Concrete Example of Sheaves

Def/ A $\mathcal{C}$ valued Presheaf on $\mathcal{T}$ is a contravariant functor $\mathcal{F}: \mathcal{T} \longrightarrow \mathcal{C}$ with certain conditions.

My purpose:

$$\mathcal{F} : \underline{\mathcal{T}op} \longmapsto \mathcal{C} \quad \text{contravariant functor s.t.}$$

$$\begin{tikzcd} U \subset X \arrow[r, mapsto] & \mathcal{F}(U) \arrow[d, "res_{U,V}"] \\ V \arrow[u, hook, "i"] \arrow[r, mapsto] & \mathcal{F}(V) \end{tikzcd} \quad \text{restriction morphism.}$$

Note: A restriction map is not necessarily an injection.

① Eg: $\mathcal{O}_{Hol} : \underline{\text{Cmplex Mfld}} \longmapsto \underline{\mathcal{H}ol \text{ functs}}$ $\quad M$ cmplx m'fld

$$\begin{tikzcd} U \subset M \arrow[r, mapsto] & \mathcal{O}_{Hol}(U) = \text{ring of Hol functs def on } U \arrow[d, "res_V"] \\ V \arrow[u, hook] \arrow[r, mapsto] & \mathcal{O}_{Hol}(V) = \text{ring of Hol functs on } V. \end{tikzcd}$$

$$\pi_1 : \underline{\mathcal{T}op_*} \longmapsto \underline{\mathcal{G}rp}$$

$$\begin{tikzcd} U \arrow[r, mapsto] & \pi_1(U, *) \arrow[d, "res_{U,V}"] \\ V \arrow[u, hook] \arrow[r, mapsto] & \pi_1(V, *) \end{tikzcd}$$

Dfn/ A Sheaf is a pre sheaf with descent condition:

For $\underline{\mathcal{T}op}$

① (Locality) if $\{U_i\}_{i \in I}$ open cover of U,
for $s, t \in F(U)$ s.t. $s|_{U_i} = t|_{U_i} \ \forall i \in I$
$\Rightarrow s = t$.

② (Glueing) $\{U_i\}_{i\in I}$ open cover of U, for $s_i \in F(U_i)$ and $s_j \in F(U_j)$, $s_i|_{U_j\cap U_i} = s_j|_{U_i\cap U_j} \Rightarrow \exists\, s \in F(U)$ s.t. $s|_{U_i} = s_i$

<u>Note:</u> $s \in F(U)$ in (2) unique. Why?

Egs: $\mathcal{O}_{Hol}$, π_1 sheaves.

$\mathcal{B}: \mathbb{R} \longrightarrow$ Bounded cont functs.

presheaf, but not a sheaf. glueing doesn't work.

Morphism of presheaves: Natural transformations.

$\mathcal{F}$ is a $\mathcal{C}$ valued Presheaf. $\mathcal{C}$ has all colimits.

<u>Dfn</u> $p \in X$ $\mathcal{F}_p := \coprod_{\substack{U\subset X\\ p\in X}} \mathcal{F}(U)$ (colim) is the stalk of $\mathcal{F}$ at p

Note: if $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$ presheaf morph $\Rightarrow$

$\varphi_p: \mathcal{F}_p \longrightarrow \mathcal{G}_p$ morph of stalks.

$+: \underline{PreSh} \longrightarrow \underline{Sh}$ Sheafification functor.

$For: \underline{Sh} \longrightarrow \underline{PreSh}$ Forgetful

} adjoint functors.

what is $+$?

$U \subset X$. Define $\mathcal{F}^+(U) = \{ s: U \longrightarrow \bigcup_{p\in U} \mathcal{F}_p \mid \forall p \in U,$ ① $s(p) \in \mathcal{F}_p$ ② $\exists V$ nbd$_p \subset U$, $t \in \mathcal{F}(V)$ s.t. $t(q) = s(q)$ $\forall q \in V. \}$

<u>Check</u> $\mathcal{F}^+$ is a sheaf:

Ex <u>Def</u> $\mathcal{F}'$ is a sub sheaf of $\mathcal{F}$ if
$\mathcal{F}'(U)$ sub obj $\mathcal{F}(U)$ $\forall U < X$.

$\varphi : \mathcal{F} \to \mathcal{G}$ sheaf morphism

<u>Check</u> ① $\operatorname{Ker}\varphi$ is a subsheaf of $\mathcal{F}$

② $\varphi(\mathcal{F})$ presheaf : $\varphi(\mathcal{F}(U)) = \varphi(\mathcal{F})(U)$ (not nec. sheaf)

<u>Def</u> $\operatorname{im}\varphi := \varphi(\mathcal{F})^+$ image sheaf

③ $\mathcal{F}'$ subsheaf of $\mathcal{F}$. $\Rightarrow$ $\mathcal{F}(U)/\mathcal{F}'(U)$ presheaf. not a sheaf.

<u>Def</u> Quotient sheaf $(\mathcal{F}/\mathcal{F}')^+$

[Hint for ③
$\mathcal{O}_{hol} \xrightarrow{exp} \underbrace{\mathcal{O}^*_{hol}}_{\text{grp non 0 hol functs}} \longrightarrow$ coker(exp) not a sheaf.]

<u>Important morphisms:</u>

$f : X \to Y$ $f \in \operatorname{Hom}_{\mathcal{T}op}(X,Y)$ $\mathcal{F}$ sheaf on X, $\mathcal{G}$ sheaf on Y

<u>Direct image sheaf:</u> $f_*\mathcal{F}$ is a sheaf on Y defined
$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ push forward in <u>$\mathcal{S}h$</u>

<u>Inverse image sheaf:</u> $f^{-1}\mathcal{G}$ sheaf on X defined
$f^{-1}\mathcal{G}(U) = \left[\coprod_{f(U)\subseteq V} \mathcal{G}(V)\right]^+$ pull back in <u>$\mathcal{S}h$</u>
(Sometimes $f^*\mathcal{G}$)

<u>Direct image w/ proper support:</u> $f_!\mathcal{F}(V) = \{ s \in f_*\mathcal{F}(V) \mid f|_{supp(s)} \text{ proper} \}$

<u>Ex:</u> $f_!\mathcal{F}$ sub sheaf of $f_*\mathcal{F}$

Schemes

R commutative ring. $\operatorname{Spec} R = X$ recall Zariski top.

Def I ideal of R. $V(I) \subseteq X$

$$V(I) = \{ \mathfrak{p} \in X \mid I < \mathfrak{p} \}$$

Note: ① I, J ideals in R. $V(IJ) = V(I) \cup V(J)$

② $\{I_\alpha\}_{\alpha \in A}$ set of ideals $V\left(\sum_{\alpha \in A} I_\alpha\right) = \bigcap_{\alpha \in A} V(I_\alpha)$

③ $V(R) = \emptyset$ $V((0)) = X$

④ $V(I) \subseteq V(J) \iff \sqrt{I} \supseteq \sqrt{J}$.

This defines a topology on X where $V(I)$ closed sets.

Alternatively: $f \in R \Rightarrow D(f) = \{ \mathfrak{p} \in X \mid f \notin \mathfrak{p} \}$

$D(f) = V((f))^c$ open sets of X.

maximal ideals are closed pts!

$\mathcal{O} : X \longrightarrow R$ Sheaf

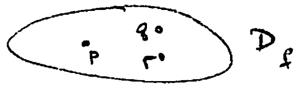
$\mathfrak{p} \in X \Rightarrow R_{\mathfrak{p}}$ = localization at $\mathfrak{p}$ = $S^{-1}R$. $S = R \setminus \mathfrak{p}$ (stalks at $\mathfrak{p}$)

$R_{\mathfrak{p}}$ | elems of $\mathfrak{p}$ not unit

$\mathcal{O}(D(f)) = \{ s : U \longmapsto \coprod_{\mathfrak{p} \in D(f)} R_{\mathfrak{p}} \mid \forall p \in D(f)$, ① $s(\mathfrak{p}) \in R_{\mathfrak{p}}$

② $\exists g$ st. $f \mid g$, $g \notin \mathfrak{p}$, $t \in \mathcal{O}(D(g))$ s.t. $t(q) = s(q)$ $\forall q \in D(g) \}$

R_f elems of (f) not units. $f = pqr$



$\mathfrak{p}$

check: $\mathcal{O}$ satisfies def of Sheaf.

Def ① $(X, \mathcal{O})$ spectrum of R.

② $R_{\mathfrak{p}}$ stalk of $\mathcal{O}$ at $\mathfrak{p}$

③ $\mathcal{O}(D(f)) =: R_f$ localized ring.

Awkward notation.

$R_f = (f)^{-1} R$

$R_{\mathfrak{p}} = S^{-1} R$ for $S = R \setminus \mathfrak{p}$

④ $(X, \mathcal{O}_X)$ (locally) ringed space if $X \in \underline{Top}$ $\mathcal{O}: \underline{Top} \to \underline{Ring}$
(s.t $\mathcal{O}_{X,p}$ local ring $\forall p \in X$.)

Morphisms:

Recall that Spec : $\underline{Rings} \longrightarrow \underline{Sets}$
contravariant functor

if. $f: X = \operatorname{Spec} R \longrightarrow Y = \operatorname{Spec} S$

$f^{\#}: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$

if $D(g) \subset Y$ open $\quad f^{\#}\ \mathcal{O}_Y(D(g)) = S_g = \mathcal{O}_X(f^{-1}(D(g)))$
$\quad = R_{(f^{-1}(g))}$

Def ① Affine Scheme: is a pair $(X, \mathcal{O}_X) \simeq (\operatorname{Spec} R, \mathcal{O})$

② Scheme is a locally ringed space $(X, \mathcal{O}_X)$ s.t
$\forall p \in X \quad \exists V$ nbd of p s.t. $(V, \mathcal{O}_X(V))$ affine.

Eg ① $R = K[X] \Rightarrow \operatorname{Spec} R = \mathbb{A}^1_K$ if K alg closed
$\mathbb{A}^1_K = \operatorname{Spec} K[X, \dots X_n]$ $\quad \operatorname{Spec} \mathbb{A}^1_K = \{p \in K\} \cup (0)$

Note! $(X-p)$ max ideal $\forall p \in K$.
(0) prime, not max. called a generic pt.
$\overline{(0)} = \operatorname{Spec} \mathbb{A}^1_K$.

What does Spec $R_{\mathfrak{p}}$ look like? What is the scheme $(R_{\mathfrak{p}}, \mathcal{O})$?

② In general, let $\mathfrak{p} \in R$ prime ideal, not maximal.
$\overline{\mathfrak{p}} = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{p} \subset \mathfrak{q}\}$

Eg $R = \mathbb{R}[x, y]$ $\eta = (X^2 + y^2 - 1)$ prime not max.

$\overline{\eta} = \eta \cup \{(x-a, y-b) \mid a^2 + b^2 = 1\}$

③ $X_1 = X_2 = \mathbb{A}^1_k$. Let $P = (x)$

$U_1 = U_2 = \mathbb{A}^1_k \setminus P$ $\qquad \varphi : U_1 \to U_2$ idenity.

$X = X_1 \sqcup X_2 / \varphi(U_1) \sim U_2$ w/ quotient topo. $\qquad i_1 : X_1 \hookrightarrow X \quad i_2 : X_2 \hookrightarrow X$

$$\mathcal{O}_X(V) = \{ (s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(i_1^{-1} V),\ s_2 \in \mathcal{O}_{X_2}(i_2^{-1} V),\ \varphi(s_1|_{i_1^{-1}(V_1) \cap U_1}) = s_2|_{i_2^{-1} V_2 \cap U_2} \}$$

$X =$ ——————— : ——————— line w/ doubled origin.

④ Projective schemes

R graded ring. $R^+ = \bigoplus_{i>0} R_{d_i}$ $\qquad R = \bigoplus_i R_i$

$\mathrm{Proj}\ R = \{ \mathfrak{p} \text{ prime ideal} \mid \mathfrak{p} \in R_i \text{ for som } i \text{ s.t. } R^1 \not\subset \mathfrak{p} \}$

$V(I) = \{ \mathfrak{p} \in \mathrm{Proj}\ R \mid \mathfrak{p} \supseteq I \}$ $\qquad f \in R^+$ homo geneous

$D_+(f) = \{ \mathfrak{p} \in \mathrm{Proj}\ S \mid f \notin \mathfrak{p} \}$

$(\mathrm{Proj}\ R, \mathcal{O})$ sheaf of rings over $\mathrm{Proj}\ R$

$$\mathcal{O}(D_+(f)) = \{ s : D_+(f) \to \coprod_{\mathfrak{p} \in D_+(f)} R_{(\mathfrak{p})} \mid \forall \mathfrak{p} \ ① s(\mathfrak{p}) \in R_{(\mathfrak{p})} \ ② \exists \text{ open n.b.d } V \text{ w/ } t \in \mathcal{O}(V) \text{ s.t. } t(q) = s(q)\ \forall q \in V \}$$

$R_{(\mathfrak{p})}$ = degree 0 elems of $T^{-1} R$

for T = homo gen elems of R not in $\mathfrak{p}$.

(graded version of Affine schemes)

<u>Ex</u> $\mathbb{P}^n_k = \mathrm{Proj}\ k[x_0 \ldots x_n]$. Formed by glueing together $n+1$ copies of $\mathbb{A}^n_k$ in the standard way.

Adjectives:

Connected Scheme: connected as a top. space.

if $X = \mathrm{Spec}\, R \Rightarrow R$ is a connected ring (no non-trivial idempotents)

finite morphism:

$f: X \to Y$ cover X w/ $\{\mathrm{Spec}\, B_i\}$

$f^{-1}(\mathrm{Spec}\, B_i) \cong \mathrm{Spec}\, A$ for some A finite B_i module.

finite type if $f^{-1}(\mathrm{Spec}\, B_i) = \bigcup_{j \in J} \mathrm{Spec}\, A_j$ w/ each A_j a finite B_i module.

reduced $\mathcal{O}_X(U)$ has no nilpotents.

irreducible X connected. $\mathcal{O}_X(U)$ irreducible rings.

integral reduced and reducible

Seperated (attempt at Hausdorff) $\Delta: Y \hookrightarrow Y \times Y$ closed

X seperated if $\Delta: X \longrightarrow X \times_{\mathrm{Spec}\, \mathbb{Z}} X$ closed imm.

f sep if $f: X \to Y$ and $\Delta: X \longrightarrow X \times_Y X$ closed imm.

proper morphism (attempt to mimick proper maps)

$f: X \to Y$ universally closed if closed, and

$$\forall \begin{array}{ccc} X' = X \times_Y Y' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

f' closed $\forall g$.

f proper if seperated, univ closed, finite type.

regular $\mathcal{O}_X(U)$ regular rings.

open subscheme $\mathcal{O}_X|_U$ for $U \subset X$ open

Open immersion $f: X \to Y$ s.t. $f(X) \cong$ open subscheme of Y.

closed immersion: $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$

$f(X) \subset Y$ closed. $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ surjective.

closed subscheme: of X equivalence class of closed immersions.

$$\begin{tikzcd} Y \arrow[r, "f"] & X \\ & Y' \arrow[ul, "i"] \arrow[u, "f'"'] \end{tikzcd}$$

equivalent $\Leftrightarrow$ $\exists\, i : Y' \to Y$ iso w/ $f' = f \circ i$