Problem sheet 7

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=39224
until 20:00, February 2.

HW 1 **[4 points]** Let I be a good rate function on E and f be a continuous function from E to S. Show that the infimum in

$$J(y) = \inf\{I(x) : \ f(x) = y\} = \inf_{f^{-1}(\{y\})} I, \quad y \in S.$$

is attained, that is, there exists $x \in E$ such that $f(x) = y$ and $J(y) = I(x)$.

1. Let $w(t)$, $t \in [0,T]$, be a Brownian motion on $\mathbb{R}$ with diffusion rate σ^2 and $w(0) = x_0$. Show that $(\sqrt{\varepsilon} w)_{\varepsilon>0}$ satisfies the LDP in $\mathrm{C}[0,T]$ and find the associated rate function.

HW 2 **[4 points]** Let $B(t) = w(t) - tw(1)$, $t \in [0,1]$, be a Brownian bridge on $\mathbb{R}$, where w is a standard Brownian motion on $\mathbb{R}$. Show that the family $(\sqrt{\varepsilon} B)_{\varepsilon>0}$ satisfy the LDP in $\mathrm{C}_0[0,1]$ with the good rate function

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 \dot{f}^2(t)dt, & \text{if } f \in H_0^2[0,1] \text{ and } f(1) = 0, \\ +\infty, & \text{otherwise} \end{cases}$$

(*Hint:* Use the contraction principle)

2. Let $\xi = (\xi_n)_{n\geq 1}$ be a sequence of i.i.d. $N(0,1)$ random variables. Use Schilder's theorem to show that the family $(\sqrt{\varepsilon}\xi)_{\varepsilon>0}$ satisfies the LDP in $\mathbb{R}^\infty$ with good rate function

$$I(x) = \begin{cases} \frac{1}{2}\sum_{n=1}^\infty x_n^2, & \text{if } x = (x_n)_{n\geq 1} \in l^2, \\ +\infty, & \text{otherwise.} \end{cases}$$

(*Hint:* Consider the map $\Phi(f) = \left(\frac{f(t_n) - f(t_{n-1})}{\sqrt{t_n - t_{n-1}}}\right)_{n \geq 1}$ for an infinite partition $0 = t_0 < t_1 < \dots < 1$ of the interval $[0,1]$ and use the contraction principle)

3. Let $\Phi : \mathrm{C}[0,T] \to \mathrm{C}[0,T]$ be defined in the proof of Theorem 8.2.
   1) Show that the function Φ is bijective.
   2) Prove that $g \in H_0^2[0,T]$ if and only if $f = \Phi(g) \in H_0^2[0,T]$.

HW 3 **[2 points]** Show that $\dot{g} = \dot{f} - a(f)$ almost everywhere for every $g \in H_0^2[0,T]$ and $f = \Phi(g)$.