand the action of fct.s not of this form. To this end, we define for $g \in \mathcal{A}^0(C,\mathbb{C}^*)$

$$W_g : \pi_1(C, c_0) \longrightarrow \mathbb{Z},$$
$$[\gamma] \longmapsto \frac{1}{2\pi i}\int_\gamma g^*\left(\frac{dz}{z}\right), \qquad z \in \mathbb{C}^*$$

which is a generalisation of the winding number in complex analysis. In that way, we obtain the group homomorphism

$$\mathcal{A}^0(C,\mathbb{C}^*) \xrightarrow{\ W\ } \mathrm{Hom}(\pi_1, \mathbb{Z})$$
$$\cong H^1(C,\mathbb{Z}).$$

$$\ker W = \{ g = e^{-f} \mid f \in \mathcal{A}^0(C,\mathbb{C}) \}$$
$$= \operatorname{im} \exp_* : \mathcal{A}^0(C,\mathbb{C}) \to \mathcal{A}^0(C,\mathbb{C}^*)$$

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This follows from the fact that a logarithm of g exists iff $w_g(\gamma) = 0 \ \forall \gamma$ & because then $f(c) = \int_{c_0}^{c} d \log g \quad \left(\frac{dg}{g}\right)$ is well-defined.

We now form the quotient in steps:

$$\mathcal{A} \Big/ \underbrace{\operatorname{im} \exp_*}_{\ker w} = \mathcal{A}^1(C,\mathbb{C}) \Big/ \ker w \cong H^1(C,\mathbb{C})$$

The action of $g \in \ker w$ is then simply given by translation via $\frac{1}{2\pi i}[g^{-1} dg] \in H^1(C,\mathbb{Z})$

$$\Longrightarrow \quad \mathcal{M}_{dR} \cong H^1(C,\mathbb{C}) \Big/ H^1(C,\mathbb{Z}).$$

$\blacksquare$

<u>Rem.:</u> This already implies that $\mathcal{M}_{Dol} \not\cong \mathcal{M}_{dR}$ as complex mf.s. Indeed, $\mathcal{M}_{Dol}$ contains the compact submf. $\mathrm{Jac}(C)$. Any hol. map

$$f : \mathcal{M}_{Dol} \longrightarrow \mathcal{M}_{dR} \cong (\mathbb{C}^*)^{2g}$$

would have to map $\mathrm{Jac}(C)$ to a point ↯

However, we next construct a natural diffeom.

First consider

$$H^1(C,\mathbb{C}) \xrightarrow{\chi} V \oplus \overline{V}, \qquad V = \overline{H^0(C,K_C)}$$

constructed as follows:

$$H^1(C,\mathbb{C}) \ni \eta = \phi + \psi = \mathrm{Re}(\phi) + i\,\mathrm{Im}(\phi)$$

$$= \tfrac{1}{2}(\eta+\overline{\eta}) + \tfrac{1}{2}(\eta-\overline{\eta})$$

Then $\phi^{1,0} = \Phi \in \overline{V}$ so that

$\phi^{0,1} = \Psi \in V$ $\qquad \eta = (\Phi + \overline{\Phi}) + (\Psi - \overline{\Psi})$.

With this notation, we define

$$\chi(\eta) := (\Psi, \overline{\Phi}).$$

Note that it is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

$$\chi(i\eta) = (i\overline{\Phi}, -i\overline{\Psi}).$$

However, it maps $H^1(C,\mathbb{Z}) \subseteq H^1(C,\mathbb{C})$ to $\Gamma_C \hookrightarrow V$ so that

we conclude

Prop.: $\chi: H^1(C,\mathbb{C}) \to V \oplus \overline{V}$ induces the diffeom.

$$\hat{\chi}: \mathcal{M}_{dR}(C,\mathbb{C}^*) \longrightarrow \mathcal{M}_{Dol}(C,\mathbb{C}^*).$$

The two different complex structures (on the same C^∞-mf.) fit into a HK-structure:

Next we give the HK-structure on $\mathcal{M}(C, \mathbb{C}^*) = V/\Gamma_c \oplus \overline{V}$.

Let us begin with

Ex.: $(V, \langle \cdot, \cdot \rangle)$ Hermitian v.s. $\qquad W := V \oplus \overline{V}$

$(V_{\mathbb{R}}, I)$ $\qquad (V_{\mathbb{R}}, I) \ (V_{\mathbb{R}}, -I)$

W carries a HK structure:

$I(v,w) = (iv, iw)$ (diagonal cplx. str.)

$J(v,w) = (ia(w), -ia(v))$ $\qquad a: V \xrightarrow{\cong} \overline{V}$ [redacted]

$\mathbb{C}$-antilin. isom.

$K := IJ$.

metric: $\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + i\omega(\cdot, \cdot)$

$g_W := g \oplus g^*$ $\qquad$ for the dual metric on $\overline{V} \cong V^*$.

Then $(W, g; I, J, K)$ is HK.

The holomorphic symplectic structure $\Omega_I = \omega_J + i\omega_K$ is given by

$$\Omega_I((v_1, w_1), (v_2, w_2))$$
$$= \langle v_2, w_1 \rangle - \langle v_1, w_2 \rangle.$$

Rem.: This is exactly Ω_c, the canonical symplectic form, on $TV = V \times V^*$.

In fact, W is simple to describe:

W is a left $\mathbb{H}$-module and $V_{\mathbb{H}} := \mathbb{H} \otimes_{\mathbb{C}} V \cong W$ as left $\mathbb{H}$-modules.

Prf: Recall $\mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C}$ (which we already used when describing Ω_I on $\mathbb{H}$). Then define $V_{\mathbb{H}} \longrightarrow W$ via

$$1 \otimes v \longmapsto (v, 0)$$
$$j \otimes v \longmapsto (0, -ia(v)).$$

$\blacksquare$

In particular, all $(W, I_{\vec{a}})$ are isomorphic as cplx. mf.s.

Back to $\mathcal{M} := \mathcal{M}(C, \mathbb{C}^*)$: $V = H^{0,1}(C)$ with Hermitian metric

$$\langle \eta_1, \eta_2 \rangle = -i \int_C \eta_1 \wedge \overline{\eta_2}.$$

Since $T\mathcal{M} \cong \mathcal{M} \times W$, $W = V \oplus \overline{V}$ as before

(*)

$\mathcal{M}$ carries a HK structure. However, there's a basic difference to $(W, g; I, J, K)$:

Prop.: $(\mathcal{M}, g; I_{\vec{a}})$, $I_{\vec{a}} \neq +/- I$.

Then $(\mathcal{M}, I_{\vec{a}}) \cong (\mathbb{C}^*)^{2g(C)}$ but

$$(\mathcal{M}, +/- I) \cong T^* Jac(C).$$

Rem.: This implies that $(\mathcal{M}, +/- I) \not\cong (\mathcal{M}, I^{+/- I}_{\vec{a}})$:
$T^* Jac(C)$ ~~has a~~ admits a compact cplx. submf.
but $(\mathbb{C}^*)^{2g}$ doesn't.

proof: Let $\vec{a}$ also denote the purely imaginary quaternion
$a_1 i + a_2 j + a_3 k \in \mathbb{H}$. Since $\vec{a} \neq +/- i$, i and $\vec{a}$ generate $\mathbb{H}$
(as an $\mathbb{R}$-algebra). Hence V generates $V_{\mathbb{H}}$ over $\langle 1, \vec{a} \rangle_{\mathbb{R}} \subseteq \mathbb{H}$

$$\Longrightarrow \quad \mathbb{C} \otimes_{\mathbb{R}} V \longrightarrow V_{\mathbb{H}} = \mathbb{H} \otimes_{\mathbb{C}} V$$

$$(x + iy) \otimes v \longmapsto (x + y\vec{a}) \otimes v$$

is an isomorphism for dimension reasons.
Now observe that $\Gamma_C \otimes_{\mathbb{Z}} \mathbb{R} = V$ so that

$$V_{\mathbb{H}} \cong \mathbb{C} \otimes_{\mathbb{R}} V \cong \mathbb{C} \otimes \Gamma, \qquad \Gamma = \Gamma_C$$

Since Γ_C is free, we obtain

$$V_{\mathbb{H}} \cong (\mathbb{C} \otimes \Gamma) / \Gamma$$

$$\cong (\mathbb{C} \otimes \Gamma) / (\mathbb{Z} \otimes \Gamma)$$

$$\cong (\mathbb{C}/\mathbb{Z}) \otimes L \cong (\mathbb{C}^*) \otimes \Gamma \cong (\mathbb{C}^*)^{2k}.$$