Higgs bdl.s in rank one (e.g. [Goldman–Xia])

We now focus on the case $r = 1$. Even though this case is classical, it shares many features of the general story, and much "easier".

In this way, we hope to convey the richness of the general theory despite ~~the~~ our time constraint.

$L \to C$ hol. l.bdl., $c_1(L) = \deg(L) = 0$

$\Downarrow$

$$(SD_1) \qquad \boxed{\begin{aligned} F^{\nabla^h} &= 0 \\ \bar\partial \Phi &= 0 \end{aligned}}$$

← note that L doesn't contribute b.c. $\mathrm{End}(L) \cong \mathbb{C}$.

In part.: A Higgs l.bdl. in rank 1 is just

Higgs l.bdl.

$$(L, \Phi), \qquad \Phi \in H^0(C, K_C) = \Omega^{1,0}(C)$$

Also: Stability is an empty statement here. But we still have:

Prop.:

$$\left\{ \begin{matrix} (L,\Phi) \text{ Higgs l.bdl.} \\ c_1(L) = 0 \end{matrix} \right\} \Big/ \text{iso.} \longleftrightarrow \left\{ \text{sol.s of } (SD_1) \right\} \Big/ \text{gauge}$$

$$\mathcal{M}_{Dol}(C, \mathbb{C}^*) \qquad\qquad \mathcal{M}(C, \mathbb{C}^*)$$

proof: We only need to go back:

Take any Hermitian metric h on L

$$\Longrightarrow \quad c_1(L) = \tfrac{1}{2\pi i}[F^{\nabla^h}] = 0 \quad \in H^2(C, \mathbb{Z})$$

$$\Longleftrightarrow \quad \exists\, \alpha : \quad d\alpha = \frac{1}{2\pi i} F^{\nabla^h}$$

$$\Longleftrightarrow \quad \exists\, u \in A^0(C, \mathbb{C}) : \quad d\alpha = \partial\bar\partial u = \frac{1}{2\pi i} F^{\nabla^h}$$

$\partial\bar\partial$-lemma

Now set $\tilde h := e^{-u} h$ and compute locally:

$$F^{\nabla^{\tilde h}} \underset{\text{locally}}{=} \bar\partial\partial \log \tilde h_0 = \bar\partial\partial (-u + \log h_0)$$

$$= -\bar\partial\partial u + F^{\nabla^h} = 0.$$

(☒)

Further: Induces a bijection on equivalence classes.

To see the HK structure, we work with $\mathcal{M}_{Dol}(C, \mathbb{C}^*)$.

$\boxed{\mathcal{M}_{Dol}(C, \mathbb{C}^*)}$ It is clear that the classification of Higgs bdl.s w/ $c_1(L) = 0$ reduces to the classification of hol bdl.s $L \to C$ w/ $c_1(L) = 0$.

In the following we make use of the Hodge decomp.

$$H^1(C, \mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C) \qquad H^{1,0} = \overline{H^{0,1}}$$

$$\begin{array}{cc} \| & \| \\ H^0(C, K_C) & H^0(C, \overline{K_C}) \\ \text{"}dz\text{"} & \text{"}d\bar z\text{"} \end{array}$$

Thm.: $\left\{ \begin{array}{l} L \to C \text{ hol. lbdl.} \\ c_1(L) = 0 \end{array} \right\} \Big/_{\text{iso}} \cong H^{0,1}(C) \Big/ H^1(C,\mathbb{Z})$

$\quad \| \quad$

$\mathrm{Jac}(C)$

$\hookleftarrow$ via $H^1(C,\mathbb{Z})$

$$\begin{array}{c} H^1(C,\mathbb{Z}) \\ \downarrow \\ H^1(C,\mathbb{C}) \\ \downarrow \text{pr} \\ H^{0,1}(C) \end{array}$$

It is an abelian variety of $\dim_{\mathbb{C}}\, g(C)$, the genus of C.

~~proof~~ (sketch) $L \to C$ is topologically trivial, hence we assume

$$\mathbb{L} = \underline{\mathbb{C}}$$

$\uparrow$ C^∞-bdl underlying $L \to C$

$$L = (\mathbb{L}, \bar\partial^L)$$

$\Rightarrow$ any hol. str. on $\mathbb{L}$ is of the form

$$\bar\partial^L = \bar\partial + \eta \,, \quad \eta \in A^{0,1}(C).$$

clear: $\bar\partial \eta = 0$, i.e. $[\eta] \in H^{0,1}(C)$.

~~Then~~ Further we have the gauge action

$$g \cdot \bar\partial^L = \bar\partial + g^{-1}\bar\partial g + \eta$$

$$\Rightarrow \quad (\mathbb{L}, \bar\partial^L) \cong (\underline{\mathbb{C}}, \bar\partial) \quad \text{iff} \quad \eta = -g^{-1}\bar\partial g.$$

It turns out that this is true iff $[\eta] \in H^1(C,\mathbb{Z})$. (Under the above inclusion) (*)

Cor.: $\mathcal{M}_{Dol}(C, \mathbb{C}^*) \cong T^* \mathrm{Jac}(C).$

$$\cong V/_{\Gamma_C} \oplus \overline{V}$$

w/ $V = H^{0,1}(C)$, $\Gamma_C = H^1(C, \mathbb{Z})$.

proof: The 2^{nd} isom. is clear because

$$\overline{H^{0,1}(C)} = H^{1,0}(C) = H^0(C, K_C).$$

1^{st} isom.: $T^* \mathrm{Jac}(C) \cong \mathrm{Jac}(C) \times H^{0,1}(C)^*.$

Hence it remains to prove that $H^{0,1}(C)^* \simeq \overline{H^{0,1}(C)}$
$= H^{1,0}(C).$

To this end, define the pairing

$$H^{0,1}(C) \otimes_{\mathbb{C}} H^{0,1}(C) \longrightarrow \mathbb{C}$$

$$(\eta_1, \eta_2) \longmapsto i \int_C \eta_1 \wedge \overline{\eta_2}$$

which is non-degenerate. $\Rightarrow H^{0,1}(C)^* \cong \overline{H^{0,1}(C)}$
$= H^{1,0}(C).$

$\blacksquare$