7-manifolds with G_2 holonomy

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What is G_2? G_2 holonomy and Ricci-flat metrics

i. the automorphism group of the octonions $\mathbb{O}$

ii. the stabilizer of a generic 3-form in $\mathbb{R}^7$

Define a vector cross-product on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$

$$u \times v = \mathrm{Im}(uv)$$

where uv denotes octonionic multiplication. Cross-product has an associated 3-form

$$\varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle$$

φ_0 is a generic 3-form so in fact

$$G_2 = \{A \in \mathsf{GL}(7, \mathbb{R}) |\ A^* \varphi_0 = \varphi \subset \mathsf{SO}(7).$$

G_2 can arise as the holonomy group of an irreducible non-locally-symmetric Riemannian 7-manifold **(Berger 1955, Bryant 1987, Bryant-Salamon 1989, Joyce 1995).** Any such manifold is automatically *Ricci-flat*.

$6+1=2\times 3+1=7$ & **SU**$(2)\subset$ **SU**$(3)\subset G_2$

$\exists$ *close relations between G_2 holonomy and Calabi-Yau geometries in* 2 *and* 3 *dimensions.*

- Write $\mathbb{R}^7=\mathbb{R}\times\mathbb{C}^3$ with $(\mathbb{C}^3,\omega,\Omega)$ the standard SU(3) structure then

  $$\varphi_0 = dt\wedge\omega + \mathrm{Re}\,\Omega$$

  Hence stabilizer of $\mathbb{R}$ factor in G_2 is SU(3) $\subset G_2$. More generally if (X,g) is a Calabi-Yau 3-fold then product metric on $\mathbb{S}^1\times X$ has holonomy SU(3) $\subset G_2$.
- Write $\mathbb{R}^7=\mathbb{R}^3\times\mathbb{C}^2$ with coords (x_1,x_2,x_3) on $\mathbb{R}^3$, with standard SU(2) structure $(\mathbb{C}^2,\omega_I,\Omega=\omega_J+i\omega_K)$ then

  $$\varphi_0 = dx_1\wedge dx_2\wedge dx_3 + dx_1\wedge\omega_I + dx_2\wedge\omega_J + dx_3\wedge\omega_K,$$

  where ω_I and $\Omega=\omega_J+i\omega_K$ are the standard Kahler and holmorphic $(2,0)$ forms on $\mathbb{C}^2$. Hence subgroup of G_2 fixing $\mathbb{R}^3\subset\mathbb{R}^3\times\mathbb{C}^2$ is SU(2) $\subset G_2$.

G_2 structures and G_2 holonomy metrics

- A G_2 structure is a 3-form ϕ on an oriented 7-manifold M such that at every point $p \in M$, $\exists$ an oriented isomorphism

$$i : T_pM \to \mathbb{R}^7, \text{ such that } i^*\varphi_0 = \phi.$$

- G_2-structures on $\mathbb{R}^7 \leftrightsquigarrow \mathrm{GL}_+(7,\mathbb{R})/G_2$.
- $\dim(\mathrm{GL}_+(7,\mathbb{R})/G_2) = 35 = \dim \Lambda^3\mathbb{R}^7$.
  $\Rightarrow$ implies small perturbations of a G_2-structure are still G_2-structures.

How to get a G_2-holonomy metric from a G_2 structure?

Theorem
Let (M, ϕ, g) be a G_2 structure on a compact 7-manifold; the following are equivalent

1. $\mathrm{Hol}(g) \subset G_2$ *and ϕ is the induced* 3*-form*
2. $\nabla\phi = 0$ *where ∇ is Levi-Civita w.r.t g*
3. $d\phi = d^*\phi = 0$*.*

Call such a G_2 structure a *torsion-free* G_2 structure.
NB (3) is nonlinear in ϕ because metric g depends nonlinearly on ϕ.

G_2 structures and G_2 holonomy metrics II

Lemma

Let M be a compact 7-manifold.

1. *M admits a G_2 structure iff it is orientable and spinnable.*
2. *A torsion-free G_2 structure (ϕ, g) on M has $\mathrm{Hol}(g) = G_2$ iff $\pi_1 M$ is finite.*
3. *If $\mathrm{Hol}(g) = G_2$ then M has nonzero first Pontrjagin class $p_1(M)$.*

Ingredients of proof for 2 and 3.

2. M has holonomy contained in G_2, implies g is Ricci-flat. Now combine structure results for non-simply connected compact Ricci-flat manifolds (application of Cheeger-Gromoll splitting theorem) with the classification of connected subgroups of G_2 that could appear as (restricted) holonomy groups of g.

3. Apply Chern-Weil theory for $p_1(M)$ and use G_2 representation theory to analyse refinement of de Rham cohomology on a G_2 manifold; full holonomy G_2 forces vanishing of certain refined Betti numbers and this leads to a sign for $\langle p_1(M) \cup [\phi], [M] \rangle$. □

Exceptional holonomy milestones

1984: (**Bryant**) locally $\exists$ many metrics with holonomy G_2 and $Spin(7)$. Proof uses Exterior Differential Systems.

1989: (**Bryant-Salamon**) explicit complete metrics with holonomy G_2 and $Spin(7)$ on noncompact manifolds.

- total space of bundles over 3 & 4 mfds
- metrics admit large symmetry groups and are asymptotically conical

1994: (**Joyce**) Gluing methods used to construct *compact* 7-manifolds with holonomy G_2 and 8-manifolds with holonomy $Spin_7$. Uses a modified Kummer-type construction.

String/M-theorists become interested in using compact manifolds with exceptional holonomy for supersymmetric compactifications.

2000: **Joyce**'s book *Compact Manifolds with Special Holonomy.*

2003: **Kovalev** uses Donaldson's idea of a *twisted connect sum* construction to find alternative constructions of compact G_2 manifolds.

The moduli space of holonomy G_2 metrics

Let M be a compact oriented 7-manifold and let $\mathcal{X}$ be the set of torsion-free G_2 structures on M. Let $\mathcal{D}$ be the group of all diffeomorphisms of M isotopic to the identity. Then $\mathcal{D}$ acts naturally on the set of G_2 structures on M and on $\mathcal{X}$ by $\phi \mapsto \Psi_*(\phi)$.

Define the *moduli space of torsion-free G_2 structures* on M to be $\mathcal{M} = \mathcal{X}/\mathcal{D}$.

Theorem (Joyce)

$\mathcal{M}$ the moduli space of torsion-free G_2 structures on M is a **smooth** *manifold of dimension $b^3(M)$, and the natural projection $\pi\colon \mathcal{M} \to H^3(M, \mathbb{R})$ given by $\pi(\phi\mathcal{D}) = [\phi]$ is a local diffeomorphism.*

Main ingredients of the proof: (a) a good choice of 'slice' for the action of $\mathcal{D}$ on $\mathcal{X}$, i.e. a submanifold S of $\mathcal{X}$ which is (locally) transverse to the orbits of $\mathcal{D}$, so that each nearby orbit of $\mathcal{D}$ meets S in a single point. (b) Some fundamental technical results about (small) perturbations of G_2 structures to yield appropriate nonlinear elliptic PDE. (c) Linearise the PDE and apply standard Hodge theory and Implicit Function Theory.

Two fundamental technical results:

Denote by Θ the (nonlinear) map sending $\phi \mapsto *\phi$.

Lemma (A)

If ϕ is a closed G_2-structure on M and χ a sufficiently small 3-form then $\phi + \chi$ is also a G_2-structure with Θ given by

$$\Theta(\phi + \chi) = *\phi + *(\textit{explicit terms linear in } \chi) - F(\chi)$$

where F is a smooth function from a closed ball of small radius in $\Lambda^3 T^* M$ to $\Lambda^4 T^* M$ with $F(0) = 0$ satisfying some additional controlled growth properties.

Lemma (B)

If (M, ϕ, g) is a compact G_2 manifold and $\tilde{\phi}$ is a closed 3-form C^0-close to ϕ, then $\tilde{\phi}$ can be written uniquely as $\tilde{\phi} = \phi + \xi + d\eta$ where ξ is a harmonic 3-form and η is a d^*-exact 2-form. Moreover, $\tilde{\phi}$ is a torsion-free G_2 structure also satisfying the "gauge fixing"/slice condition if and only if

$$(*) \quad \Delta\eta = *dF(\xi + d\eta).$$

The latter gives us the *nonlinear elliptic PDE* (for the coexact 2-form η) we seek.

How to construct compact G_2 manifolds

Meta-strategy to construct compact G_2 manifolds

I. Find a closed G_2 structure ϕ with sufficiently small torsion on a 7-manifold with $|\pi_1| < \infty$

II. Perturb to a torsion-free G_2 structure ϕ' close to ϕ.

- II was understood in great generality by Dominic Joyce using an extension of Lemma (B) to G_2 structures ϕ that are closed and sufficiently close to being torsion-free.
- Condition that the perturbed G_2 structure $\phi + d\eta$ be torsion-free still becomes a nonlinear elliptic PDE (*)' for the 2-form η; get extra terms on RHS of (*) coming from failure of background G_2-structure ϕ to be torsion-free.
- Joyce solves (*)' by iteratively solving a sequence of linear elliptic PDEs together with a priori estimates (of appropriate norms) on the iterates to establish their convergence to a limit satisfying (*)'.

Q: How to construct closed almost torsion-free G_2 structures?!

Degenerations of compact G_2-manifolds I

Q: How to construct closed almost torsion-free G_2 structures?!

- Key idea: Think about possible ways a family of G_2 holonomy metrics on a given compact 7-manifold might *degenerate*.
- Find instances in which the singular "limit" G_2 holonomy space X is simple to understand.
- Try to construct a smooth compact 7-manifold M which resolves the singularities of X; use the geometry of the resolution to build by hand a **closed** G_2-structure on M that is *close enough to torsion-free*.

M has holonomy $G_2 \Rightarrow M$ is Ricci-flat; so think about how families of compact Ricci-flat manifolds (more generally Einstein manifolds or just spaces with lower Ricci curvature bounds) can degenerate.

Degenerations of compact G_2-manifolds II

Case 1. Neck stretching degeneration.

- A degeneration in which (M, g_i) develops a long "almost cylindrical neck" that gets stretched longer and longer.
- In the limit we decompose M into a pair of noncompact 7-manifolds M_+ and M_-; $M_\pm$ should each be asymptotically cylindrical G_2 manifolds.

Given such a pair $M_\pm$ with appropriately compatible cylindrical ends we could try to *reverse* this construction, i.e. to build a compact G_2 manifold M by truncating the infinite cylindrical end sufficiently far down to get a G_2-structure with small torsion and a long "almost cylindrical" neck region.

Big disadvantage: doesn't seem any easier to construct asymptotically cylindrical G_2 manifolds than to construct compact G_2 manifolds.

Advantage: maintain good geometric control throughout, e.g. lower bounds on injectivity radius, upper bounds on curvature etc.
$\Rightarrow$ perturbation analysis remains relatively simple technically.

Donaldson suggested a way to circumvent the problem above.

Degenerations of compact G_2-manifolds III

Case 2. Diameter bounded with lower volume control

How can sequences of compact Ricci-flat spaces degenerate with bounded diameter and lower volume bounds?

Simplest answer: they could develop orbifold singularities in codimension 4.

Simplest model is a metric version of the *Kummer construction* for K3 surfaces.

Choose a lattice $\Lambda \simeq \mathbb{Z}^4$ in $\mathbb{C}^2$ and form 4-torus $T^4 = \mathbb{C}^2/\Lambda$. Look at involution $\sigma : T^4 \to T^4$ induced by $(z_1, z_2) \mapsto (-z_1, -z_2)$.

- σ fixes $2^4 = 16$ points $\{[z_1, z_2] : (z_1, z_2) \in \frac{1}{2}\Lambda\}$.
- $T^4/\langle\sigma\rangle$ is a flat hyperkähler orbifold with 16 singular points modelled on $\mathbb{C}^2/\{\pm 1\}$.
- S the blow-up of $T^4/\langle\sigma\rangle$ is a smooth K3 surface: a *Kummer surface*.
- Pulling back flat orbifold metric g_0 from T^4 to S gives a *singular* Kähler metric on S, degenerate at the 16 $\mathbb{P}^1$ introduced by blowing-up.

The metric Kummer construction

Want to build a family of smooth metrics g_t on S which converges as $t \to 0$ to this singular flat orbifold metric.

Key is the *Eguchi-Hanson metric*, which gives a hyperkähler metric on the blowup of $\mathbb{C}^2/\langle \pm 1 \rangle$ (which is biholomorphic to $T^*\mathbb{P}^1$).

To get a nonsingular Kahler metric on S near each $\mathbb{P}^1$ we replace the degenerate metric with a suitably scaled copy of Eguchi-Hanson metric and interpolate to get ω'_t on S, where parameter t controls the diameter of the 16 $\mathbb{P}^1$.

- Page observed that ω'_t is close to Ricci-flat.
- Topiwala, LeBrun-Singer then proved that it can be perturbed to a Ricci-flat Kahler metric ω_t.
- ω_t converges to the flat orbifold metric as $t \to 0$ and the size of each $\mathbb{P}^1$ goes to 0.

Could try similar thing using other ALE hyperkähler 4-manifolds constructed by Gibbons-Hawking, Hitchin, Kronheimer for all the ADE singularities $\mathbb{C}^2/\Gamma$, i.e. where Γ is a finite subgroup of $SU(2)$.

Joyce's orbifold resolution construction of compact G_2 manifolds

Basic idea: seek a G_2 analogue of the metric Kummer construction above.

- look at finite subgroups $\Gamma \subset G_2$ and consider singular flat orbifold metrics $X = T^7/\Gamma$.
- analyse the singular set of T^7/Γ; this is never an isolated set of points and often can be very complicated with various strata.
- look for Γ for which the singular set is particularly simple, e.g. a disjoint union of smooth manifolds.
- find appropriate G_2 analogues of Eguchi-Hanson spaces, i.e. understand how to find resolutions of $\mathbb{R}^7/G$ and put (Q)ALE G_2 holonomy metrics on them.
- Use these ingredients to find a smooth 7-manifold M resolving the singularities of X, admitting a 1-parameter family of closed G_2 structures ϕ_t with torsion sufficiently small compared to lower bounds for injectivity radius and upper bound for curvature; apply the general perturbation theory for closed G_2 structures with small torsion; analysis is delicate because induced metric is nearly singular.

Simplest generalised Kummer construction

If $G \subset SU(2)$ is a finite group and Y an ALE hyperkahler manifold then $\mathbb{R}^3 \times Y$ is naturally a (Q)ALE G_2-manifold, e.g. Y could be Eguchi-Hanson space for $G = \mathbb{Z}_2$.

Simplest Kummer construction:

- find finite $\Gamma \simeq \mathbb{Z}_2^3 \subset G_2$ so that singular set S of T^7/Γ is a disjoint union of 3-tori for which some open neighbourhood of each torus is isometric to $T^3 \times B^4/\langle \pm 1 \rangle$.
- Replace $B^4/\langle \pm 1 \rangle$ by its blowup U and (using explicit form of Kähler potential) put a 1-parameter family of triples of 2-forms $\omega_i(t)$ on U that interpolates between the hyperkähler structures of Eguchi-Hanson and of $\mathbb{C}^2/\langle \pm I \rangle$
- Obtain a compact smooth 7-manifold M by replacing a neighbourhood of each component of singular set S by $T^3 \times U$
- The triple of 2-forms $\omega_i(t)$ on U gives rise to a closed G_2 structure on $T^3 \times U$ for t sufficiently small and which is flat far enough away from T^3; so M has a 1-parameter family of closed G_2-structures ϕ'_t with small torsion supported in some "annulus" around the T^3.

Now apply the perturbation theory to get a 1-parameter family of torsion-free G_2 structures ϕ_t and verify that M has finite (actually trivial) fundamental group so that g_t all have full holonomy G_2. Can also compute Betti numbers of M: $b^2 = 12$, $b^3 = 43$.

$\mathbf{SU}(3) + \mathbf{SU}(3) + \epsilon = G_2$

Donaldson suggested constructing compact G_2 manifolds from a pair of asymptotically cylindrical Calabi-Yau 3-folds via a *neck-stretching* method.

i. Use noncompact version of Calabi conjecture to construct asymptotically cylindrical Calabi-Yau 3-folds V with one end $\sim \mathbb{C}^* \times D \sim \mathbb{R}^+ \times \mathbb{S}^1 \times D$, with D a smooth $K3$.

ii. $M = \mathbb{S}^1 \times V$ is a 7-manifold with $\operatorname{Hol} g = \mathrm{SU}(3) \subset G_2$ with end $\sim \mathbb{R}^+ \times T^2 \times K3$.

iii. Take a *twisted connected sum* of a pair of $M_\pm = \mathbb{S}^1 \times V_\pm$

iv. For $T >> 1$ construct a G_2-structure w/ small torsion (exponentially small in T) and prove it can be corrected to torsion-free.

Kovalev (2003) carried out Donaldson's proposal for AC CY 3-folds arising from Fano 3-folds. However the paper contains two serious mistakes.

Twisted connected sums & hyperkähler rotation

Product G_2 structure on $M_\pm = \mathbb{S}^1 \times V_\pm$ asymptotic to

$$d\theta_1 \wedge d\theta_2 \wedge dt + d\theta_1 \wedge \omega_I^\pm + d\theta_2 \wedge \omega_J^\pm + dt \wedge \omega_K^\pm$$

$\omega_I^\pm$, $\omega_J^\pm + i\,\omega_K^\pm$ denote Ricci-flat Kähler metric & parallel $(2,0)$-form on $D_\pm$.

To get a well-defined G_2 structure using

$$F : [T-1, T] \times \mathbb{S}^1 \times \mathbb{S}^1 \times D_- \to [T-1, T] \times \mathbb{S}^1 \times \mathbb{S}^1 \times D_+$$

given by

$$(t, \theta_1, \theta_2, y) \mapsto (2T - 1 - t, \theta_2, \theta_1, f(y))$$

to identify end of M_- with M_+ we need $f : D_- \to D_+$ to satisfy

$$f^*\omega_I^+ = \omega_J^-, \quad f^*\omega_J^+ = \omega_I^-, \quad f^*\omega_K^+ = -\omega_K^-.$$

- Constructing such hyperkähler rotations is nontrivial and a major part of the construction.
- Some problems in Kovalev's original paper here.

Twisted connected sum G_2-manifolds

1. Construct suitable ACyl Calabi-Yau 3-folds V;
2. Find sufficient conditions for existence of a *hyperkähler rotation* between D_- and D_+;
   - Use global Torelli theorems and lattice embedding results (e.g. Nikulin) to find hyperkähler rotations from suitable initial pairs of (deformation families of) ACyl CY 3-folds.
3. Given a pair of ACyl CY 3-folds $V_\pm$ and a HK-rotation $f : D_- \to D_+$ can *always* glue M_- and M_+ to get a 1-parameter family of closed manifolds M_T with holonomy G_2.
   - in general for the same pair of ACyl CY 3-folds different HK rotations can yield different 7-manifolds (e.g. different Betti numbers b^2 and b^3).

$\Rightarrow$ have reduced solving nonlinear PDEs for G_2-metric to two problems about complex projective 3-folds.

ACyl Calabi-Yau 3-folds

Theorem (H-Hein-Nordström JDG 2015)

Any simply connected ACyl Calabi-Yau 3-fold X with split end $\mathbb{S}^1 \times K3$ is quasiprojective, i.e. $X = \overline{X} \setminus \overline{D}$ for some smooth projective variety $\overline{X}$ and smooth anticanonical divisor $\overline{D}$. Moreover $\overline{X}$ fibres holomorphically over $\mathbb{P}^1$ with generic fibre a smooth anticanonical K3 surface. Conversely, the complement of any smooth fibre in any such $\overline{X}$ admits (exponentially) ACyl CY metrics with split end.

Builds on previous work of Tian-Yau and Kovalev; HHN proved more general compactification for ACyl CY manifolds (ends need not split; compactification can be singular).

3 main sources of examples of such K3 fibred 3-folds:

- Fano 3-folds, K3 surfaces with nonsymplectic involution (Kovalev); gives several hundred examples.
- weak or semi-Fano 3-folds (Corti-H-Nordström-Pacini); gives at least several hundred thousand examples!

Simple example of a semi-Fano 3-fold

Example 1: start with a (singular) quartic 3-fold $Y \subset \mathbb{P}^4$ containing a projective plane Π and resolve. If $\Pi = (x_0 = x_1 = 0)$ then eqn of Y is

$$Y = (x_0 a_3 + x_1 b_3 = 0) \subset \mathbb{P}^4$$

where a_3 and b_3 are homogeneous cubic forms in $(x_0, \ldots, x_4)$. Generically the plane cubics

$$(a_3(0,0,x_2,x_3,x_4) = 0) \subset \Pi,$$

$$(b_3(0,0,x_2,x_3,x_4) = 0) \subset \Pi$$

intersect in 9 distinct points, where Y has 9 ordinary double points. Blowing-up $\Pi \subset Y$ gives a smooth 3-fold X such that $f : X \to Y$ is a *projective* small resolution of all 9 nodes of Y.

X is a smooth (projective) semi-Fano 3-fold; it contains 9 smooth rigid rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$; X has genus 3 and Picard rank 2.

G_2-manifolds and toric semi-Fano 3-folds

Theorem (Corti-Haskins-Nordström-Pacini (Duke 2015)+CHK)

There exist over 900 million matching pairs of ACyl CY 3-folds of semi-Fano type for which the resulting G_2-manifold is 2-connected.

Main ingredients of proof.

- Use a pair of ACyl CY 3-folds with one of toric semi-Fano type and the other a semi-Fano (or Fano) of rank at most 2.
- Use further arithmetic information about polarising lattices (discriminant group information) to prove there are over 250,000 toric semi-Fanos that can be matched to *any* ACyl CY 3-fold of Fano/semi-Fano type of rank at most 2. Over 250,000 rigid toric semi-Fanos arise from only the 12 most "prolific" polytopes.
- There are over 200 deformation types of Fanos/semi-Fanos of rank at most 2.

□