(34)

SITES

Def^n: A GROTHENDIECK TOPOLOGY on a category $\mathcal{T}$ is a function J which assigns to $X \in \mathcal{T}$, a family $J(X)$ of sieves on X s.t. 1) the maximal sieve $t_X = \{\text{all morphisms into } X\}$ is in $J(X)$

2) (STABILITY) if $S \in J(X)$, then $\forall h: Y \to X$, $h^*(S) \in J(Y)$

3) (TRANSITIVITY) If $S \in J(X)$, and R is ANY sieve on X s.t. $h^*(R) \in J(Y)$ $\forall h: Y \to X$ then $R \in J(X)$ also.

A GROTHENDIECK SITE is a pair $(\mathcal{T}, J)$

e.g. • Top w/ "open covers" (i.e. $J(X) = \{$all collections $\{U_i \to X\}$ which form an open cover, together with all composites$\}$)

Lecture (4') Starts Here

• (Manifolds, Schemes, etc. with appropriate maps which are topological open covers)

• Measure spaces w/ "almost everywhere covers"

Note: All of these really start with generators of the sieves in $J(X)$

Def^n: A BASIS for a Grothendieck topology on $\mathcal{T}$ (if $\mathcal{T}$ has pullbacks!) is

K: objects of $\mathcal{T}$ $\to$ collections of morphisms to X

1) containing all isos

2) if $\{f_i: X_i \to X\}_I \in K(X)$, $g: Y \to X$ $\Rightarrow$ $\{\pi_2: X_i \times_X Y \to Y\}_I \in K(Y)$

"stability"

(3) If $\{f_i : x_i \to x\}_{I} \in K(x)$, and $\forall i \in I$,

(coverall objects in a cover) $\to$ $\exists \{g_{ij} : y_{ij} \to x_i\} \in K(x_i)$ then

$\{f_i \circ g_{ij} : y_{ij} \to x\} \in K(x)$ ("transitivity")

- It is enough to specify a BASIS for a Grothendieck topology

cf. generators for a nt. ideal.

eg: The Zariski site (on a category of commutative k-algebras)

(See more details next time.)

SHEAVES ON SITES

Sites have exactly the right structure needed to define sheaves: the sieves $S \in J(x)$ define "covers" of x.

To express the gluing condition:

Def²: If $(\mathcal{T}, J)$ a G. site, $P : \mathcal{T}^{op} \to Sets$ a presheaf, and $S \in J(x)$ for $x \in \mathcal{T}$

a MATCHING FAMILY for S of P-elements is an assignment

$$(y \xrightarrow{f} x) \longmapsto \omega_f \in P(y)$$

$$\in S$$

such that $\omega_f \cdot g = \omega_{fg} \quad \forall (g : z \to y) \in \mathcal{T}$

An AMALGAMATION of a matching family is

$\omega \in P(x)$ s.t. $\omega \cdot f = \omega_f \quad \forall f \in S$.

Note: SHEAF condition means exactly "every matching family has a unique amalgamation"

Intuition: Think of P = {cts functions on a space}

Then a matching family is: a family of functions on spades mapping into X, which is "stable under restriction".

Sheaf condition means this can (uniquely!) be got from a cts function on X.

(Could fail by - nonexistence
- nonuniqueness)

Diagrammatically:

Prop: A sheaf is exactly a presheaf P such that $\forall x \in \mathcal{J}$, $\forall S \in J(x)$

$$P(x) \xrightarrow{e} \prod_{f \in S} P(\text{domain}(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{(f,g)\ \text{composable}} P(\text{domain}(g))$$

~~is~~ is an equalizer

(i.e. $p \circ e = a \circ e$, and any other e' with $p \circ e' = a \circ e'$ factors thru e).

where: e is the restriction map
p comes from composition with g
a comes from the action by $\mathcal{J}$ on P

(i.e: when restricted from X, p & a act the same)

[Equivalently: A sheaf satisfies the DESCENT condition!
(More on descent later!)]

~~Next: Examples & Properties~~

Def: A GROTHENDIECK TOPOS is any category equivalent to some $Sh(\mathcal{J}, J)$, a cat. of sheaves.

Examples of Sites

- **Top**: $\mathcal{J}$ has: $\begin{cases} \underline{Obj}: \text{Topological Spaces} \\ \underline{Mor}: \text{Continuous Maps} \end{cases}$

  $J(X)$: (Generated by) (good) open covers

  i.e. $\{U_i\}$ with $\bigcup_i U_i = X$

  (and $U_i \cap U_j$ contractible)

- **Diff**: $\mathcal{J} = Diff$ has $\begin{cases} \underline{Obj}: \text{Smooth manifolds} \\ \underline{Mor}: \text{Smooth maps} \end{cases}$

  $J(X)$: Good open covers

- **CartSp**$_{\text{top/smooth}}$: $\begin{cases} \underline{Obj}: \text{Cartesian spaces } \mathbb{R}^n \\ \underline{Mor}: (\text{cts/smooth}) \text{ maps} \end{cases}$

  $J(X)$: good open covers

  [**OP**: Same except objects are open subsets of Cartesian spaces]

- **Meas**: $\underline{Obj}$: Measure spaces (or Measurable spaces)

  $\underline{Mor}$: Measurable functions (Meas-0-preserving)

  $J(X)$: almost-everywhere covers (generators)

  i.e. $\{U_i\}$ with $X \setminus \bigcup U_i$ measure 0.

  (G. Topology properties follow since all h are meas-0 preserving)

- **Zariski Site**:
  - Assume a fixed base ring (commutative) k
  - **Zar** has: $\begin{cases} \underline{Obj}: \text{Finitely presented } k\text{-algebras} \\ \quad A = k[x_1, \dots, x_n]/(f_1, \dots, f_m) \\ \quad \text{for } f_j \text{ a polynomial in } x_1, \dots, x_n \text{ over } k \\ \underline{Mor}: (\text{Algebra homomorphisms})^{op} \end{cases}$

Covers for the Zariski Site

Q: What is $J(A)$ in the Zariski site?

Idea: Model open sets in the Zariski topology

[Recall: Zariski topology on a variety $V \subset \mathbb{C}^n$ determined by that on $\mathbb{C}^n$
- Open sets and complements of varieties]

So: A cover for $X \in \underline{Zar}$, is determined by

$\{a_1, \ldots, a_n\} \subset A$, s.t. $1 \in (a_1, \ldots, a_n)$ ← (ideal generated by the a_i)

It is given by the generators for $J(X)$:

$$\{(A \longrightarrow A[a_i^{-1}])^{op}\}_{i=1\ldots n}$$

These form a basis for a Grothendieck topology J on $\underline{Zar}$.

Def^n: The ZARISKI SITE is $(\underline{Zar}, J)$.

Proof this forms a basis:

1) $(A \xrightarrow{id} A)^{op}$ is clearly a cover $(a_1 = 1)$

Precomposing with any iso $(A \xrightarrow{f} B)^{op}$ gives all isos in $J(A)$.

2) (Stability) If $(a_1, \ldots, a_n) = 1$ (so the a_i make a cover) and $h: A \to B$ an algebra homomorphism (i.e. $h^{op}: B \to A$ in $\underline{Zar}$) then we have for each i that

$$\begin{array}{ccc} A & \longrightarrow & A[a_i^{-1}] \\ {\scriptstyle h}\downarrow & & \downarrow \\ B & \longrightarrow & B[h(a_i)^{-1}] \cong B \otimes_A A[a_i^{-1}] \end{array}$$

is a pushout

(Thus this gives a pullback in $\underline{Zar}$)

So we get a cover given by $\{h(a_i)\}_{i\in I}$ since $h(1)=1$.

3) (Transitivity) Suppose $\{a_1,\dots,a_n\}$ give a cover and $\forall i=1\dots n$, we have $\{c_{ij}\}_{j=1\dots n_i} \subset A[a_i^{-1}]$.

determines a cover of $A[a_i^{-1}]$ by:

$$\{(A[a_i^{-1}] \longrightarrow A[a_i^{-1}][c_{ij}^{-1}])^{op}\}_{j=1\dots n_i}$$

Since $c_{ij}\in A[a_i^{-1}]$ means $c_{ij} = b_{ij}\cdot(a_i^{-1})^{m_{ij}}$

for some $m_{ij}\in\mathbb{Z}$ and $b_{ij}\in A$

we can write these as:

$$\{(A[a_i^{-1}] \longrightarrow A[a_i^{-1}][b_{ij}^{-1}] \xrightarrow{\simeq} A[(a_i b_{ij})^{-1}])^{op}\}_{i,j}$$

But $1\in(b_{i1},\dots,b_{in_i})$ in $A[a_i^{-1}]$

So $\exists k_i$ with $a_i^{k_i}\in(a_i b_{i1},\dots a_i b_{in_i})$ in A

(by clearing denominators)

Since $1\in(a_1,\dots a_n)$, we have $1=\sum d_i a_i$

So $1=(\sum d_i a_i)^K \in (a_1^{k_1}\dots a_n^{k_n}) \subset (\{a_i b_{ij}\})$

for some sufficiently large k.

So the $\{a_i b_{ij}\}_{i,j}$ generate a cover in $J(A)$! ∎

So: (Zar, J) is indeed a site!

Examples of Sheaves

1) STRUCTURE SHEAVES

- A class of functions can specify geometric structure on a space

eg (i) For X a topological space, the locale $\mathrm{Top}(X)$ is a site with the usual covers
The structure sheaf is: $\mathcal{O}_{cts}: \mathrm{Top}(X) \to [\mathrm{Rings}]$
$\mathcal{O}_{cts}(U) = C(U, \mathbb{R})$, the ring of cts $\mathbb{R}$'s on $U \subset X$

(ii) For X a smooth manifold, the structure sheaf is:
$\mathcal{O}_{smooth}: \mathrm{Top}(X) \to [\mathrm{Rings}]$
$\mathcal{O}_{smooth}(U) = C^\infty(U, \mathbb{R})$ (smooth $\mathbb{R}$'s)

Both can be extended to [Top]

Note: over the site (Top, open covers)
$\mathcal{O}_{cts}$ is representable: $\mathcal{O}_{cts} = \mathrm{Hom}(-, \mathbb{R})$
$\mathcal{O}_{smooth}$ is NOT representable
"No classifying space for smooth functions"

☆ Over ([Smooth Mflds], open covers), $\mathcal{O}_{smooth} = \mathrm{Hom}(-, \mathbb{R})$

2) BUNDLE SHEAVES

(These can't easily be extended to [Top])

If $X \in [\mathrm{Top}]$ (or [Smooth Mfld]), and there is a given bundle (cts. or smooth)

$$\begin{array}{c} M \\ \pi \downarrow \\ X \end{array}$$

(i.e. a surjective map s.t. $\forall x \in X, \exists U \ni x$ s.t. $\pi^{-1}(U) \simeq F \times U$ for fibre F)

then there is a SHEAF OF CONTINUOUS (or SMOOTH) SECTIONS ⤵

For $\mathcal{U} \subset X$, $\underline{S}(\mathcal{U}) = \{ s : \mathcal{U} \to M \mid \pi \circ s = id_{\mathcal{U}} \}$

ex: check the sheaf conditions.

3) A DIFFEOLOGICAL SPACE is a sheaf on $\mathcal{X}$ on ([CartSp], J) satisfying some conditions ("CONCRETENESS") which ensure it has an underlying set $X = \mathcal{X}(\{*\}) = \mathbb{R}^0$.

MORE ON THIS LATER, but...

Idea: $\mathcal{X}(\mathcal{U})$ defines the "plots" for $\mathcal{U} (\subset \mathbb{R}^k)$ which are defined to be the smooth maps $\mathcal{U} \longrightarrow X$

("Concreteness" ensures this makes sense)

This defines smoothness for [DiffSpc] the way a collection of CHARTS defines smoothness for manifolds

BUT: [DiffSp] will be a topos

ASSOCIATED SHEAF FUNCTOR ("sheafification")

Idea: We saw in Lecture 2 the "sheafification" functor $+ : PSh(X) \to Sh(X)$

$$\mathcal{F} \longmapsto \mathcal{F}^+$$

for sheaves on spaces. We want a version for any site, not just $TOP(X)$.

Basic Claim: The (generalized) sheafification functor a is a left adjoint to the inclusion

$$Sh(\mathcal{C}) \hookrightarrow PSh(\mathcal{C})$$

i.e. $\mathrm{Hom}_{Sh}(a(\mathcal{F}), \mathcal{G}) \cong \mathrm{Hom}_{PSh}(\mathcal{F}, \mathcal{G})$