Sheaves As Generalized Spaces

The Yoneda Lemma says all (pre-) sheaves are determined by relation to the representable ones:

Recall: The YONEDA EMBEDDING $y : \mathcal{T} \longrightarrow PSh(\mathcal{T})$
$x \longmapsto Hom(-, x)$

Lemma If $F \in PSh(\mathcal{T})$, $x \in \mathcal{T}$, then
$$F(x) \cong Hom_{PSh(\mathcal{T})}(y(x), F)$$

i.e.: Any presheaf F "looks representable" in $PSh(\mathcal{T})$ (but only to objects $y(x)$).
So F is a "generalized object of $\mathcal{T}$".

Concrete Sheaves

To get "nice" generalizations of SPACES
(eg. $\mathcal{T}$ = Top, Diff, Schemes, CartSp, etc...)

- F should be a SHEAF
- F should be CONCRETE
  (roughly: have a well-behaved underlying set of pts)

Def: A site $(\mathcal{T}, J)$ is SUBCANONICAL if $\forall x \in \mathcal{T}$, the representable presheaf $y(x)$ is a sheaf: $y(x) \in Sh(\mathcal{T}, J)$

- A (subcanonical) site is CONCRETE if:
  - $\mathcal{T}$ has a terminal object $\mathbb{1}$ such that
  - $Hom_{\mathcal{T}}(\mathbb{1}, -) : \mathcal{T} \longrightarrow \underline{Set}$ is FAITHFUL
  - $\forall$ covers $\{U_i \xrightarrow{f_i} U\} \in J(U)$, the map
    $$\coprod_i Hom_{\mathcal{T}}(\mathbb{1}, f) : \coprod_i Hom_{\mathcal{T}}(\mathbb{1}, U_i) \to Hom_{\mathcal{T}}(\mathbb{1}, U)$$
    is SURJECTIVE

e.g.)

(50) Interpretation:

- $\mathbb{1}$ is "the point" for $\mathcal{T}$ as a category of spaces.
- $\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathcal{U})$ is the UNDERLYING SET OF POINTS for the object $\mathcal{U}$.

Faithful $\Rightarrow$ (all morphisms $\mathcal{U} \to \mathcal{V}$ are determined by the underlying set maps)

- $\coprod_i \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, f_i)$ SURJECTIVE

$\Rightarrow$ "covers" are actual covers of underlying sets.

eg// Subcanonical sites include Top, Diff, etc...

- These are also CONCRETE!
- Note: The site Meas is NOT concrete...

(covers are not set-theoretic surjections)

Def^n: A CONCRETE (PRE)SHEAF is a (pre) sheaf $F: \mathcal{T}^{op} \to \underline{\mathrm{Set}}$ such that the map $\tilde{F}_{\mathcal{U}}(-) : F(\mathcal{U}) \to \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathcal{U}), F(\mathbb{1}))$ is injective.

Note: $\tilde{F}_{\mathcal{U}}$ is the map coming from:

$\mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathcal{U}) \to \mathrm{Hom}_{\mathrm{Set}}(F(\mathcal{U}), F(\mathbb{1}))$ ← by contravariance

$\Rightarrow F(\mathcal{U}) \times \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, \mathcal{U}) \to F(\mathbb{1})$ ← Hom-x duality

$\Rightarrow \tilde{F}_{\mathcal{U}}$ ← Hom-x duality

Motivation: A concrete (pre) sheaf is one which is characterized by set maps to $F(\mathbb{1})$

i.e. $\tilde{F}_U$ takes $F(U)$ to "underlying maps" from the underlying set of U to $F(\mathbb{1})$

injective: each element of $F(U)$ corresponds to just one map.

Thus: think of $F(\mathbb{1})$ as "the underlying set of points" for the generalized space F. Think of $F(U)$ as the set of "valid" maps from U. (Analog of Yoneda lemma for concrete sheaves).

Diffeological Spaces: (see Baez & Hoffnung, Iglesias-Zimmour)

- These are a class of spaces which allow differential geometry, but also have limits, etc.
- They are concrete sheaves on the following site:

Def°: The site **Diffeo** (extends $\mathbf{CartSp}$) is:

- The category with Obj: open subsets of $\mathbb{R}^n$
  Mor: C^∞ maps
- The topology J is generated by the usual open covers (i.e. jointly surjective morphisms)

Def°: A DIFFEOLOGICAL SPACE is a concrete sheaf on $(\mathbf{Diffeo}, J)$. Let $\mathbf{ConcSh}(\mathbf{Diffeo}, J)$ be $\mathbf{DiffSpc}$.

Idea: If $X \in \mathbf{DiffSpc}$, we think of $X(U)$ as defining the collection of smooth maps into X.

- In particular, $X(\mathbb{1}) =$ (underlying set)

(S2) Claim: The above is equivalent to the following definition:

Def^{n}: A diffeological space is a set X with a "SET OF PLOTS" (functions) from each $\mathcal{U} \subseteq_{op} \mathbb{R}^n$ (for all n)
s.t. D1) If $\varphi : \mathcal{U} \to X$ is a plot in X, and
$f : \mathcal{U}' \to \mathcal{U}$ a smooth f^{n} of opensets
Then $\varphi f : \mathcal{U}' \to X$ is a plot in X
D2) If $\{\mathcal{U}_i \xrightarrow{i} \mathcal{U}\}$ are a cover for $\mathcal{U}$
and $\varphi \cdot i_j$ is a plot in X $\forall j$, so is φ
D3) Every map from $\mathbb{R}^0 = \{0\}$ into X is a plot in X.

a.k.a. a DIFFEOLOGY

Proof Sketch of Equivalence: Take $X = F(\mathbb{1})$, $F(\mathcal{U}) = \{\text{plots } \mathcal{U} \to X\}$

Note: points in $\mathcal{U}$ are given by $\mathbb{1} \to \mathcal{U}$
and restriction along these maps make $\varphi \in F(\mathcal{U})$ into a function.

(Concreteness of the site Diffeo).

This definition can work either way (i.e. on $F \Longleftrightarrow X$)
Then roughly:
(D1) $\Leftrightarrow$ Presheaf (functoriality) condition
(D2) $\Leftrightarrow$ Sheaf condition
(D3) $\Leftrightarrow$ Concreteness condition

(For details: Baez & Hoffnung "Convenient Categories of Smooth Spaces"

eg1. Any smooth manifold gives a diffeological space
- plots $\mathcal{U} \to X$ are exactly smooth functions in the usual sense.

- Any set has DISCRETE diffeologies: {constant f^{n}s
INDISCRETE . all f^{n}s.

- Any subspace, quotient etc. of diffeological spaces; mapping space's, pullbacks, etc.

Quotients

If X a diffeological space $\sim$ a equivalence rel^{n} on X
$Y = X/\sim$ is a diffeological space w.r.t plots

$\varphi : U \to Y$ is a plot if:

- $\exists$ cover $\{U_j \xrightarrow{i_j} U\}$ and plots $\{\varphi_j : U_j \to X\}$ such that

$$\begin{array}{ccc} U_j & \xrightarrow{\varphi_j} & X \\ {\scriptstyle i_j}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[\varphi]{} & Y \quad [x] \end{array}$$

commutes.

Exercise: Check this makes Y a diffeological space.

General Fact: Categories of CONCRETE SHEAVES on a concrete site, as full subcat. of $Sh(\mathbb{T})$ may not be toposes, but are close...

Def^{n} A QUASITOPOS is a category $\mathcal{E}$ with all finite limits & colimits, which is LOCALLY Cartesian closed and has a WEAK subobject classifier.

- LOCALLY C.C.: All "slice categories" $\mathcal{E}/x$: $\{ \text{dom} : y \to x \}$ are cartesian closed (i.e. have exponentials)

- WEAK Subobject Classifier Ω

(Same def^{n} as subobject classifier, except Ω only classifies "strong monomorphisms"

In this case, "subspaces" of X are $Y \rightarrowtail X$, but not just ANY subset counts. Only the strong monos:

A STRONG MONO $Y \xrightarrow{i} X$ must have "enough plots"

i.e. $\varphi : U \to Y$ is a plot $\iff i\varphi : U \to X$ is a plot

Theorem: The category $\mathrm{ConcSh}(\hat{T}, J)$ is a quasitopos. (See Baez & Hoffnung)

Moral: The base site Diff_0 (or, generally, any base site) provides "model spaces". The category of concrete sheaves (or generally of any sheaves, but less intuitively) is a "category of generalized spaces" whose structure is defined by maps into them from model spaces

Internal Constructions in Toposes

Idea: We can define "internal" structures of various kinds in any topos.

eg.: • Group objects, ring objects etc...
• Simplicial objects (e.g. "simplicial sets") etc.

Roughly: The CATEGORY OF INTERNAL ___ in Sh(X) is the same as the CATEGORY OF ___-valued SHEAVES

First Example: Group objects

In a category $\mathcal{C}$ with finite products (eg. $\mathrm{Sh}(\hat{T})$) a GROUP OBJECT in $\mathcal{C}$ is an object $G \in \mathcal{C}$ equipped with:

$$m : G \times G \to G$$
$$e : 1 \to G$$
$$i : G \to G$$