Introduction to Higgs bundles

Lecture 1

Source
"Lectures on Kähler Geometry"
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1.) Manifolds, tensor fields and derivatives

1.1 Smooth manifolds

Definition 1.1.1: A smooth (differentiable) manifold of dimension n is a topological manifold $(M, \mathcal{U})$ whose atlas $\{\phi_U\}_{U \in \mathcal{U}}$ satisfies the following compatibility condition, for every intersecting $U, V \in \mathcal{U}$, the map between open sets of $\mathbb{R}^n$

$$\phi_{UV} := \phi_U \circ \phi_V^{-1} \quad \text{is a diffeomorphism.}$$

If this condition holds, the atlas $\{\phi_U\}_{U \in \mathcal{U}}$ is also called a smooth structure on M.

Def 1.1.2: Let $(M, \{\phi_U\}_{U \in \mathcal{U}})$, $(N, \{\psi_V\}_{V \in \mathcal{V}})$ be two smooth manifolds. A continuous map $f: M \to N$ is said to be smooth if $\psi_V \circ f \circ \phi_U^{-1}$ is a smooth map for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$. A homeomorphism which is smooth, together with its inverse, is called a diffeomorphism.

Def 1.1.3 Let M be a smooth manifold. A local coordinate system around some $x \in M$ is a diffeomorphism btw an open neighborhood of x and an open set in $\mathbb{R}^n$

1.2] The tangent space:

Recall that if $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function, its differential at any point $x \in U$ is the linear map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix in the canonical basis is

$$(df_x)_{ij} := \frac{\partial f_i}{\partial x_j}(x)$$

Prop. 1.2.1: (Chain rule) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be two open sets and let $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^k$ be two smooth maps. Then for every $x \in U \cap f^{-1}(V)$ we have

$$d(g \circ f)_x = dg_{f(x)} \circ df_x \qquad (1.2.1)$$

in particular when $m = n = k$, $f : U \to V$ is a diffeomorphism and $g = f^{-1}$ the previous relation reads

$$(d(f^{-1}))_{f(x)} = (df_x)^{-1} \qquad (1.2.2)$$

Let x be a point of some manifold $(M, \mathcal{U})$ of dim n. We denote by I_x the set of all $U \in \mathcal{U}$ containing x. On $I_x \times \mathbb{R}^n$ we define the relation "$\sim_x$" by

$$(U, u) \sim_x (V, v) \iff u = (d\,\phi_{UV})_{\phi_V(x)}(v)$$

$\sim_x$ is an equivalence rel. by (1.2.1) and (1.2.2)

An equivalence class is called a tangent vector of M at x.
By linearity of $d\phi_{UV}$ we see that the quotient $I_x \times \mathbb{R}^n / \sim_x$ is an n-dimensional vector space. This vector space is called the tangent vector space of M at x and is denoted by $T_x M$. The tangent vector defined by the pair (U, x) is denoted by $[U, x]_x$.

For each $U \in \mathcal{U}$ containing x, a tangent vector $X \in T_x M$ has a unique representative (U, u) in $\{U\} \times \mathbb{R}$.

Definition 1.2.2: The union of all tangent spaces $TM := \bigsqcup_{x \in M} T_x M$ is called the tangent bundle of M.

If M and N are smooth manifolds, $f : M \to N$ is a smooth map and $x \in M$, one can define the differential

$df_x : T_x M \to T_{f(x)} N$ in the following way:

choose local charts ϕ_U and ψ_V around x and $f(x)$ and define

$$df_x([U, u]) := [V, d(\psi_V \circ f \circ \phi_U^{-1})_{\phi_U(x)}(u)].$$

If $f : M \to N$ is a smooth map, the collection $(df_x)_{x \in M}$ defines a map $df : TM \to TN$ called the differential of f which will sometimes be denoted by f_*.

Def 1.2.3: A smooth map $f : M \to N$ is called a submersion if its differential df_x is onto (surj.) for every $x \in M$

Theorem 1.2.4: If $f : M \to N$ is a submersion then $f^{-1}(y)$ is a smooth submanifold of M for every $y \in N$.

1.3 Vector fields.

Let M be a smooth manifold. Every map $X : M \to TM$ such that $X(x) \in T_xM$ for all $x \in M$ defines for every local chart $\phi : U \to \hat{U} \subset \mathbb{R}^n$ a map $X_\phi : \hat{U} \to \mathbb{R}^n$ by

$$X_\phi(\phi(x)) := d\phi_x(X_x)$$

If all these maps are smooth, we say that X is a (smooth) vector field on M. For $x \in M$, $X(x)$ (X_x) is a vector in the tangent space T_xM. The set of all vector fields on M is a module over the algebra of smooth fcts $C^\infty(M)$ and is denoted by $\mathcal{X}(M)$.

Ex.: Let e_i denote the constant vector field on $\mathbb{R}^n$ defined by the i-th element of the canonical basis.

If $\phi_U : U \to \hat{U}$ is a local chart on M, we define the local vector field $\frac{\partial}{\partial x_i}$ on U by $\frac{\partial}{\partial x_i}(x) := [U, e_i]_x$ i.e.

$$(d\phi_U)_x\left(\frac{\partial}{\partial x_i}(x)\right) = e_i \quad \forall \ x \in U.$$

Since for every $x \in U$, $(d\phi_U)_x$ is an isom. between T_xM and $\mathbb{R}^n$, $\left\{\frac{\partial}{\partial x_i}(x)\right\}_{i=1,\dots,n}$ is a basis of T_xM.

We call $\{\partial/\partial x_i\}$ a local frame on U.

Rem 1.3.1 i) Let X be a smooth vector field on a smooth mfd M and $f \in C^\infty(M)$ we define $(\partial_X f)(x) := df_x(X)$

ii) $\exists$ isom of $C^\infty(M)$-modules $\phi : \mathcal{X}(M) \to \mathcal{D}(C^\infty(M))$

"Lie algebra of derivations of the alg of smooth fct"

1.4) Integral curves

1.4.1 Def A path on a manifold M is a smooth map $c: \mathbb{R} \to M$.
The tangent vector to c at t, denoted by $\dot{c}(t)$ is by definition the image of the canonical vector $\frac{\partial}{\partial t} \in T_t\mathbb{R}$ through the differential of c at t.

$$\dot{c}(t) := dc_t\left(\frac{\partial}{\partial t}\right)$$

1.4.2) Let X be a vector field on M and let x be some pt of M. A local integral curve of X through x is a local path $c: (-\varepsilon, \varepsilon) \to M$ such that $c(0) = x$ and $\dot{c}(t) = X_{c(t)}$ for every $t \in (-\varepsilon, \varepsilon)$

1.4.3 Proposition
Let $X \in \mathcal{X}(M)$ be a smooth vector field on the manifold M.

i) For every $x \in M$ there exists an open neighbourhood U_x of x and $\varepsilon > 0$ st. the integral curve of X through every $y \in U_x$ is defined for $|t| < \varepsilon$.

ii) For $x \in M$, let U_x and ε be given by (i). If $t < \varepsilon$ we define the map

$$\varphi_t : U_x \to M \quad \text{by} \quad \varphi_t(y) := c_y(t)$$

where c_y is the integral curve of X through y, then we have

$$\varphi_t \circ \varphi_s = \varphi_{s+t} \qquad \forall\ |t|, |s|, |s+t| < \varepsilon$$

iii) for every $t < \varepsilon$ the local map φ_t is a local diffeo.

$\{\varphi_t\}$ is called the <u>local flow</u> of V.

1.5) Tensor fields

Let V denote a real vector space of dim n with dual vector space V^*. Let $V^{k,\ell} := V^{\otimes k} \otimes (V^*)^{\otimes \ell}$ denote the space of tensors of type (k,ℓ)

<u>Definition 1.5.1</u>: A (smooth) tensor field of type (k,ℓ) on an open set of $\mathbb{R}^n$ is a smooth map defined on that open set with values in $(\mathbb{R}^n)^{k,\ell}$

Let now M be a manifold with smooth atlas $(U, \phi_U)_{U \in \mathcal{U}}$ for $x \in M$ let $T_x^{k,\ell} M := (T_x M)^{k,\ell}$ be the space of tensors of type (k,ℓ), we denote by $T^{k,\ell} M$ the disjoint union $\bigsqcup_{x \in M} T_x^{k,\ell} M$ the tensor bundle of type (k,ℓ)

$T^{1,0}M$ is the tangent bundle $T^{0,1}M$ its dual is usually denoted by T^*M

<u>Def 1.5.2</u>: A smooth tensor field K of type (k,ℓ) on M is a map $K : M \to T^{k,\ell} M$ s.t.

- $K(x) \in T_x^{k,\ell} M$ for all $x \in M$ and
- K is smooth in the sense that for every local chart $\phi : U \to \tilde{U}$ the map $d\phi(K) : \tilde{U} \to (\mathbb{R}^n)^{k,\ell}$ given by $d\phi(K)(y) := d\phi_{\phi^{-1}(y)}(K)$ is smooth

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<u>Def 1.5.3</u>: If $\varphi : M \to N$ is a diffeomorphism between two manifolds M and N, its differential maps tensor fields of type (k,ℓ) on M to tensor fields of type (k,ℓ) on N by the formula

$$K \mapsto \varphi_*(K), \text{ where}$$

$$\varphi_*(K)_y := d\varphi(K_{\varphi^{-1}(y)})$$

The tensor field $\varphi_*(K)$ is called the <u>push forward</u> of K.

1.6) The Lie derivative of tensor fields

1.6.1) Definition. We introduce the Lie derivative of a tensor field K with respect to a vector field X. Let φ_t denote the local flow of X. We define

$$\mathcal{L}_X K := \lim_{t \to 0} \frac{1}{t} \left(K - (\varphi_t)_* (K) \right)$$

1.6.2) Def. The Lie bracket of two vector fields X and Y on a smooth manifold M is the vector field denoted by $[X, Y]$ corresponding to the derivation $\partial_X \circ \partial_Y - \partial_Y \circ \partial_X$ on $C^\infty(M)$

1.7 The exterior derivative:

Let M be a smooth manifold and $x \in M$. A p-form is an element of $\Lambda^p(T_x^* M)$ which is canonically isomorphic to the space of p-linear alternated maps $(T_x M)^p \to \mathbb{R}$. The p-th exterior bundle of M is the disjoint union

$$\Lambda^p M = \bigsqcup_{x \in M} \Lambda^p(T_x^* M).$$

A p-form is a map $\omega : M \to \Lambda^p M$ s.t. $\omega_x \in \Lambda^p(T_x^* M)$ for all x.

We denote the space of smooth p-forms $\Gamma(\Lambda^p M)$ by $\Omega^p M$ and the space of exterior forms $\Gamma(\Lambda M)$ by $\Omega^* M$.

Theorem 1.7.1: Let M be a smooth mfld of dim n, there exists a unique $\mathbb{R}$-linear endomorphism d of $\Omega^* M$ called the exterior derivative satisfying

i) d maps p-forms to $(p+1)$-forms
ii) d is the usual differential on functions or elements of $\Omega^0 M$
iii) $d(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^p \theta \wedge d\omega$ for θ a p-form, ω a q-form
iv) $d \circ d = 0$

Theorem 1.7.2: For every vector field X and exterior form ω on M, the following relation holds:

$$\mathcal{L}_X \omega = d(X \lrcorner \omega) + X \lrcorner d\omega$$

Notation: $(X \lrcorner \omega)(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1})$ interior product of ω p-form