2. Bundles and connections

2.1 Lie groups:

<u>Definition 2.1.1</u>: A group G which has a smooth manifold structure such that the multiplication $G \times G \longrightarrow G$ and the inverse map $G \longrightarrow G$ are smooth is called a <u>Lie group</u>.

For $g \in G$ we denote by $L_g : G \to G$ and $R_g : G \to G$ the left and right multiplication by g, called left and right translations which are clearly smooth diffeomorphisms of G with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$.

A vector field X on G is called left invariant if $dL_g(X) = X$ for every $g \in G$, i.e.

$$(dL_g)_{g^{-1}a}\left(X_{g^{-1}a}\right) = X_a \qquad \forall\, a, g \in G$$

in particular $X_g = (dL_g)_e X_e$.

We denote by $\mathfrak{g}$ the vector space of left invariant vector fields on G, called the <u>Lie algebra</u> of G.

Let φ_t denote the local flow of some $X \in \mathfrak{g}$, and let $\varepsilon > 0$ st the curve $a_t := \varphi_t(e)$ is defined for $t < \varepsilon$.

a_t is a local integral curve of X, i.e. $\dot{a}_t = X_{a_t}$.

If g is any element of G we have

$$\frac{d}{dt}(g a_t) = \frac{d}{dt}(L_g a_t) = dL_g(X_{a_t}) = X_{g a_t}$$

So $g a_t$ is a local integral curve for X near g

This means that the local flow of X is $\varphi_t = R_{a_t}$

In particular $a_t a_s = a_{t+s}$ $\quad t, s, s+t < \varepsilon$ which

allows us to define a_t for every real t by requiring

$t \mapsto a_t$ to be group morphism from $(\mathbb{R}, +)$ to G.

The image of this morphism is called the 1-parameter

subgroup generated by X and one usually denotes $\exp(tX) = a_t$

<u>Examples</u>: $Gl_n(\mathbb{R})$ and $Gl_m(\mathbb{C})$

the Lie algebra structure of on $gl_n(\mathbb{R}) = T_{I_n} Gl_n(\mathbb{R}) \simeq M_n(\mathbb{R})$

is given by $[A, B] = AB - BA$.

<u>Definition 2.1.2</u>:

A group action (to the right) of a Lie group G on a manifold

M is a smooth map $M \times G \to M$, denoted by $(m, g) \longmapsto mg$

such that $\quad$ i) $me = m$ $\quad \forall m \in M$

$\quad$ ii) $m(gh) = (mg)h$ $\quad \forall m \in M, g, h \in G$

In part. $R_g : M \to M$ defined by $R_g(m) = mg$ are all diffeo.

The action is called <u>free</u> if R_g has no fixed point for all $g \neq e$

and transitive if for every $m, m' \in M$ there exists $g \in G$

s.t. $m' = mg$.

2.2| Principal bundles

Let G be a Lie group and let M be a smooth manifold.

Definition 2.2.1. A G-principal bundle (also called a G-structure) over M is a smooth manifold P together with a smooth submersion $\pi : P \to M$ and a group action of G on P to the right which restricts to a free transitive action on each fibre $\pi^{-1}(x)$. G is called the structure group of P. A section of P is a smooth map $\sigma : M \to P$ s.t. $\pi \circ \sigma = Id_M$

2.3| Vector bundles

Definition 2.3.1. Let M be a smooth manifold. A rank k vector bundle over M is a smooth manifold E together with a submersion $\pi : E \to M$ such that

i) Each fibre $E_x := \pi^{-1}(x)$ has a structure of a k-dim real vector space.

ii) (Local triviality) For every $x \in M$ there exists an open neighbourhood U of x and a diffeomorphism $\varphi_U : \pi^{-1}U \to U \times \mathbb{R}^k$ whose restriction to E_y is a vector space isomorphism onto $\{y\} \times \mathbb{R}^k$ for every $y \in U$.

A section of E is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = Id_M$. The space of all sections of E is denoted by $\Gamma(E)$.

examples: the tensor bundles $T^{k,l}M$

2.4] Correspondence between principal and vector bundles

To any G-principal bundle P over a manifold M and k-dimensional representation $\rho : G \to Gl_k(\mathbb{R})$ of G one can associate a rank k vector bundle

$$E := P \times_\rho \mathbb{R}^k := P \times \mathbb{R}^k / \sim$$

where $\sim$ is the equivalence relation on $P \times \mathbb{R}^k$

$$(u, \xi) \sim (v, \zeta) \Leftrightarrow \exists g \in G \text{ s.t. } v = ug$$

$$\text{and } \zeta = \rho(g^{-1})\xi .$$

The equivalence class defined by a pair $(u, \xi) \in P \times \mathbb{R}^k$ is denoted by $[u, \xi]$, and has to be understood as the element in E represented by ξ in the frame u

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Conversely, if E is a rank k vector bundle, we define the frame bundle $Gl(E)$, whose fibre at x is the set of all isomorphisms $u : \mathbb{R}^k \to E_x$. $Gl(E)$ is a $Gl_k(\mathbb{R})$ principal bundle over M with the group action

$$Gl(E) \times Gl_k(\mathbb{R}) \to Gl(E) \text{ defined by}$$

$$(u, A) \longmapsto u \circ A$$

2.5 Covariant derivatives on vector bundles:

Let $\pi: E \to M$ be a vector bundle.

Definition 2.5.1: A covariant derivative on E is an $\mathbb{R}$-linear operator $\nabla: C^\infty(M) \times \Gamma(E) \to \Gamma(E)$ denoted by $(Y, \sigma) \mapsto \nabla_Y \sigma$ such that for all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ the following conditions are satisfied:

i) $\nabla_{fX} \sigma = f \nabla_X \sigma$

ii $\nabla_X (f\sigma) = f \nabla_X \sigma + (\partial_X f)\, \sigma$.

Given a vector $X \in T_pM$, a local section σ is called parallel in the direction of X at p if $\nabla_{X_p} \sigma = 0$. More generally, σ is called parallel along a curve γ_t in M if $\nabla_{\dot\gamma_t} = 0$ for all t.

Remark 2.5.2: For any $e \in E$ we can obtain a linear map $h: T_{\pi(e)} M \to T_e E$ denoted by

$X \longmapsto \tilde{X} := \sigma_*(X)$, where σ is any local section of E which is parallel in the direction of X at p and satisfies $\sigma(p) = e$. By construction we have

$$\pi_* \circ h = Id$$

The vector $\tilde{X}$ is called the <u>horizontal lift</u> of X at e and the image of h, denoted by $T_e^h E$ is called the <u>horizontal subspace</u> of $T_e E$. Since π_* is surjective, h is an isomorphism from $T_{\pi(e)} M$ to $T_e^h E$.

If we denote the tangent space to the fibre of E at e by $T_e^v E$, we have $T_e E = T_e^v E \oplus T_e^h E$.

The union of all subspaces $T_e^h E$ forms a smooth vector subbundle of rank n of TE, called the <u>horizontal distribution</u>.

2.6 <u>Connections on principal bundles and linear connections</u>

<u>Definition 2.6.1.</u> Let P be a G-principal bundle over a manifold M with vertical distribution (tangent to the fibre) denoted by $\mathcal{V}$. A connection on P is a smooth distribution $\mathcal{H}$ such that $TP = \mathcal{H} \oplus \mathcal{V}$ and $(R_a)_*(\mathcal{H}_u) = \mathcal{H}_{ua}$ for all $u \in P$ and $a \in G$.

<u>Definition 2.6.2.</u> A linear connection on M is a connection in the frame bundle of M, or equivalently, a covariant derivative on TM.

Lemma 2.63. Let ∇ be the covariant derivative of a linear connection on M. Then the expression

$$T^{\nabla}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \quad \forall X,Y \in \mathfrak{X}(M)$$

defines a tensor field of type $(2,1)$ called the torsion of ∇.

A linear connection is called torsion-free if its torsion vanishes.