# 4. Hermitian, Kähler and hyper Kähler structures

## 4.1) Holomorphic vector bundles

Def 4.1.1: Let $M$ be a complex manifold and let $\pi: E \to M$ be a complex vector bundle over $M$ (i.e. each fibre $\pi^{-1}(x)$ is a complex vector space. $E$ is called a holomorphic vector bundle if there exists a trivialization with holomorphic transition functions. More precisely, there exists an open cover $\mathcal{U}$ of $M$ and for each $U \in \mathcal{U}$ a diffeomorphism $\psi_U: \pi^{-1}(U) \to U \times \mathbb{C}^r$ s.t.

- The following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & U \times \mathbb{C}^r \arrow[dl, "pr_U"] \\ U & \end{tikzcd}$$

- For every intersecting $U$ and $V$ one has

$$\psi_U \circ \psi_V^{-1}(x,v) = (x, g_{UV}(x)\, v) \quad \text{where}$$

$$g_{UV}: U \cap V \to Gl_r(\mathbb{C}) \subset \mathbb{C}^{r^2} \text{ are hol. fcts.}$$

Ex: The tangent bundle of a hol. manifold

For every holomorphic bundle $E$ one defines the bundles $\Lambda^{p,q}(E) := \Lambda^{p,q} M \otimes E$ of $E$-valued forms on $M$ of type $(p,q)$.

The space of sections of $\Lambda^{p,q}(E)$ is denoted by $\Omega^{p,q}(E)$ ②

We define the $\bar\partial$-operator $\bar\partial : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ in the following way. If a section $\sigma$ of $\Lambda^{p,q}(E)$ is given by $\sigma = (\omega_1, --, \omega_k)$ in some local trivialization we define $\bar\partial\sigma = (\bar\partial\omega_1, -- \bar\partial\omega_k)$. Suppose $\sigma$ is written as $\sigma = (\tau_1, -- \tau_k)$ in some other trivialization, then one has $\tau_j = \sum_{\ell=1}^{k} g_{j\ell}\,\omega_\ell$ for some hol. fcts $g_{j\ell}$

thus $\bar\partial \tau_j = \sum_{\ell=1}^{k} g_{j\ell}\,\bar\partial\omega_\ell$ so $\bar\partial\sigma$ does not depend on the chosen trivialization. By construction one has $\bar\partial^2 = 0$ and $\bar\partial$ satisfies the Leibniz rule

$$\bar\partial(\omega \wedge \sigma) = (\bar\partial\omega)\wedge\sigma + (-1)^{p+q}\,\omega \wedge(\bar\partial\sigma)$$

$\forall\, \omega \in \Omega^{p,q}M$, $\sigma \in \Omega^{r,s}(E)$

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## 4.2) Holomorphic structures:

**Def. 4.2.1:** A pseudo-holomorphic structure on a complex vector bundle $E$ is an operator $\bar\partial : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying the Leibniz rule. If, moreover, $\bar\partial^2 = 0$ the $\bar\partial$ is called a hol. structure.

A section $\sigma$ in a pseudo hol. vector bundle $(E, \bar\partial)$ is called hol. if $\bar\partial\sigma = 0$.

**Lemma 4.2.2:** A pseudo hol. vector bundle $(E, \bar\partial)$ of rank $k$ is hol. iff each $x \in M$ has an open neighborhood $U$ and $k$ hol. sections $\sigma_i$ of $E$ over $U$ s.t. $\{\sigma_i(x)\}$ form a basis of $E_x$

## 4.3 The curvature of a connection:

**Def 4.3.1:** A connection on $E$ is a $\mathbb{C}$-linear differential operator $\nabla : \Gamma(E) \to \Omega^1(E)$ satisfying the Leibniz rule

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla\sigma \quad \forall f \in C^\infty(M), \sigma \in \Gamma(E)$$

where $\Omega^1(E)$ denotes the space of $E$-valued 1-forms, i.e. sections of $\Lambda^1 M \otimes E$.

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The curvature operator of $\nabla$ is the $\mathrm{End}(E)$-valued 2-form

$$R^\nabla(\sigma) := \nabla(\nabla\sigma), \quad \forall \sigma \in \Gamma(E)$$

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More explicitly, if $\{\sigma_1 \ldots \sigma_n\}$ are local sections of $E$ which form a basis of each fibre over some open set $U$ we define local connection forms $\omega_{ij} \in \Omega^1(U)$ by

$$\nabla \sigma_i = \omega_{ij} \otimes \sigma_j$$

also the local curvature 2-forms $R^\nabla_{ij}$ are def. by

$$R^\nabla(\sigma_i) = R^\nabla_{ij} \otimes \sigma_j$$

$$R^\nabla_{ij} \otimes \sigma_j = R^\nabla(\sigma_i) = \nabla(\nabla\sigma_i) = \nabla(\omega_{ij} \otimes \sigma_j)$$

$$= (d\omega_{ij}) \otimes \sigma_j - \omega_{i\ell} \wedge \omega_{\ell j} \otimes \sigma_j$$

$$R^\nabla_{ij} = d\omega_{ij} - \omega_{i\ell} \wedge \omega_{\ell j}.$$

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## 4.4) Hermition structures and connections

Let $E \to M$ be a complex rank $k$ bundle over a smooth mfd $M$.

**Def. 4.4.1:** A Hermitian structure $H$ on $E$ is a smooth field of Hermitian products on the fibres of $E$, that is, for every $x \in M$ the map $H: E_x \times E_x \to \mathbb{C}$ satisfies

- $H(u,v)$ is $\mathbb{C}$-linear in $u$ for every $v \in E_x$
- $H(u,v) = \overline{H(v,u)} \quad \forall\, u,v \in E_x$
- $H(u,u) > 0 \quad \forall\, u \neq 0$
- $H(u,v)$ is a smooth fct. on $M$ for every smooth sections $u$ and $v$ of $E$

A complex vector bundle endowed with a Hermitian structure is called a Hermitian vector bundle.

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Suppose now that $M$ is a complex manifold. Consider the proj. $\pi^{1,0}: \Lambda^1(E) \to \Lambda^{1,0}(E)$ and $\pi^{0,1}: \Lambda^1(E) \to \Lambda^{0,1}(E)$

For every connection $\nabla$ on $E$ one can consider its $(1,0)$ and $(0,1)$ components $\nabla^{1,0} := \pi^{1,0} \circ \nabla$ and $\nabla^{0,1} := \pi^{0,1} \circ \nabla$

we can extend these operators to $\Omega^{p,q}(E)$

then $\nabla^{0,1}$ is a pseudo-holomorphic structure on $E$ for every connection $\nabla$.

For every section $\sigma$ of $E$ one can write

$$R^{\nabla}(\sigma) = \nabla^2 \sigma = (\nabla^{1,0} + \nabla^{0,1})^2 (\sigma)$$

$$= (\nabla^{1,0})^2(\sigma) + (\nabla^{0,1})^2(\sigma) + (\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0})(\sigma)$$

the the $(0,2)$ type component of the curvature is given by

$$(R^{\nabla})^{0,2} = (\nabla^{0,1})^2$$

if this vanishes then $E$ is a hol. bundle with $\bar\partial = \nabla^{0,1}$

Def. 4.4.2 We call $D$ an $H$-connection (or Hermitian connection) if $H$, viewed as a field of $\mathbb{C}$-valued bilinear forms on $E$ is parallel with respect to $D$.

Theorem 4.4.3: For every Hermitian structure $H$ in a hol. bundle $E$ with hol. structure $\bar\partial$, there exists a unique $H$-connection $D$ (called the Chern connection) such that $D^{0,1} = \bar\partial$.

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Rem. The $(0,2)$-component of the Chern connection vanishes.

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## 4.5 Hermitian and Kähler metrics:

Definition 4.5.1: A Hermitian metric on an almost complex manifold $(M, J)$ is a Riemannian metric $h$ such that $h(X,Y) = h(JX, JY)$ for all $X, Y \in TM$. The fundamental 2-form of a Hermitian metric is defined by

$$\Omega(X,Y) := h(JX, Y).$$

The extension by $\mathbb{C}$-linearity (also denoted by $h$) of the Hermitian metric to $TM^{\mathbb{C}}$ satisfies

- $h(\bar Z, \bar W) = \overline{h(Z, W)} \quad \forall\, Z, W \in TM^{\mathbb{C}}$
- $h(Z, \bar Z) > 0 \quad$ for $Z \neq 0$
- $h(Z, W) = 0 \quad \forall\, Z, W \in T^{1,0}M$ and $Z, W \in T^{0,1}M$

Rem. The tangent bundle of an almost complex manifold is in particular a complex vector bundle. If $h$ is a Hermitian metric on $M$, then

$$H(X,Y) := h(X,Y) - i\,h(JX,Y) = (h - i\Omega)(X,Y)$$ defines a Hermitian structure on the complex vector bundle $(TM, J)$ as defined in 4.4.1.

Conversely any Hermitian structure $H$ on $TM$ as a complex vector bundle defines a Hermitian metric $h$ on $M$ by $h := \mathrm{Re}(H)$

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Let $z_\alpha$ be hol. coordinates on a complex Hermitian manifold $(M^{2n}, h, J)$ and denote by $h_{\alpha\bar\beta}$ the coefficients of the metric tensor in these local coordinates

$$h_{\alpha\bar\beta} := h\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar z_\beta}\right)$$

The fundamental form is given by

$$\Omega = i \sum_{\alpha,\beta=1}^{m} h_{\alpha\bar\beta}\, dz_\alpha \wedge d\bar z_\beta$$

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Suppose that the fundamental form $\Omega$ of a complex Hermitian manifold is closed then there exists in the neighbourhood of each point a real function $u$ such that $\Omega = i\partial\bar\partial u$

which reads in local coordinates

$$h_{\alpha\bar\beta} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar z_\beta}$$

**Definition 4.5.2:**

A Hermitian metric $h$ on an almost complex manifold $(M,J)$ is called a Kähler metric if $J$ is a complex structure and the fundamental form $\Omega$ is closed

$$h \text{ Kähler} \iff \begin{cases} N^J = 0 \\ d\Omega = 0 \end{cases}$$

A local real function $u$ satisfying $\Omega = i\partial\bar{\partial}u$ is called a local Kähler potential of the metric $h$.

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**Definition 4.5.3:** A Riemannian manifold $(M^n, g)$ is called hyperkähler if there exist three complex structures $I, J, K$ on $M$ satisfying $K = IJ$ such that $g$ is a Kähler metric w.r.t. each of these complex structures.

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