I am interested in geometric and algebraic questions motivated by high energy physics, particularly renormalization. I study renormalization using combinatorial Hopf algebra in the program established by Connes and Kreimer in [11]. The study of combinatorial Hopf algebras leads me to work on problems in many different fields, such as non-commutative geometry, number theory, and even control theory. I am also interested motives, particularly as they apply to multiple polylogarithms and field theories in configuration space.

Combinatorial Hopf algebras have led me to study the process of renormalization certain types of field theories [2, 1, 5]. These algebras have applications in the study of multiple polylogarithms [18, 3]. They have an important role to play in non-commutative geometry [15]. In ongoing work, I am studying their relationship with gauge structures found in control theory. In more speculative work, I am investigating whether these Hopf algebras and their related structures may shed light on the Fundamental Lemma, at least in the case of $GL_2(k)$.

Combinatorial Hopf algebras can be expressed as the Hopf algebra of rooted trees, $\mathcal{H}_{rt}$. These are non-planar oriented trees with one marked vertex that all edges are oriented away from. The coproduct structure is defined by making a certain type of cuts on these trees. These structures first appeared in the problem of regularization of a toy model of perturbative Quantum Field Theory (QFT) in [11], where it was also noticed that a sub Hopf algebra of $\mathcal{H}_{rt}$ is isomorphic to the Hopf algebra $\mathcal{H}(1)$ of linear maps on the group of diffeomorphism on a one dimensional manifold, defined in [15]. There is a large body of work studying both the non-commutative geometry applications of $\mathcal{H}(1)$ and the QFT applications of $\mathcal{H}_{rt}$.

1 Geometrizing the renormalization group flow

My work on QFTs uses the Hopf algebra of Feynman diagrams, $\mathcal{H}_{Fd}$, and a related Hopf algebra of rooted trees $\mathcal{H}_{rt}$, that captures the divergence structure of scalar theories [12], QED [25], and QCD [21]. In much of the literature on the subject, when the authors study regularization, they restrict themselves to dimensional regularization. My contribution to the field has been to incorporate other regularization schemes into this Hopf algebraic approach to renormalization, and use the tools developed in [13] to relate different regularization schemes to each other. In the process, I show that the β function that appears in this context can be written in terms of the Maurer-Cartan connection on a group defined by the QFT and the renormalization scheme [4].

In this literature, the Feynman rules for regularized QFTs are viewed as maps from $\mathcal{H}_{Fd}$ to some algebra specific to the regularization scheme. The Hopf algebra $\mathcal{H}_{Fd}$, being commutative, has an associated affine group scheme G. For an appropriate algebra $\mathcal{A}$, the group $G(\mathcal{A}) = \mathrm{Hom}_{k\ alg}(\mathcal{H}_{Fd}, \mathcal{A})$ contains regularized Feynman integrals, [14], chapters 3-7. The regularized Feynman integrals are not invariant under the scaling of the variables of integration $p_i \to \mu p_i$, for $\mu \in \mathbb{C}^\times$. The action of the renormalization scale describes this dependence. A key object of study in [12, 13] is the element of $\mathfrak{g}(\mathcal{A}) = \mathrm{Lie}(G(\mathcal{A}))$, called the β function, that defines this renormalization scale dependence for dimensional regularization. This β function is closely related to the expression

$$\beta(\lambda) = \frac{1}{\mu}\frac{d\lambda}{d\mu} \tag{1}$$

which measures the dependence of the the coupling constant of a theory, λ, on the energy scale, μ. In [13], the authors show that the geometric β function defines a flat connection on a particular principle fiber bundle in the case of dimensional regularization.

I study the geometric aspects of this β function. In [2], I generalize the geometric β function of [14] to show

Theorem 1. *The geometric β function for dimensional regularization can be generalized to define a global connection on the renormalization bundle.*

In this paper, I show that scalar field theories on curved Euclidean backgrounds can be encoded in the same structure. Theorem 1 is key to relating the different regularization schemes. If two regularization schemes can be expressed as element of the same group $G(\mathcal{A})$, then their geometric β functions can be related by a gauge transformation on the appropriate renormalization bundle. There is little known about how to relate different renormalization schemes. Dimensional regularization is gauge invariant. Operator regularization, which gives the same results in the case of scalar field theories, is not gauge invariant. Neither is momentum cutoff, which is by far the easiest to calculate. In this case, the physical β function (equation (1)) agrees with that of dimensional regularization only up to three loop orders for QED.

In [1] and [5], I show

Theorem 2. *Operator regularization and dimensional regularization are both elements of $G(\mathcal{A})$, for $\mathcal{A} = \mathbb{C}[z^{-1}][[z]]$. Momentum cutoff regularization is an element of $G(\mathcal{A}')$ for a larger algebra $\mathcal{A} \subset \mathcal{A}'$.*

I show that the geometric β function for these regularization schemes can be written as a connection on an appropriate regularization bundle. In all of these cases, the geometric β function uniquely defines the regularized QFT as an element of $G(\mathcal{A})$. Therefore, the global connection defined in theorem 1 gives an explicit relationship between regularization schemes.

Theorem 3. *Operator, dimensional and momentum cutoff regularization are related by a gauge transformation of the connection defined by the geometric β function.*

In [1], I also show that

Theorem 4. *The geometric β function for operator regularization can be defined for scalar theories on closed compact boundary less space-times, and for conformal field theories on the same.*

However in this case, the target algebra is very different than those discussed in the previous cases.

In [4], I show that the β function can be written in terms of fundamental geometric object.

Theorem 5. *If the action of the renormalization scale defines a one parameter family of diffeomorphism of $G(\mathcal{A})$, for some algebra $\mathcal{A}$, it defines a geometric β function, which can be written in terms of the Maurer-Cartan connection on $G(\mathcal{A})$.*

For example, dimension, operator and momentum cutoff regularization all define a one parameter family of diffeomorphism on $G(\mathcal{A})$, for appropriate $\mathcal{A}$. This result is closely related to a relationship between the Dynkin operator and map, as defined in [23, 17] respectively. In [4] I show that the β function for QFTs is an application of these generalizations.

Theorem 6. *Let σ be a renormalization scale action as in Theorem 5. It defines a vector field on $G(\mathcal{A})$, X_σ. Writing the geometric β function in terms of X_σ gives exactly the generalizations of the Dynkin operator and map defined in [23, 17].*

This interpretation of the geometric β function has applications to control theory as discussed in section 5.

2 Generalization of the Connes-Moscovici's Hopf algebras

In non-commutative, the natural extension of deRham cohomology is either Hochschild cohomology, or Hopf Cyclic cohomology, which are calculated by studying the action of a Hopf algebra of a module. Both of these are well understood for the family of Hopf algebras constructed by Connes and Moscovici in [15], $\mathcal{H}(n)$, over n dimensional manifolds. One of the Hopf algebra of interest in renormalization, $\mathcal{H}_{rt}$, is closely related to $\mathcal{H}(1)$, as shown in [11]. The the only Hochschild cocycle for $\mathcal{H}_{rt}$ currently known is crucial to understanding the combinatorial underpinnings of the Dyson-Schwinger equations for QFTs [8]. Finding other cocycles for $\mathcal{H}_{rt}$ is an open problem.

In a paper with an undergraduate student, [6], we define a Hopf algebra $\mathcal{H}_{rt}(1)$.

Theorem 7. *The Hopf algebra $\mathcal{H}_{rt}(1)$ contains both $\mathcal{H}_{rt}$ and $\mathcal{H}(1)$. The $\mathcal{H}(1)$ module defined in [15] is also an $\mathcal{H}_{rt}(1)$ module.*

This gives a context for calculating the Hochschild cohomology of $\mathcal{H}_{rt}$. I am very interested in investigating relations between newly discovered cocylces and QFTs generated along the way. This investigation is left for on going work, and has strong potential for involvement of masters or PhD level students.

3 Motives and curvature

Staring with [7], there has been a large amount of work on the connection between QFTs and motives. In [1], I noticed

Theorem 8. *On a closed, compact boundaryless space-time, discretized versions of the graph polynomials found in [7] appear.*

The objects that appear in this class of curved space-time are not periods that give multiple zeta values, but sums. If the motives associated to Feynman integrals have physical meaning, they should behave in a controlled way as one introduces a background curvature. However, the Feynman rules on a background curvature do not necessarily define a rational form. Thus, one cannot talk of periods of Feynman integrals in the general case. Inspired by [10], in ongoing work with Özgür Ceyhan, we identify a class of background metrics for which such periods can be defined.

Conjecture 1. *Let $\phi : X \to \mathbb{R}^{2n}$ be a rational map of even dimensional space-times. If the critical locus of ϕ is mixed Tate, the Feynman integrals on X give rise to multiple zeta values.*

Thus far, all the successes in studying motives and Feynman integrals have occurred for massless theories. We hope that by incorporating curvature, and looking at conformal field theories, we will understand the motivic content of certain massive field theories by changing the curvature, or conformal mass of the theory.

4 Multiple polylogarithms

Multiple polylogarithms appear in calculations of regularized Feynman integrals, and symbols of multiple polylogarithms play a key role in amplitude calculations of SYM N=4 [19]. In [18], the authors show a coalgebra homomorphism between the Hopf algebra of polygons with labeled sides and the Hopf algebra of multiple logarithms. This paper is a first step towards creating a graphical representation of a subcategory of mixed Tate motives.

In [3] I use the polygon representation of iterated integrals to study the dihedral symmetries of multiple polylogarithms. In [3],

Theorem 9. *I explicitly calculate the action of the dihedral group on the arguments of the iterated integrals representation of multiple polylogarithms.*

Furthermore, I identify a family of bar complexes on the tensor algebra defined by these polygons.

Theorem 10. *The several bar complexes that exist on the algebra of decorated polygons can be related, up to certain primitive co-ideals, to the bar complex that corresponds to the bar complex of iterated integrals.*

In writing this paper, I found that many of the calculations performed in this paper can only be done up to a primitive coideal. This raised questions about the nature of these coideals, and the structure of the bar complex of polygons. I explore this in ongoing work with Owen Patashnick.

Conjecture 2. *The Hopf algebra of iterated integrals can be written as a quotient of the Hopf algebra associated to Hopf algebra of polygons.*

Under the quotient map, some of the primitive coideals disappear.

Conjecture 3. *The primitive coideals that do not disappear define relationships between multiple polylogarithms.*

If this line of research is successful, then working with the polygon avatars of the iterated integrals may prove to be a simpler tool for determining multiple polylogarithm relations. The calculations for understanding multiple polylogarithms in terms of trees and polygons involves a new type of symbolic calculus. Creating a program in SAGE or other suitable language to deal with these calculations would greatly aid the in the discovery of more of these interested in co-ideals, and thus multiple polylogarithm relations. This has suitable aspects for undergraduate original research or master theses.

Current research on symbols of multiple polygons has applications in calculating amplitudes SYM N=4 [19]. I am curious about the relationship between these amplitudes and properties of iterated integrals, and calculations using their polygon avatars [16]. I have spent some time at Caltech studying SYM N=4 from both the symbols point of view and by trying to understand the twistor calculus involved in computing amplitudes.

5 Associations with control theory

In control theory, one often studies connections on principle bundles over a space of controls of a system, with fibers corresponding to the group of outputs. The renormalization bundle is a simple case of such principle bundles. The system is the quantum field theory, the space of controls is given by the renormalization scale, and the group of outputs is $G(\mathcal{A})$. In control theory, one is interested in twisted Maurer-Cartan connections on the Lie group $G(\mathcal{A})$. In ongoing work with Kurusch Ebrahimi-Fard, Joris Vankerschaver and Maria Barbero Liñan, we specify this relationship between QFTs and control theory.

Theorem 11. *If the action of the renormalization scale on a QFT induces a one parameter diffeomorphism on $G(\mathcal{A})$, then the geometric β function is the contraction of the complete vector field defined by the family of diffeomorphisms with the Maurer-Cartan connection.*

Conversely, the dynamics at the heart of control theory problems can be written in terms of Hopf algebras and generalized Dynkin operators.

Theorem 12. *Every finite dimensional group of outputs G is associated to the Hopf algebra $k[G]$, for some field k of characteristic 0. The(twisted) Maurer-Cartan connections correspond to (twisted) generalized Dynkin operators on $k[G]$.*

While the β function for QFTs over flat space time is a simple example of a control system, the real gains from this point of view lies in studying conformal field theories. At least in the scalar case, I have constructed a principal fiber bundle that encodes the β function for a conformal theory over an Euclidean space time. The intersection with control theory is a completely new way of understanding conformal scaling.

6 The Langlands' fundamental lemma

The combinatorial proof for the Fundamental Lemma in the case of $Gl_2(k)$ (where k is a field with discrete valuation) [20], depends on the combinatorics of paths on a Bruhat-Tits tree. Counting the number of paths satisfying certain properties gives the values of orbital integrals associated to elements of $Gl_2(k)$. In preliminary work, I have shown that these paths can be encoded in a Hopf algebra of trees that is a generalization of the one used for renormalization theory. I have also shown that the set of recursive equations for counting paths in [20] can be written in terms of a pre-Lie product. I hope that this technology will let me identify a character on the Hopf algebra that is related to the orbital integrals discussed above.

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