AKS, R-matrices, group factorizations

1

Lie Poisson structure : $\mathfrak{g}$ Lie alg. G connected $\rightsquigarrow (\mathfrak{g}^*, \{\cdot,\cdot\})$

$$\{f,g\}(\alpha) = \alpha([df_\alpha, dg_\alpha]) \qquad f,g \in C^\infty(\mathfrak{g}^*)$$

Ham. vector field of f : $dg_\alpha(X_f) = X_f|_\alpha(g) = \{f,g\}(\alpha) = \alpha([df_\alpha, dg_\alpha])$

$$= -\mathrm{ad}^*_{df_\alpha}(\alpha)(dg_\alpha). \Rightarrow \boxed{X_f|_\alpha = -\mathrm{ad}^*_{df_\alpha}(\alpha)}$$

If $f \in C^\infty(\mathfrak{g}^*)^G$, i.e. $f(\mathrm{Ad}^*_g \alpha) = f(\alpha) \rightsquigarrow \underline{\underline{0}} = \frac{d}{dt} f(\mathrm{Ad}^*_{\exp(tL)}\alpha)$

$$= df_\alpha(\mathrm{ad}^*_L \alpha) = \alpha([L, df_\alpha]) = -\mathrm{ad}^*_{df_\alpha}(\alpha)(L) \quad \forall L \in \mathfrak{g}$$

$\Rightarrow \underline{X_f = 0}$ if $f \in C^\infty(\mathfrak{g}^*)^G$. Have shown: $\{f,g\} = 0 \;\; \forall f,g \in C^\infty(\mathfrak{g}^*)^G$

Aim: modify the setup $\{\cdot,\cdot\} \rightsquigarrow \{\cdot,\cdot\}_R$ s.t. still $\{f,g\}_R = 0$ but ${}^R X_f \neq 0$.

R-matrices : idea $R : \mathfrak{g} \xrightarrow{\text{linear}} \mathfrak{g} \rightsquigarrow [L,M]_R = [RL,M] + [L,RM]$

②

Jacobi-identity ~~is~~ is satisfied if R solves the modified Yang-Baxter eq.

$$R[L,M]_R - [\overset{L}{R\cancel{*}}, \overset{M}{R\cancel{*}}] = \underset{\substack{\uparrow \\ c \in \mathbb{R}, \mathbb{C}}}{c}[L,M] \qquad \Big(\underset{=0}{[L_1,[L_2,L_3]_R]_R + \text{cyclic}} \Big)$$

$$\rightsquigarrow (\mathfrak{g}, [\cdot,\cdot]_R) \rightsquigarrow (\mathfrak{g}^*, \{\cdot,\cdot\}_R) \qquad f \in C^\infty(\mathfrak{g}^*) \rightsquigarrow {}^R X_f|_\alpha = -{}^R\mathrm{ad}^*_{df|_\alpha}(\alpha)$$

Explicitly: $-{}^R\mathrm{ad}^*_{df_\alpha}(\alpha)(L) = \alpha([df_\alpha, L]_R) = \underbrace{\alpha([Rdf_\alpha, L]}_{} + [df_\alpha, RL])$

$$= -\mathrm{ad}^*_{Rdf_\alpha}(\alpha)(L) - \mathrm{ad}^*_{df_\alpha}(\alpha)(RL)$$

$$(\text{if } f \in C^\infty(\mathfrak{g}^*)^G) \overset{\curvearrowright}{=} -\mathrm{ad}^*_{Rdf_\alpha}(\alpha)(L) \quad \Rightarrow \boxed{{}^R X_f = -\mathrm{ad}^*_{Rdf_\alpha}(\alpha).}$$

If $f, g \in C^\infty(\mathfrak{g}^*)^G$, then $\{f,g\}_R = -\mathrm{ad}^*_{Rdf_\alpha}(\alpha)(dg_\alpha) = \mathrm{ad}^*_{dg_\alpha}(\alpha)(Rdf_\alpha) = 0$

$\Rightarrow$ This shows: **Prop**: Let $R : \mathfrak{g} \to \mathfrak{g}$ be a sol'n of mYB, then the alg. $C^\infty(\mathfrak{g}^*)^G$ is Poisson commutative on $(\mathfrak{g}^*, \{\cdot,\cdot\}_R)$

The Ham. flow of $f \in C^\infty(\mathfrak{g}^*)^G$ is

$$\dot{\alpha} = X_f|_\alpha = -\mathrm{ad}^*_{Rdf_\alpha}(\alpha).$$

If $\mathfrak{g}$ comes with $\langle\cdot,\cdot\rangle$, inv't inner product $\left(\langle L,[M,N]\rangle = -\langle [M,L],N\rangle\right)$

then $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ $\quad L \longmapsto \langle L, \cdot\rangle$ equivariant iso $\quad \left(\mathfrak{g} = \mathfrak{sl}_n\mathbb{C},\ \langle L,M\rangle = \mathrm{Tr}(LM)\right)$

$$f,g \in C^\infty(\mathfrak{g}) \rightsquigarrow \{f,g\}_{(R)}(L) := \{f\circ\phi^{-1}, g\circ\phi^{-1}\}_{(R)}(\phi(L))$$

$$= \langle L, [\nabla f|_L, \nabla g|_L]_{(R)}\rangle$$

$\nabla f = \phi^{-1} df$ gradient

$$X_f(L) = [\nabla f_L, L], \quad {}^{R}X_f(L) = [R\nabla f_L, L] \quad f \in C^\infty(\mathfrak{g})^G.$$

eq. of motion $\rightsquigarrow$

$$\dot{L} = [R\nabla f_L, L] = [M, L], \quad (M(L) = R\nabla f_L)$$

Rmk: ~~Example~~:

$$R : \mathfrak{g} \longrightarrow \mathfrak{g} \rightsquigarrow \underbrace{R \in \mathfrak{g}^* \otimes \mathfrak{g} \overset{\langle,\rangle}{\cong}}_{\longmapsto\ r} \underline{\mathfrak{g} \otimes \mathfrak{g}} \qquad \eta_{\alpha\beta} = \langle E_\alpha, E_\beta\rangle$$

$R = \mathrm{id} \rightsquigarrow \{E_\alpha\}$ Basis of $\mathfrak{g}$ $\qquad r = \sum_{\alpha,\beta} \eta^{\alpha\beta} E_\alpha \otimes E_\beta$ "tensor Casimir"

" Pedit, Burstall "harmonic maps via AKS theory"

<u>AKS</u> $\mathfrak{g}$, $\langle\cdot,\cdot\rangle \leadsto \phi: \mathfrak{g}\to\mathfrak{g}^*$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ as vector space
$\mathfrak{k}, \mathfrak{b}$ subalg.

$\Rightarrow \quad \mathfrak{k}^\perp = \{ L \in M : \langle L, M\rangle = 0 \ \forall M \in \mathfrak{k} \} \cong \mathfrak{b}^*$

$\boxed{\text{Idea: use } \phi: \mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}^* \text{ to get a Poisson str. on } \mathfrak{k}^\perp \text{ from } \mathfrak{b}^*.}$

<u>Example:</u> • $\mathfrak{g} = \mathfrak{sl}(N,\mathbb{C})$ $\quad \mathfrak{k} = \mathfrak{su}(n)$, $\mathfrak{b} = \left\{ \begin{pmatrix} b_1 & & * \\ & \ddots & \\ 0 & & b_N \end{pmatrix} \middle| \, b_i \in \mathbb{R} \right\}$
$\langle L, M\rangle = \operatorname{Im}(\operatorname{Tr}(LM))$ $\quad \mathfrak{k}^\perp = \mathfrak{k}$, $\mathfrak{b}^\perp = \mathfrak{b}$

• $\hat{L\mathfrak{g}} = \mathbb{C}[[\lambda]][\lambda^{-1}]$ = Laurent series in $\mathfrak{g}$ with finite ppal part. $\quad \mathfrak{k} = L\mathfrak{g}_-$ ppal parts
$\mathfrak{b} = L\mathfrak{g}_+$

$$\langle L(\lambda), M(\lambda)\rangle = \operatorname{Res}_{\lambda=0} \operatorname{Tr}(L(\lambda) M(\lambda))$$

$\leadsto \quad \hat{LG} \cong LG_+ \times LG_-$ open dense subset.

$SL(N,\mathbb{C}) = SU(N)\cdot B$ "Iwasawa decomp": (Gram-Schmidt)

Let $f, g \in C^\infty(\mathfrak{k}^\perp)$, then $\boxed{\{f,g\}(L)} = \{ f\circ\phi^{-1}, g\circ\phi^{-1}\}_{\mathfrak{b}^*}(\phi(L))$

choose $\tilde{f} \in C^\infty(\mathfrak{g})$ $\tilde{f}|_{\mathfrak{k}^\perp} = f$ $\quad = \langle L, [d(f\circ\phi^{-1})_{\phi(L)}, d(g\circ\phi^{-1})_{\phi(L)}]\rangle$

$df_L(X) = \frac{d}{dt}\Big|_{t=0} f(L+tX) = \frac{d}{dt}\Big|_{t=0} \tilde{f}(L+tX)$ $\quad \boxed{= \langle L, [\pi_{\mathfrak{b}} \nabla \tilde{f}_L, \pi_{\mathfrak{b}} \nabla \tilde{g}_L]\rangle}$

$\Rightarrow \cancel{\ } df_L \simeq \pi_{\mathfrak{b}} \nabla\tilde{f}_L$ $\quad = \langle \nabla\tilde{f}_L, X\rangle = \langle \pi_{\mathfrak{b}} \nabla\tilde{f}_L, X\rangle$ $\quad \tilde{f}, \tilde{g}$ extensions of f, g to $\mathfrak{g}$

$$\{f,g\}(L) = \langle L, [\pi_{\mathfrak{b}} \nabla \tilde{f}_L, \pi_{\mathfrak{b}} \nabla \tilde{g}_L] \rangle = \langle [L, \pi_{\mathfrak{b}} \nabla \tilde{f}_L], \underbrace{\pi_{\mathfrak{b}} \nabla \tilde{g}_L} \rangle$$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b} \qquad = \langle \pi_{\mathfrak{k}^\perp} [L, \pi_{\mathfrak{b}} \nabla \tilde{f}_L] , \nabla g|_L \rangle$$

$$\simeq \mathfrak{k}^\perp \oplus \mathfrak{b}^\perp$$

$$\Rightarrow \quad X_f|_L = -\pi_{\mathfrak{k}^\perp} [\pi_{\mathfrak{b}} \nabla \tilde{f}_L , L]$$

If $\tilde{f} \in C^\infty(\mathfrak{g})^G$, then $\tilde{f}(\mathrm{Ad}_g L) = \tilde{f}(L) \overset{\text{diff}}{\rightsquigarrow} [\nabla \tilde{f}_L, L] = 0$

$$0 = [\pi_{\mathfrak{b}} \nabla \tilde{f}_L + \pi_{\mathfrak{k}} \nabla \tilde{f}_L , L] \quad (*)$$

$\Rightarrow$ $f = \tilde{f}|_{\mathfrak{k}^\perp}$ has $X_f|_L = \pi_{\mathfrak{k}^\perp} [\underbrace{\pi_{\mathfrak{k}} \nabla \tilde{f}_L}_{\in \mathfrak{k}}, \underbrace{L}_{\in \mathfrak{k}^\perp}] = [\pi_{\mathfrak{k}} \nabla \tilde{f}_L, L]$

$[\mathfrak{k}, \mathfrak{k}^\perp] \subset \mathfrak{k}^\perp$ Lax form!

Ham. flow: $\dot{L} = [\pi_{\mathfrak{k}} \nabla \tilde{f}_L , L]$

$\tilde{f}, \tilde{g} \in C^\infty(\mathfrak{g})^G \rightsquigarrow f = \tilde{f}|_{\mathfrak{k}^\perp}, g = \tilde{g}|_{\mathfrak{k}^\perp} \rightsquigarrow \{f,g\}(L) = X_f|_L(g) = dg_L(X_f)$

$$= -\langle \pi_{\mathfrak{b}} \nabla \tilde{g}_L, [L, \pi_{\mathfrak{b}} \nabla \tilde{f}_L] \rangle$$

$$= \langle \underbrace{[L, \pi_{\mathfrak{b}} \nabla \tilde{g}_L]}_{\in \mathfrak{k}^\perp}, \underbrace{\pi_{\mathfrak{k}} \nabla \tilde{f}_L}_{\in \mathfrak{k}} \rangle = 0.$$

Thm (AKS) let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a Lie alg, $\langle\cdot,\cdot\rangle$ inv't inner product,

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ $\mathfrak{k}, \mathfrak{b}$ subalgebras. Then ~~the Poisson~~ consider

$\mathfrak{k}^\perp$ with $\{\cdot,\cdot\}$ induced from $\underbrace{\mathfrak{b}^* \simeq \mathfrak{k}^\perp}$. If $\tilde{f}, \tilde{g} \in C^\infty(\mathfrak{g})^G$

then $f = \tilde{f}|_{\mathfrak{k}^\perp}$, $g = \tilde{g}|_{\mathfrak{k}^\perp}$ Poisson commute and have Ham.

eq. of Lax form: $\dot{L} = [\pi_{\mathfrak{k}} \nabla \tilde{f}|_L , L] = X_f(L)$

Example: $\mathfrak{g} = \widehat{L\mathfrak{g}}$ $\mathfrak{k} = L\mathfrak{g}_-$ $\mathfrak{b} = L\mathfrak{g}_+$ $\pi_{\mathfrak{k}}(L(\lambda)) = L(\lambda)_-$

$\pi_{\mathfrak{b}}(L(\lambda)) = L(\lambda)_+$

$\tilde{f} \in C^\infty(\hat{L\mathfrak{g}})$ inv't function, then on $L\mathfrak{g}_-$:

$$\dot{L} = [(\nabla f_L)_- , L]$$

Toda $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{k} = \mathfrak{su}(n)$, $\mathfrak{b} = \left\{ \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \right\}$ $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_i \end{pmatrix}$

coadjoint orbits: $b \in B$, $x \in \mathfrak{k}^\perp$ $Ad^*_b(x) - \pi_{\mathfrak{k}^\perp}(Ad_b x)$ $\boxed{2n-2}$

coadjoint orbits in $\mathfrak{k}^\perp$: $\pi_{\mathfrak{k}^\perp}(Ad_B(x)) = \mathcal{O}_x$ $\dim = \underset{r = \mathrm{rk}\,\mathfrak{g}}{2r}$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$

Solution

$\tilde{f} \in C^\infty(\mathfrak{g})$ $\quad \tilde{f}|_{\mathfrak{k}^\perp} = f$ $\quad \boxed{X_f(L) = [\underbrace{\pi_{\mathfrak{k}} \nabla \tilde{f}_L}, L] = \dot{L}}$ $\quad L(0) = \underline{L_0}$

$g : \mathbb{R} \longrightarrow G = \underline{KB}.$

Put: $g(t) = \exp(-t\, \underbrace{\nabla \tilde{f}_{L_0}})$

$= \boxed{k(t)} \cdot b(t)$ $\qquad \mathfrak{g} = \mathfrak{sl}_n \mathbb{C}$ $\quad \mathfrak{k} = \mathfrak{su}(n)$, $\quad \mathfrak{b} = \left\{ \begin{pmatrix} \ddots & * \\ & \ddots \end{pmatrix} \right\}$

Claim: $L(t) = Ad_{k^{-1}(t)} L_0$ is the solution. $\qquad \nabla \tilde{f} : \mathfrak{g} \to \mathfrak{g}$

proof: $\tilde{f}$ invariant: $f(Ad_g L) = f(L) \rightsquigarrow \nabla \tilde{f}_{Ad_g L} = Ad_g (\nabla \tilde{f}_L)$

$$\to \nabla \tilde{f}_{\underbrace{Ad_{k^{-1}} L_0}_{L}} = Ad_{k^{-1}} \nabla \tilde{f}_{L_0} = -Ad_{k^{-1}}(\dot{g} g^{-1})$$

$$= -Ad_{k^{-1}}\left((\dot{k} b + k \dot{b}) b^{-1} k^{-1}\right) = -Ad_{k^{-1}}\left(\dot{k} k^{-1} + Ad_k \dot{b} b^{-1}\right)$$

$$= -k^{-1}\dot{k} + \underbrace{\dot{b} b^{-1}}_{\in \mathfrak{b}}$$

$$\text{OTOH: } \dot{L} = \frac{d}{dt}(k^{-1} L_0 k) = -k^{-1}\dot{k}\, \underbrace{k^{-1} L_0 k}_{L} + k^{-1} L_0 \overset{kk^{-1}}{\vee} \dot{k} = \underline{-[k^{-1}\dot{k}, L]}$$

$$= [\pi_{\mathfrak{k}} \nabla \tilde{f}_L, L].$$

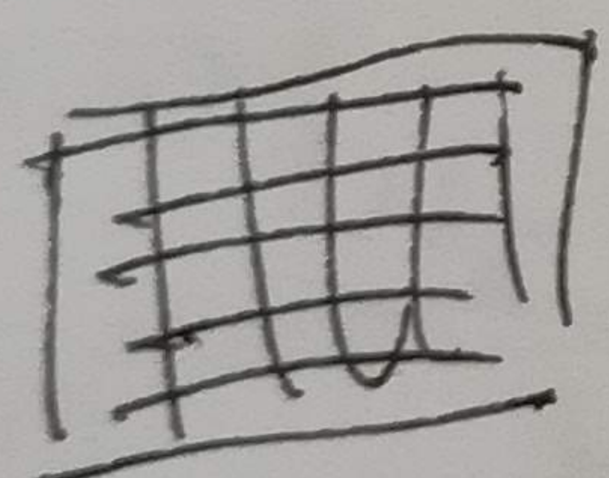
Example: $\mathfrak{g} - \hat{L\mathfrak{g}}$

$$\dot{L} = [(\nabla \tilde{f}_L)_-, L] \qquad \nabla \tilde{f}_L = M$$
$$= [M_-, L]$$

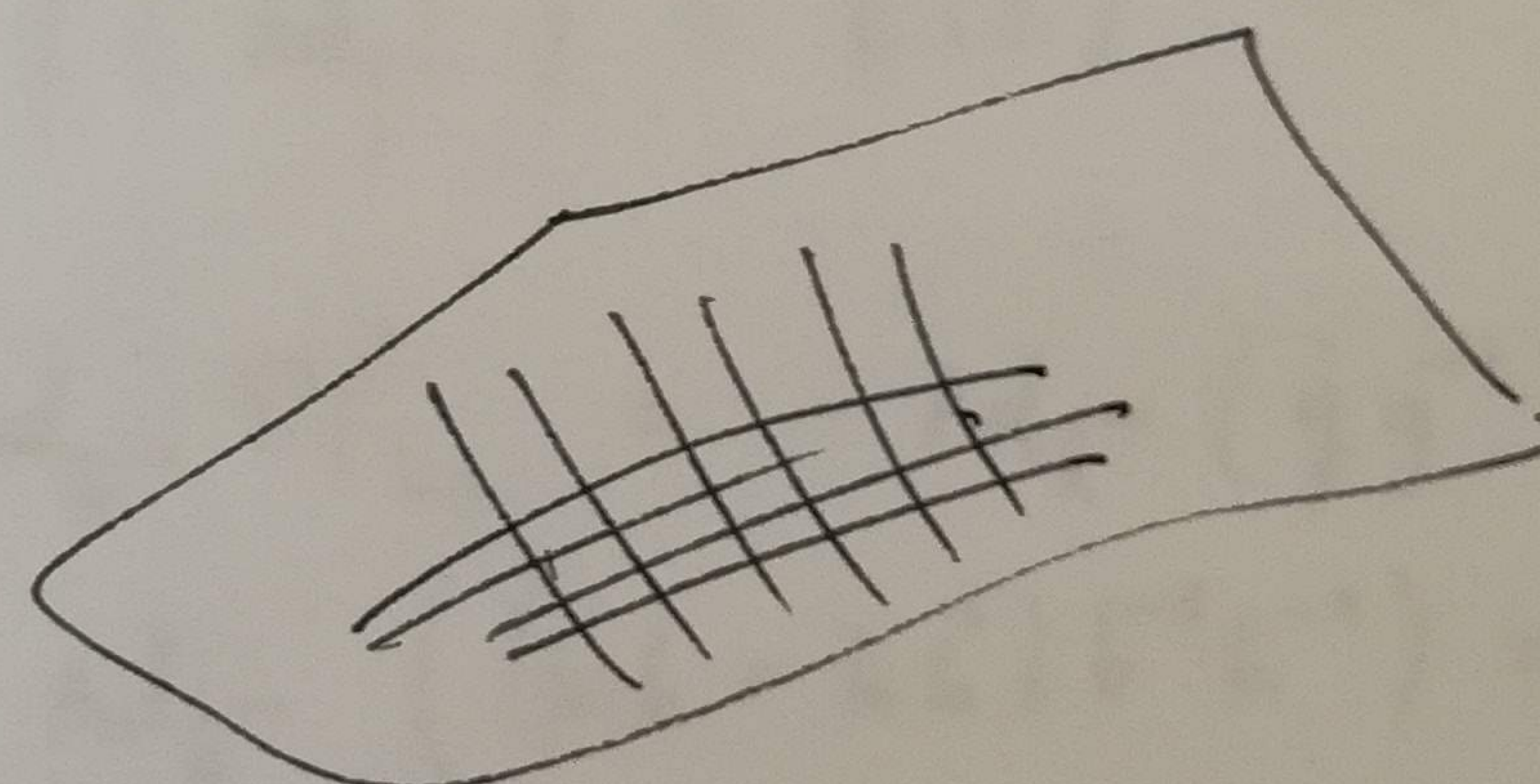
$$\rightsquigarrow g(t) = \exp(-t M(\lambda, 0)) = g_-(t) \cdot g_+(t)$$

solution $L(t) = \mathrm{Ad}_{g_-^{-1}} L(\lambda, 0) = g_-^{-1}(\lambda, t) L(\lambda, 0) g_-(\lambda, t)$

$\tilde{f}_1, \dots, \tilde{f}_N \in C^\infty(\mathfrak{g})^G \qquad \{f_i, f_j\} = 0 \quad \forall i, j.$



$(-\varepsilon, \varepsilon)^N$



$\underline{N = 2}$

$$\underline{t} \; (t_1, \dots, t_N) \longmapsto \Phi_{t_1}^{f_1} \circ \dots \circ \Phi_{t_n}^{f_N} (L_0)$$

$$g(t) = \exp(-\sum M_i t_i) \qquad M_i = \nabla \tilde{f}_i$$
$$= k(t) \, b(t) \, ?$$

The end.