We pick up (HK) hyperkähler mf.s $(M, g; I, J, K)$ from yesterday and deduce a few properties.

<u>Lem.:</u> $(M, g; I, J, K)$ HK mf.

Define $I_{\hat{a}} := a_1 I + a_2 J + a_3 K$, $\hat{a} \in \mathbb{R}^3$ "purely imaginary quaternions" of norm 1

Then $(M, g; I_{\hat{a}})$ is a Kähler mf. for all $\hat{a} \in S^2$.

<u>Ex.:</u> Let $M = \mathbb{H}$ be the quaternions w/ norm

$$g(q,q) = \|q\|^2$$ from which we get a metric g on $\mathbb{H}$.

Clearly $I = i$, $J = j$, $K = k$ give ~~this~~ $(\mathbb{H}, g)$ an HK-structure.

HK mf.s are closely related to hol. symplectic mf.s:

<u>Lem.:</u> $(M, g; I, J, K)$ HK mf.

Define $\Omega_I = \omega_J + i\,\omega_K$ for the Kähler forms

$$\omega_J = g(J\cdot, \cdot)$$
$$\omega_K = g(K\cdot, \cdot).$$

Then $((M, I), \Omega_I)$ is hol. symplectic, i.e. Ω_I is a hol. $(2,0)$-form wrt. which is non-degenerate.

<u>proof:</u> (I, J, K) satisfy the $\mathbb{H}$-relations, in particular

$$JI = -K, \qquad KI = J.$$

Then we compute

$$\Omega_I(Iv,w) = g(JIv,w) + ig(KIv,w)$$
$$= -g(Kv,w) + ig(v,w)$$
$$= i\,\Omega_I(v,w).$$

$$\Omega_I(v,Iw) = i\,\Omega_I(v,w). \Rightarrow (2,0)$$

closedness implies hol.

Non-degeneracy: Use non-deg. of $\omega_{I,J,K}$.

$(\square)$

Ex.: $M = \mathbb{H}$ as above. Then $\left(\mathbb{H} \cong \overset{(x_0,\dots,x_3)}{\mathbb{R}^4}\right)$

$$\Omega_I = dw \wedge dz \qquad \text{for } w = x_0 + ix_1$$
$$z = x_2 + ix_3.$$

In particular, $(\mathbb{H}, I) \cong (\mathbb{C}^2, i)$.

Similarly: $\mathbb{H}^n$.

Self-duality equations

We start by deducing Hitchin's self-duality eq.s on Riemann surfaces, following the very first part of [Hit 87].

Interlude: Self-duality eq.s in (real) 4 dimensions.

(M, g) $\underbrace{\text{oriented}}_{\text{compact}}$ Riemannian 4-mf.

$A^k(M)$ smooth k-forms on M

$* : A^k(M) \longrightarrow A^{4-k}(M)$ Hodge $*$-operator

locally: $*(dx_1 \wedge \dots \wedge dx_k) = dx_{k+1} \wedge \dots \wedge dx_n$ $n=4 \dots$

oriented

$V \longrightarrow M$ smooth vbdl. of rank r (more generally any G-bdl. w/ G compact Lie group)

The $*$-operator extends to

$$* : \Omega^k(M, \operatorname{End} V) \longrightarrow \Omega^{4-k}(M, \operatorname{End} V).$$

$k=2$: $*^2 = \mathrm{id}$ so that

orthogonal wrt. induced metric

$$\Omega^2(M, \operatorname{End} V) = \Omega^2_+ \oplus \Omega^2_-$$

"+1" — "−1" — eigenspace of $*$.

Let $\mathcal{A}(M,\mathbb{V}) = \left\{ \begin{array}{l} \nabla : TM \times \mathbb{V} \longrightarrow \mathbb{V} \\ \quad\quad \mathbb{V} \longrightarrow \mathcal{A}^1(\mathbb{V}) \end{array} \right\}$

be the affine space of connections on $\mathbb{V}$ (affine for $\mathcal{A}^1(M, \operatorname{End}\mathbb{V})$). We define the Yang-Mills functional

$$S_{YM} : \mathcal{A}(M,\mathbb{V}) \longrightarrow \mathbb{R}, \quad \nabla \longmapsto \int_M |F^\nabla|^2$$

$$\Longrightarrow \quad S_{YM}(\nabla) = \int_M \underbrace{|F_+^\nabla|^2}_{\mathcal{A}^2_+} + \underbrace{|F_-^\nabla|^2}_{\mathcal{A}^2_-} .$$

Clearly: $S_{YM}(\nabla) \geq |k(\underset{\nabla}{\mathbb{V}})|$, $\quad k(\mathbb{V},\nabla) := \int_M |F_+^\nabla|^2 - |F_-^\nabla|^2$

<u>fact:</u> $k(\mathbb{V},\nabla)$ only depends on $\mathbb{V}$, <u>not</u> on ∇.

$\Longrightarrow$ if ∇ is a self-dual connection, $\boxed{F^\nabla = * F^\nabla}$, then

$$S_{YM}(\nabla) = k(\mathbb{V})$$

and it is a global minimum, in part. a critical point of S_{YM}.

We next only look at the self-duality eq.s, $F^\nabla = * F^\nabla$, on $M = \mathbb{R}^4$ w/ standard coord.s (x_1, x_2, x_3, x_4) and $\mathbb{V} = \underline{\mathbb{C}}^r := M \times \mathbb{C}^r \longrightarrow M$.

Let $d + A$, $\quad A = \sum_i A_i \, dx_i$, $A_i : M \to sL(r, \mathbb{R})$

be an $SL(r,\mathbb{R})$ - connection on V.

<u>dimensional reduction:</u> We next assume that A is invariant under the translations $(x_1, \ldots, x_4) \longmapsto (x_1, x_2, x_3 + a, x_4 + b)$, $(a,b) \in \mathbb{R}$.

Then $A_i = A_i(x_1, x_2)$ and we define

$$\nabla := d + A_1 dx_1 + A_2 dx_2 \quad , \quad \nabla_1 := \nabla_{\frac{\partial}{\partial x_1}} \, , \quad \nabla_2 := \nabla_{\partial/\partial x_2}$$

$$\phi_1 := A_3 \quad , \quad \phi_2 := A_4 .$$

The self-duality eq.s (SD1) for $d + A$ are equivalent to

$$F^\nabla = [\phi_1, \phi_2] \, , \qquad \nabla_1 \phi_1 = - \nabla_2 \phi_2$$

$$\nabla_1 \phi_2 = \nabla_2 \phi_1 .$$

Introducing $z = x_1 + i x_2$

$$\Phi = \tfrac{1}{2}(\phi_1 - i \phi_2) \, dz$$

$$\Phi^* = \tfrac{1}{2}(\phi_1 + i \phi_2) \, d\bar{z} \, ,$$

$sL(r,\mathbb{R}) \oplus i \, sL(r,\mathbb{R})$
$\shortparallel$
$sL(r,\mathbb{C})$ - valued

we arrive at the equivalent

$$\boxed{\begin{aligned} F^\nabla &= - [\Phi, \Phi^*] \\ \bar{\partial}^\nabla \Phi &= 0 . \end{aligned}} \qquad (SD_r)$$

for the rank

These equations are invariant under conformal transformations and hence make sense on any (compact) Riemann surface C

We will look at two ~~special~~ slightly modified cases:

1.) $r = 2$: $\quad E \to C$ hol. rk. 2 bdl, $\Lambda^2 E \overset{C^\infty}{\cong} \underline{\mathbb{C}}$

$(\to \quad E \underset{C^\infty}{\cong} \underline{\mathbb{C}}^2)$

h Hermitian metric

$\Phi \in \mathcal{A}^{1,0}(C, \mathrm{End}_0 E)$ — trace-free

locally: $A_z\, dz$, $\operatorname{tr} A_z = 0$

$$\boxed{\begin{aligned} F^{\nabla^h} &= -[\Phi, \Phi^{*_h}] \\ \bar\partial^{\nabla^h} \Phi &= 0 \end{aligned}}$$

(SD_2)

here: $h(\Phi v, w) = h(v, \Phi^{*_h} w)$

These are the self-duality eq.s that Hitchin considers in [Hit 87].

2.) $r = 1 \quad \to$ next lecture.

<u>(SD_2) and Higgs bdl.s:</u>

Assume $((E,h),\Phi)$ solves (SD_2). Then it defines the $SL(2,\mathbb{C})$-Higgs bundle (E,Φ),

- $E \to C$ hol. rk. 2 bdl., $\Lambda^2 E \overset{C^\infty}{\cong} \underline{\mathbb{C}}$
- $\Phi \in H^0(C, K_C \otimes \mathrm{End}_0 E)$

$K_C := T^*C$ hol. cotangent bdl.

eq.: $\Phi : E \to K_C \otimes E$ morph. of hol. bdl.s

So any sol. of (SD_2) gives a Higgs bdl. What about the converse?

$g(C) = \frac{1}{2} \dim H^1(C,\mathbb{C})$

<u>Ex.:</u> Let C be of genus $g(C) > 1$.

$\exists$ lbdl. $L \to C$ s.t. $L^{\otimes 2} \cong K_C$

$$\Updownarrow$$

$$(L^*)^{\otimes 2} \cong TC$$

Set $E := L \oplus L^*$ and $\Phi \in H^0(C, \mathrm{End}_0 E)$, locally given by

$$\phi_z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dz \quad \text{wrt.} \quad (\underset{L}{dz^{1/2}}, \underset{L^*}{dz^{-1/2}})$$

We want to find a metric h s.t. $((E,h),\Phi)$ satisfies (SD_2).

<u>ansatz:</u> $h = \begin{pmatrix} h_0^{-1} & \\ & h_0 \end{pmatrix}$ in particular, $\underline{h_0^2 =: g}$ gives a metric on $L^{-2} = TC$.

$$\longrightarrow \quad F^{\nabla^h} = \begin{pmatrix} -\partial\bar\partial \log h_0 & 0 \\ 0 & \partial\bar\partial \log h_0 \end{pmatrix}$$

$$\overset{!}{=} -[\Phi, \Phi^{*h}] = -\begin{pmatrix} -h_0^2 & 0 \\ 0 & h_0^2 \end{pmatrix}$$

$$\Leftrightarrow \quad \boxed{\partial\bar\partial \log g = -2g}$$

this is the eq. for a constant curv. metric on C which always exists.

Observe: $L^* \subset E$ is the only lbdl. which is preserved by Φ.

More generally, we define:

<u>def.</u>: A $SL(2,\mathbb{C})$-Higgs bdl. (E,Φ) is <u>stable</u> if

$$\frac{\deg L}{\mathrm{rk}\, L} < \frac{\deg E}{\mathrm{rk}\, E} = 0$$

for all holomorphic lbdls $L \subsetneq E$ s.t. $\Phi(L) \subseteq K_C \otimes L$.

The relevance of this def. is the following

<u>Thm.</u> (Hitchin)

$$\underbrace{\left\{\begin{matrix}\text{stable Higgs}\\ \text{bdls}\end{matrix}\right\}\Big/_{\text{iso.}}}_{\substack{\parallel \\ \mathcal{M}_{Dol}(C,SL(2,\mathbb{C}))}} \xleftrightarrow{1:1} \underbrace{\left\{\begin{matrix}\text{irred. sol.s}\\ \text{of } (SD_2)\end{matrix}\right\}\Big/_{\text{gauge}}}_{\substack{\parallel \\ \mathcal{M}(C,SL(2,\mathbb{C}))}}$$

$\uparrow$
Dolbeault

In fact, $\mathcal{M} = \mathcal{M}(C,SL(2,\mathbb{C}))$ carries a HK structure (I,J,K)

and $\mathcal{M}_{Dol}(C,SL(2,\mathbb{C}))$ the structure of a cplx. mf. s.t.

$$\mathcal{M}_{Dol}(C,SL(2,\mathbb{C})) \cong (\mathcal{M}, I)$$

as cplx. mf.s.

<u>Relation to flat $SL(2,\mathbb{C})$-conn.s</u> ("de Rham")

Observe that if $((E,h),\Phi)$ solves (SD_2), then

$$D := \nabla^h + \Phi + \Phi^{*_h}$$

is a flat $SL(2,\mathbb{C})$-connection on C. Hitchin & Donaldson extends the previous theorem to:

$$\mathcal{M}(C, SL(2,\mathbb{C}))$$

$\boxed{\text{Dolbeault}}$ $\quad$ "non-abelian Hodge correspondence" $\quad$ $\boxed{\text{de Rham}}$

$$\mathcal{M}_{Dol}(C, SL(2,\mathbb{C})) \longleftrightarrow \mathcal{M}(C, SL(2,\mathbb{C})) \longleftrightarrow \mathcal{M}_{dR}(C, SL(2,\mathbb{C}))$$

$$\mathcal{M}_{dR}(C, SL(2,\mathbb{C})) = \{\text{flat } SL(2,\mathbb{C})\text{-conns}\}/\text{gauge}$$

$\boxed{\text{Betti}}$

$$\mathcal{M}_{dR}(C, SL(2,\mathbb{C})) \longleftrightarrow \mathcal{M}_{Betti}(C, SL(2,\mathbb{C}))$$

$$\mathcal{M}_{Betti}(C, SL(2,\mathbb{C})) = \{\text{rep.s } \pi_1(C, c_0) \longrightarrow SL(2,\mathbb{C})\}/\text{conjugation}$$

$\mathcal{M}_{dR}$ & $\mathcal{M}_{Betti}$ carry complex structures s.t.

$$(\mathcal{M}, J) \cong \mathcal{M}_{dR} \cong \mathcal{M}_{Betti}$$
$$K$$

as complex mf.s.