Problem sheet 6

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=37532
until 20:00, January 21.

1. Let $X(t)$, $t \geq 0$, be a process and $\mathcal{F}_t^X = \sigma(X(s),\ s \leq t)$. Let also $\mathbb{E}\,|X(t)| < \infty$ for all $t \geq 0$. Show that X is a martingale if and only if for every $s < t$, $n \in \mathbb{N}$, partitions $s_1 < s_2 < \dots < s_n \leq s$ and bounded continuous functions $h_1, \dots, h_n$ one has

$$\mathbb{E}\,(X(t)h_1(X(s_1))\dots h_n(X(s_n))) = \mathbb{E}\,(X(s)h_1(X(s_1))\dots h_n(X(s_n))).$$

HW 1 **[4 points]** Let $X_n(t)$, $t \in [0,T]$, be a family of continuous square-integrable martingales such that

$$\mathbb{E} \sup_{t\in[0,T]} (X_n(t) - X(t))^2 \to 0, \quad n \to \infty,$$

where $X(t)$, $t \in [0,T]$, is a continuous process. Show that X is a square-integrable martingale.

HW 2 **[1 points]** Let $X(t)$, $t \geq 0$, be a continuous square integrable martingale with quadratic variation $\langle X \rangle_t$, $t \geq 0$. Show that

$$\mathbb{E}\, X^2(t) = \mathbb{E}\, X^2(0) + \mathbb{E}\,\langle X \rangle_t$$

for all $t \geq 0$.

2. Let $f \in C_0[0,T]$.

   (i) Let f be absolutely continuous. Show that for every $h \in \mathrm{C}^1[0,T]$

$$h(T)f(T) - \int_0^T h'(t)f(t)dt = \int_0^T h(t)\dot{f}(t)dt.$$

   (*Hint:* Check first the equality if $\dot{f} \in \mathrm{C}[0,T]$. Then, in the general case, approximate $\dot{f}$ in $L_1[0,T]$ by continuous functions)

   HW 3 **[4 points]** Let $g \in L_2[0,T]$ and for every $h \in \mathrm{C}^1[0,T]$

$$h(T)f(T) - \int_0^T h'(t)f(t)dt = \int_0^T h(t)g(t)dt.$$

   Show that f is absolutely continuous with $\dot{f} = g$.
   (*Hint:* Consider the function $\tilde{f}(t) = \int_0^t g(s)ds$ and apply to $\int_0^T h(t)g(t)dt$ the integration by parts formula)