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Loss systems in a random environment - steady state analysis

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We consider a single server system with infinite waiting room in a random environment. The service system and the environment interact in both directions. Whenever the environment enters a specific subset of its state space the service process is completely blocked: Service is interrupted and newly arriving customers are lost. We prove an if-and-only-if-condition for a product form steady state distribution of the joint queueing-environment process. A consequence is a strong insensitivity property for such systems.

We discuss several applications, e.g. from inventory theory and reliability theory, and show that our result extends and generalizes easily several theorems found in the literature, e.g. of queueing-inventory processes.

We investigate further classical loss systems, where due to finite waiting room loss of customers occurs. In connection with loss of customers due to blocking by the environment and service interruptions new phenomena arise.

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1 Introduction

Product form networks of queues are common models for easy to perform structural and quantitative first order analysis of complex networks in Operations Research applications.

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The most prominent representatives of this class of models are the Jackson [Jac57] and Gordon-Newell [GN67] networks and their generalizations as BCMP [BCMP75] and Kelly [Kel76] networks, for a short review see [Dad01].

The mathematical description of this class of models are time homogeneous Markovian vector processes, where each coordinate represents the behaviour of one of the queues. Product form networks are characterized by the fact that in steady state (at any fixed time t) the joint distribution of the multi-dimensional (over nodes) queueing process is the product of the marginal distributions of the isolated (non Markovian) queues in steady state. With respect to the research described in this note the key point is that the coordinates of the vector process represent objects of the same class, namely queueing systems.

In Operations Research applications queueing systems constitute an important class of models in very different settings. But in many situations those parts of a complex production system which are modeled by queues interact with other subsystems which usually can not be modeled by queues. Typically, there is a manufacturing system, (machine, modeled by a queueing system) which assembles delivered raw material to a final product, consuming in the production process some further material (we will call these additional pieces of material ”items" henceforth) which is hold in inventories which provide from stock these items.

In classical Operations Research the fields of queueing theory and inventory theory are almost disjoint areas of research, but with increasing complexity of integrated production and inventory systems (think of large Supply Chains) the need for integrated models that encompass queueing and inventory processes and their interaction is obvious.

Indeed, there is a large number of papers on integrated queueing-inventory models, see the review in [KLM11], but only recently, Schwarz, Sauer, Daduna, Kulik, and Szekli [SSD$^+$06] discovered product forms for the steady state distributions of an $M/M/1/\infty$ under (r, S) and (r, Q) policies with lost-sales. Further contributions to product form results in this field are by Vineetha [Vin08], Saffari, Haji, and Hassanzadeh [SHH11], and Saffari, Asmussen, and Haji [SAH10]. An early paper of Berman and Kim [BK99] can be considered to contribute to integrated models with product form steady state. Due to zero lead time for replenishment the queue length can develop independent of the inventory size.

A similar observation was made for other vector process describing (possibly many) queues as parts of a system, which furthermore encompasses coordinates which describe the reliability status of the system as a vector process or as a compound reliability component information. In [SD03] it is proved that under certain conditions a product form equilibrium for such networks of unreliable servers exists.

Our present research is motivated by the observation that in both of the described situations we have a construction of an integrated model

- for production processes modeled by queueing systems and

- an additional relevant part of the system, e.g., inventory control or availability control.

The important observation in those models is that both of the components strongly interact, but that in equilibrium the coordinates at fixed time points decouple, which means that a product form equilibrium emerges.

An important common aspect of the interaction in both applications leads to the term **loss system**: Whenever, respectively,

- the inventory is depleted,
- the machine is broken down,

service is interrupted due to stock out, resp. no production capacity available. Additionally, during the time the interruption continues no new arrivals are admitted to both systems, due to lost sales and because customers prefer to enter some other working server.

This loss of customers is different from what is usually termed loss systems in pure queueing theory, where loss of customers happens, when the finite waiting space is filled up to maximal capacity.

Following the above description, in our present investigation of complex systems we always start with a queueing system as one subsystem and a general attached other subsystem which imposes side constraints on the queueing process and in general interacts in both directions with the queue. For ease of discussion in the general model we will call this second component an "environment" which influences the production process and which is influenced by the production vice versa. In typical cases we assume that there will be a part of the environment's state space, the states of which we shall call "blocking states", with the following property: Whenever the environment enters a blocking state, the service process will be interrupted and no new arrivals are admitted to enter the system and are lost to the system forever.

Our main result (Theorem 2) is that in these systems where production and environment strongly interact, asymptotically and in equilibrium (at fixed time instants) the production process and the environment process seem to decouple, which means that a product form equilibrium emerges.

This shows that the mentioned independence results in [SSD+06], [SD03], [SHH11], [SAH10], and [Vin08] do not depend on the specific properties of the attached second subsystem. Furthermore, the theorem can be interpreted as a strong insensitivity property of the system: As long as ergodicity is maintained, the environment can change drastically without changing the steady state distribution of the queue length.

And, vice versa, it can be seen that the environment's steady state will not change when the service capacity of the production will change.

We shall discuss this with related problems and some complements to the theorem in more detail in Section 3 after presenting our main result there.

In a more abstract setting, we can our present work describe as to develop a framework for a birth-death process in a random environment, where the birth-death process' development is interrupted from time to time by some configurations occurring in the environment.

On the other side, the birth-death process influences the development of the environment.

There are many investigations on birth-death processes in random environments, we shall cite only some selected references. Best to our knowledge our results below are complementary to the literature.

A stream of research on birth-death processes in a random environment exploits the interaction of birth-death process and environment as the typical structure of a quasi-birth-death process. Such "QBD processes" have two dimensional states, the "level" indicates the population size, while the "phase" represents the environment. For more details see Chapter 6 (Queues in a Random Environment) in [Neu81], and Example C in [Neu89][p. 202, 203].

Related models are investigated in the theory of branching processes in a random environment, see Section 2.9 in [HJV05] for a short review.

An early survey with many references to old literature is [Kes80].

Another branch of research is optimization of queues under constraints put on the queue by a randomly changing environment as described e.g. in [HW84].

While the most of the annotated sources are concerned with conventional steady state analysis, the work [Fal96] is related to ours two-fold: A queue (finite classical loss system) in a random environment shows a product form steady state.

In Section 5 we turn to more concrete problems. We extend and generalize with the help of Theorem 2 the known product form results for queueing-inventory and queueing-availability processes from the literature. We present additionally a promising application in modeling nodes in a wireless sensor network.

We investigate furthermore in Section 4 classical loss systems, where due to finite waiting room loss of customers occur. In connection with loss of customers due to blocking by the environment and service interruptions new phenomena arise. We identify specific environments which produce in this setting product form steady states. It seems remarkable to us that, different from conventional queueing theory, the restriction from infinite waiting room to finite waiting room poses additionally more restrictions on the environment's behavior to generate the product form steady state.

2 The general model

We consider a two-dimensional process $(X, Y) = ((X(t), Y(t)) : t \in [0, \infty))$ with state space $E = \mathbb{N}_0 \times K$. K is a countable set, the environment space of the process, whereas the queueing state space is $\mathbb{N}_0$.

We assume throughout that (X, Y) is a homogeneous strong Markov jump process with cadlag paths which is non-explosive in finite times. (X, Y) is assumed throughout to be irreducible unless specified otherwise.

According to our introductory example the environment space of the process is partitioned into disjoint components $K := K_W + K_B$. In the framework of K describing the inventory size K_B describes the status "stock out", in the reliability problem K_B describes the status "server broken down". So accordingly K_W indicates for the inventory that there is stock on hand for production, and "server is up" in the other system.

The general interpretation is that whenever the environment process enters K_B the service process is **b**locked, which is resumed whenever the environment process returns to K_W, the server **w**orks again.

Whenever the environment process stays in K_B arrivals are lost.

Obviously, it is natural to assume that the set K_W is not empty, while in certain frameworks K_B may be empty, e.g. no break down of the server in the second example.

The server in the system is a single server under First-Come-First-Served regime (FCFS) with an infinite waiting room.
The arrival stream of customers is Poisson with rate $\lambda^{(n)} > 0$, when there are n customers in the system.

The system develops over time as follows.

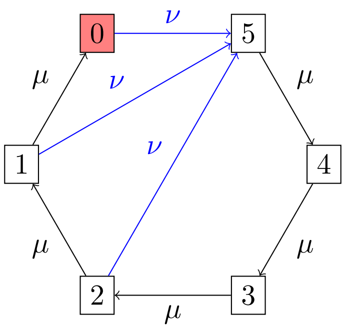
1) If the environment at time t is in state $Y(t) = k \in K_W$ and if there are $X(t) = n$ customers in the queue then service is provided to the customer at the head of the queue with rate $\mu^{(n)} > 0$. The queue is organized according First-Come-First-Served regime (FCFS). As soon as his service is finished he leaves the system and the environment changes with probability R_{km} to state $m \in K$, independent of the history of the system, given k. We consider

$$R \in \mathbb{R}^{|K| \times |K|}, \quad \text{with} \quad \sum_{m \in K} R_{km} = 1 \wedge R \geq 0.$$

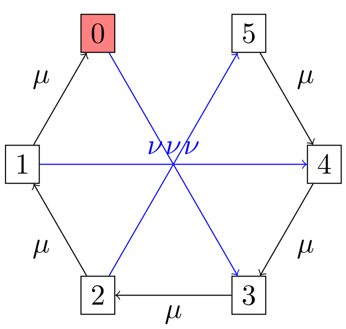
as a stochastic routing matrix for the environment driven by the departure process.

2) If the environment at time t is in state $Y(t) = k \in K_B$ no service is provided to customers in the queue and arriving customers are lost.

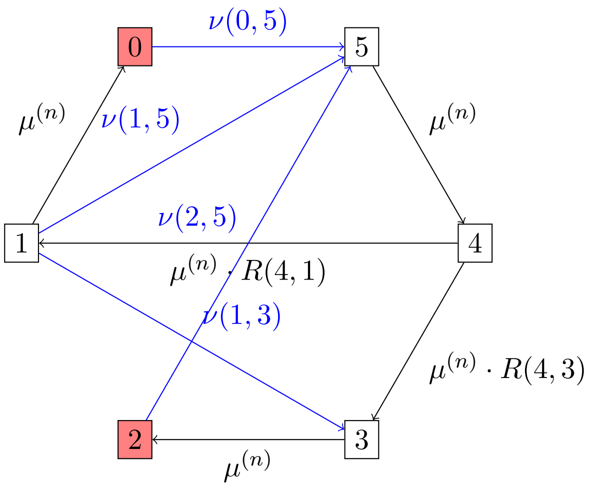
3) Whenever the environment at time t is in state $Y(t) = k \in K$ it changes with rate $\nu(k, m)$ to state $m \in K$, independent of the history of the system, given k.
Note, that such changes occur independent from the service and arrival process, while the changes of the environment's status under **1)** are coupled with the service process.



(a) K under $(r = 2, S = 5)$ policy. In state 0 the arrival stream is interrupted.



(b) K under $(r = 2, Q = 3)$ policy. In state 0 the arrival stream is interrupted.



(c) K for a complex Loss-System with product form steady state. In states $0, 2$ the arrival stream is interrupted.

Figure 2.1: Examples of environments for a queue: inventory under (r, S)-, resp. (r, Q)-policy , and a more general model. All lead to product form steady states. Blocking states are red: No service is provided, arrivals are lost. Black arrows point to states where the system changes as soon as a service expires. Service rates $\mu^{(n)}$ depend on the queue length n. Blue arrows point to states where the system can change with rate $\nu(i, j)$ independent of service activities and queue length.

From the above description we conclude that the non negative transition rates of (X, Y) are for $(n, k) \in E$

$$\begin{aligned}
q((n,k) \to (n+1,k)) &= \lambda^{(n)}, \qquad k \in K_W, \\
q((n,k) \to (n-1,m)) &= \mu^{(n)} R_{km}, \qquad k \in K_W, n \geq 1, \\
q((n,k) \to (n,m)) &= \nu(k,m) \in \mathbb{R}_0^+, \quad k \neq m, \\
q((n,k) \to (i,m)) &= 0\,, \qquad \text{otherwise for } \ (n,k) \neq (i,m)\,.
\end{aligned}$$

Note, that the diagonal elements of $Q := (q((n,k) \to (i,m)) : (n,k),(i,m) \in E)$ are determined by the requirement that row sum is 0.

Remark 1. It is allowed to have positive diagonal entries R_{kk}. R needs not be irreducible, there may exist closed subsets in K.

$\nu(k,k) = -\sum_{m \in K \setminus \{k\}} \nu(k,m)$ is required for all $k \in K$ such that

$$\Upsilon = (\nu(k,m) : k,m \in K)$$

is a generator matrix.

The Markov process associated with Υ may have absorbing states, i.e., Υ then has zero rows.

Following our motivating remarks, in figures 2.1 we sketch the way from inventories (under (r,S) and (r,Q) policies) as environment K towards more general environments which still fit into our framework. For more details on inventory theory in our framework, see Section 5.1.

3 Steady state solution

Our aim is to compute for an ergodic system explicitly the steady state distribution of (X, Y). We can not expect that this will be possible in the general system as described in Section 2, but fortunately enough we will be able to characterize those systems which admit a product form equilibrium.

Theorem 2. **(a)** *Denote for* $n \in \mathbb{N}_0$

$$\begin{aligned}
\tilde{q}_{kk}^{(n)} &= -(1_{[k \in K_W]} \lambda^{(n)} (1 - R_{kk}) + \sum_{m \in K \setminus \{k\}} \nu(k,m)) \\
\tilde{q}_{km}^{(n)} &= \lambda^{(n)} R_{km} 1_{[k \in K_W]} + \nu(k,m) \qquad k \neq m
\end{aligned} \tag{3.1}$$

and

$$\tilde{Q}^{(n)} = (\tilde{q}_{km}^{(n)} : k,m \in K)\,.$$

Then the matrices $\tilde{Q}^{(n)}$ *are generator matrices for some homogeneous Markov processes.*

(b) *If the process* (X, Y) *is ergodic denote its unique steady state distribution by*

$$\pi = (\pi(n,k) : (n,k) \in E := \mathbb{N}_0 \times K).$$

Then the following properties are equivalent:

(i) (X,Y) *is ergodic with product form steady state*

$$\pi(n,k) = \underbrace{C^{-1}\prod_{i=0}^{n-1}\frac{\lambda^{(i)}}{\mu^{(i+1)}}}_{=:\xi(n)}\theta(k) \tag{3.2}$$

(ii) *The summability condition*

$$C \quad := \quad \sum_{n=0}^{\infty}\prod_{i=0}^{n-1}\frac{\lambda^{(i)}}{\mu^{(i+1)}} < \infty \tag{3.3}$$

holds, and the equation

$$\theta\cdot\tilde{Q}^{(0)} = 0 \tag{3.4}$$

admits a stochastic solution $\theta = (\theta(k) : k \in K)$ *which solves also*

$$\forall n \in \mathbb{N} : \theta\cdot\tilde{Q}^{(n)} = 0\,. \tag{3.5}$$

(c) *If* $(\hat{X},\hat{Y})$ *is distributed according to the (1-dimensional) stationary distribution* π *of* $(X(t),Y(t))$ *(any* t*), then*

$$\xi(n) := P(\hat{X}=n) \quad \theta(k) := P(\hat{Y}=k), \quad \pi(n,k) : = P(\hat{X}=n,\hat{Y}=k) = \xi(n)\cdot\theta(k)\,,$$

Before proving the theorem some remarks seem to be in order.

Remark 3.

$$\xi = (\xi(n) := C^{-1}\prod_{i=0}^{n-1}\frac{\lambda^{(i)}}{\mu^{(i+1)}} : n \in \mathbb{N}_0) \tag{3.6}$$

is the steady state distribution of an ergodic birth-death process with birth rates $\lambda^{(n)}$ and death rates $\mu^{(n)}$. But $X = (X(t) : t \geq 0)$ is in general not a birth-death process because in general it is not even Markov.

The observation (3.6) is remarkable not only because X is is not a birth-death process, but also because neither the $\lambda^{(i)}$ are the effective arrival rates (expected number of arrivals per time unit) for queue length i nor the $\mu^{(i+1)}$ the effective service rates (expected maximal number of departures per time unit) for queue length $i+1$. In case of a pure birth-death process without an environment $\lambda^{(i)}, \mu^{(i+1)}$ **are** the respective rates.

The conclusion is that both rates are diminished by the influence of the environment by the same portion. It seems to be contra intuition to us that the reduction of $\lambda^{(i)}$ goes in parallel to that of $\mu^{(i+1)}$, while in the running system under queue length i due to Y entering K_B arrivals at rate $\lambda^{(i)}$ are interrupted in parallel to services of rate $\mu^{(i)}$.

The similar problem was noticed already for the case of queueing-inventory processes with state independent service and arrival rates in Remark 2.8 in [SSD+06], but in this setting clearly the problem of $\lambda^{(i)}$ versus $\mu^{(i+1)}$ is still hidden.

Remark 4. The statement of the theorem can be interpreted as a strong **insensitivity property** of the system: As long as ergodicity is maintained, the environment can change drastically without changing the **steady state of the queue length at any fixed time point**. An intuitive interpretation of this result seems to be hard. Especially, this insensitivity can not be a consequence of the form of the control of the inventory or the availability.

We believe that there is intuitive explanation of a part of the result. The main observation with respect to this is:

Whenever a customer is admitted to the queue, i.e. not lost, he observes the service process as that in a conventional $M/M/1/\infty$ queue with state dependent service and arrival rates, as long as the blocking periods are skipped over.

Saying it the other way round, whenever the environment enters K_B and blocks the service process, the arrival process is blocked as well, i.e. the queueing system is completely frozen and is revived immediately when the environment enters K_W next.

Skipping the problem of i versus $i+1$ discussed in Remark 3 this observation might explain the form of the marginal stationary distribution of the customer process X, but it does by no means explain the independence of the marginal variables of $(\hat{X}, \hat{Y})$, i.e. the product form structure.

A similar observation was utilized in [SAH10] in a queueing-inventory system (with state independent service and arrival rates) to construct a related system which obviously has the stationary distribution of $\hat{X}$ and it is argued that from this follows that the original system shows the same marginal queue length distribution.

Remark 5. The proven insensitivity does not mean, that the time development of the queue length processes with fixed $\lambda^{(n)}, \mu^{(n)}$ is the same under different environment behavior. This can be seen by considering the stationary sojourn time of admitted customers, which is strongly dependent of the interruption time distributions (= sojourn time distribution of Y in K_B).

Similarly, multidimensional stationary probabilities for $(X(t_1), X(t_2), \ldots, X(t_n))$ will clearly depend on the occurrence frequency of the event $(Y \in K_B)$.

Proof. of Theorem 2 **(a)** We have $\tilde{q}_{kj}^{(n)} \geq 0 \ \forall k \neq j$ and $\tilde{q}_{kk}^{(n)} \leq 0$, and, furthermore, it holds (for all $k \in K$) :

$$\begin{aligned}\sum_{m \in K} \tilde{q}_{k,m}^{(n)} &= -(1_{[k \in K_W]} \lambda^{(n)} (1 - R_{kk}) + \sum_{m \in K \setminus \{k\}} \nu(k,m)) \\ &\quad + \underbrace{\left(\lambda^{(n)} \sum_{m \in K \setminus \{k\}} R_{km} \right) 1_{[k \in K_W]}}_{= \lambda^{(n)} (1 - R_{kk}) 1_{[k \in K_W]}} + \sum_{m \in K \setminus \{k\}} \nu(k,m) = 0 \,.\end{aligned}$$

which shows that the $\tilde{Q}^{(n)}$ are generator matrices.

(b) (ii) ⇒ (i):
The global balance equations of the Markov process (X, Y) are for $(n, k) \in E$

$$\begin{aligned} & \pi(n,k)\left(1_{[k\in K_W]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}} \nu(k,m) + 1_{[k\in K_W]}1_{[n\geq 1]}\mu^{(n)}\right) \\ = \; & \pi(n-1,k)1_{[k\in K_W]}1_{[n\geq 1]}\lambda^{(n-1)} + \sum_{m\in K_W} \pi(n+1,m)R_{mk}\mu^{(n+1)} \\ & + \sum_{m\in K\setminus\{k\}} \pi(n,m)\nu(m,k) \end{aligned} \tag{3.7}$$

Inserting the proposed product form solution (3.2) for $\pi(n, k)$ into the global balance (3.7) equations, canceling C^{-1} yields

$$\begin{aligned} & \theta(k)\prod_{i=0}^{n-1}\frac{\lambda^{(i)}}{\mu^{(i+1)}}\left(1_{[k\in K_W]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}} \nu(k,m) + 1_{[k\in K_W]}1_{[n\geq 1]}\mu^{(n)}\right) \\ = \; & \theta(k)\prod_{i=0}^{n-2}\frac{\lambda^{(i)}}{\mu^{(i+1)}}1_{[k\in K_W]}1_{[n\geq 1]}\lambda^{(n-1)} + \sum_{m\in K_W}\theta(m)\prod_{i=0}^{n}\frac{\lambda^{(i)}}{\mu^{(i+1)}}R_{mk}\mu^{(n+1)} \\ + & \sum_{m\in K\setminus\{k\}}\theta(m)\prod_{i=0}^{n-1}\frac{\lambda^{(i)}}{\mu^{(i+1)}}\nu(m,k)\,, \end{aligned} \tag{3.8}$$

and multiplication with $\prod_{i=0}^{n-1}\left(\frac{\lambda^{(i)}}{\mu^{(i+1)}}\right)^{-1}$ yields

$$\begin{aligned} & \theta(k)\left(1_{[k\in K_W]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}} \nu(k,m) + 1_{[k\in K_W]}1_{[n\geq 1]}\mu^{(n)}\right) \\ = \; & \theta(k)\frac{\mu^{(n)}}{\lambda^{(n-1)}}1_{[k\in K_W]}1_{[n\geq 1]}\lambda^{(n-1)} + \sum_{m\in K_W}\theta(m)\frac{\lambda^{(n)}}{\mu^{(n+1)}}R_{mk}\mu^{(n+1)} \\ & + \sum_{m\in K\setminus\{k\}}\theta(m)\nu(m,k) \end{aligned}$$

$$\begin{aligned} \iff \quad & \theta(k)\left(1_{[k\in K_W]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}} \nu(k,m) + 1_{[k\in K_W]}1_{[n\geq 1]}\mu^{(n)}\right) \\ = \; & \theta(k)\mu^{(n)}1_{[k\in K_W]}1_{[n\geq 1]} + \sum_{m\in K_W}\theta(m)\lambda^{(n)}R_{mk} \\ & + \sum_{m\in K\setminus\{k\}}\theta(m)\nu(m,k) \end{aligned}$$

$$\iff 0 = -\theta(k)\left(1_{[k\in K_W]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}} \nu(k,m)\right)$$
$$+ \sum_{m\in K_W} \theta(m)\lambda^{(n)}R_{mk} + \sum_{m\in K\setminus\{k\}} \theta(m)\nu(m,k)$$

$$\iff 0 = \underbrace{\theta(k)\left\{-\left(1_{[k\in K_W]}\lambda^{(n)}(1-R_{kk}) + \sum_{m\in K\setminus\{k\}} \nu(k,m)\right)\right\}}_{=:\tilde{q}^{(n)}_{kk}} \tag{3.9}$$
$$+ \sum_{m\in K_W\setminus\{k\}} \theta(m)\underbrace{\left(\lambda^{(n)}R_{mk}1_{[m\in K_W]} + \nu(m,k)\right)}_{=:\tilde{q}^{(n)}_{mk}} + \sum_{m\in K_B\setminus\{k\}} \theta(m)\underbrace{\nu(m,k)}_{=:\tilde{q}^{(n)}_{mk}}$$

By definition this is (for all $n \in \mathbb{N}_0$) the condition (3.4) and (3.5).

By assumption (3.4) there exists a stochastic solution to $\theta^{(0)}\cdot\tilde{Q}^{(0)} = 0$, which according to requirement (3.5) is a solution of $\theta^{(n)}\cdot\tilde{Q}^{(n)} = 0$ with $\sum_{k\in K}\theta^{(n)}(k) = 1$ and $\theta^{(n)}(k) \geq 0$ as well.

Setting $\theta^{(0)} =: \theta$ in (3.8) provides a solution of the global balance equations (3.7). Therefore, the steady state equations of (X, Y) admit a stochastic solution, and so (X, Y) is ergodic and we have identified the unique stochastic solution of (3.7).

(b) (i) ⇒ (ii):
Because π is stochastic, summability (3.3) holds. Insert the stochastic vector of product form (3.2) into (3.7). As shown in the part **(ii) ⇒ (i)** of the proof this leads to (3.9) and we have found a solution of (3.4) which solves (3.5) for all $n \in \mathbb{N}$ as well. □

Corollary 6. *If in the framework of Theorem 2 the arrival stream is a Poisson-λ stream (which is interrupted when the environment process stays in K_B) then the stationary distribution in case of ergodic (X, Y) is of product form*

$$\pi(n,k) = C^{-1}\frac{\lambda^n}{\prod_{i=0}^{n-1}\mu^{(i+1)}}\theta(k) \quad (n,k)\in E, \tag{3.10}$$

with normalization constant C.

Proof. Because of $\lambda^{(n)} = \lambda$ for all n holds $Q^{(0)} = Q^{(n)}$ and the condition (3.5) is trivially valid. □

Corollary 7. *If in the framework of Theorem 2 the environment state space K is finite, then the equations (3.4) and (3.5) always admit stochastic solutions, and the stationary distribution of (X, Y) is of product form*

$$\pi(n,k) = C^{-1}\prod_{i=0}^{n-1}\frac{\lambda^{(i)}}{\mu^{(i+1)}}\theta(k)\,, \tag{3.11}$$

whenever (3.4) *and* (3.5) *have a common solution.*

Remark 8. The proof of Theorem 2 reveals that the solution of the equation $\theta \cdot \tilde{Q}^{(0)} = 0$ (see (3.4)) does not depend on the values $\mu^{(n)}$. So, changing the service capacity of the queueing system will not change the the steady state of the environment, as long as the system remains stable (ergodic).

The next examples comment on different forms of establishing product form equilibrium which may arise in the realm of Theorem 2.

Example 9. There exist non trivial loss systems with non constant (i.e., state dependent) arrival rates $\lambda^{(n)}$ in a random environment which have a product form steady state distribution. This is verified by the following example. We have environment

$$K = \{1, 2\} = K_W ,$$

and for some $c, d \in (0, 1)$ the routing matrix

$$R = \begin{pmatrix} 1-c & c \\ d & 1-d \end{pmatrix} ,$$

whereas Υ is the matrix of only zeros. It follows

$$\tilde{Q}^{(n)} = \begin{pmatrix} -\lambda^{(n)}c & \lambda^{(n)}c \\ \lambda^{(n)}d & -\lambda^{(n)}d \end{pmatrix} = \lambda^{(n)} \begin{pmatrix} -c & c \\ d & -d \end{pmatrix}$$

It follows that $Q^{(n+1)} = c^{(n+1)} Q^{(n)}$ holds for suitable $c^{(n+1)}, n \in \mathbb{N}_0$, which immediately shows that (3.4) and (3.5) have a common solution, which is

$$\theta = \left(\frac{d}{d+c}, \frac{c}{d+c} \right) .$$

Example 10. There exist non trivial ergodic loss systems in a random environment which have a product form steady state distribution if and only if the arrival rates $\lambda^{(n)} = \lambda$ are independent of the queue lengths. This is verified by the following example (which describes a queueing-inventory system under (r, S) policy with $(r = 1, S = 2)$, as will be seen in Section 5.1, Definition 18). We have an environment

$$K = \{0, 1, 2\}, \quad \text{with blocking set} \quad K_B = \{0\} ,$$

"routing matrix" R and and the generator matrix Υ given as

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \qquad \Upsilon = \begin{pmatrix} -\nu & 0 & \nu \\ 0 & -\nu & \nu \\ 0 & 0 & 0 \end{pmatrix} .$$

It follows

$$\tilde{Q}^{(n)} = \begin{pmatrix} -\nu & 0 & \nu \\ \lambda^{(n)} & -(\lambda^{(n)} + \nu) & \nu \\ 0 & \lambda^{(n)} & -\lambda^{(n)} \end{pmatrix}$$

Clearly, if $\lambda^{(n)} = \lambda$ are equal, the equations

$$\theta \cdot \tilde{Q}^{(n)} = 0, \quad n \in \mathbb{N}_0\,, \tag{3.12}$$

have a common stochastic solution.

On the other hand, the solutions of (3.12) are

$$\theta^{(n)} = (\theta^{(n)}(0), \theta^{(n)}(1), \theta^{(n)}(2)) = C^{(n)-1}(\frac{\lambda^{(n)}}{\nu}, 1, \frac{\lambda^{(n)} + \nu}{\lambda^{(n)}})\,, \quad n \in \mathbb{N}_0\,. \tag{3.13}$$

We conclude

$$\forall n \in \mathbb{N}_0 : \theta^{(n)} = \theta^{(n+1)} \Longrightarrow \frac{\theta^{(n)}(0)}{\theta^{(n)}(1)} = \frac{\theta^{(n+1)}(0)}{\theta^{(n+1)}(1)} \Longleftrightarrow \lambda^{(n)} = \lambda^{(n+1)}\,.$$

Remark 11. In Section 4 we will show in the course of proving a companion of Theorem 2 for loss systems with finite waiting room that more restrictive conditions on the environment are needed. It turns out that the construction in the proof of the Theorem 12 will provide us with more general constructions for examples as those given here, see Remark 16 below.

4 Finite capacity loss systems

In this section we study the systems from Section 2 under the additional restriction that the capacity of the waiting room is finite. That is, we now consider loss systems in the traditional sense with the additional feature of losses due to the environment restrictions on customers' admission.

Recall, that for the pure exponential single server queueing system with state dependent rates and $N \geq 0$ waiting places the state space is $E = \{0, 1, \ldots, N, N+1\}$ and the queueing process X is ergodic with stationary distribution $\pi = (\pi(n) : n \in E$ of the form

$$\pi(n) = C^{-1} \prod_{i=0}^{n-1} \frac{\lambda^{(i)}}{\mu^{(i+1)}}\,, \quad n \in E\,. \tag{4.1}$$

If the queueing system with infinite waiting room and the same rates $\lambda^{(i)}, \mu^{(i)}$ is ergodic, the stationary distribution π in (4.1) is simply obtained by conditioning of the stationary distribution of this infinite system on E. (Note, that ergodicity in the finite waiting room case is granted by free, without referring to the infinite system.)

We will show, that a similar construction by conditioning is in general not possible for the loss system in a random environment. The structure of the environment process will play a crucial role for enabling such a conditioning procedure.

We take the interaction between the queue length process X and the environment process Y of the same form as in Section 2, with R and Υ of the same form, and

$\lambda^{(i)} > 0$ for $i = 0, \ldots, N$, and $\mu^{(i)} > 0$ for $i = 1, \ldots, N+1$. The state space is $E := \{0, \ldots, N+1\} \times K$. The non negative transition rates of (X, Y) are for $(n, k) \in E$

$$\begin{aligned} q((n,k) \to (n+1,k)) &= \lambda^{(n)} \qquad k \in K_W, n < N+1 \\ q((n,k) \to (n-1,m)) &= \mu^{(n)} R_{km} \qquad k \in K_W, n \geq 1 \\ q((n,k) \to (n,m)) &= \nu(k,m) \in \mathbb{R}_0^+, \quad k \neq m \\ q((n,k) \to (i,m)) &= 0 \qquad \text{otherwise for } (n,k) \neq (i,m) \in E \end{aligned}$$

The first step of the investigation is nevertheless completely parallel to Theorem 2.

Theorem 12. **(a)** *Denote for* $n \in n \in \{0, \ldots, N+1\}$

$$\begin{aligned} \tilde{q}_{kk}^{(n)} &= -(1_{[k \in K_W]} \cdot 1_{[n \in \{0,\ldots,N\}]} \lambda^{(n)} (1 - R_{kk}) + \sum_{m \in K \setminus \{k\}} \nu(k,m)) \\ \tilde{q}_{km}^{(n)} &= \lambda^{(n)} R_{km} 1_{[k \in K_W]} \cdot 1_{[n \in \{0,\ldots,N\}]} + \nu(k,m) \qquad k \neq m \end{aligned} \tag{4.2}$$

and

$$\tilde{Q}^{(n)} = (\tilde{q}_{km}^{(n)} : k, m \in K).$$

Then the matrices $\tilde{Q}^{(n)}$ *are generator matrices for some homogeneous Markov processes.*

(b) *If the process* (X, Y) *is ergodic denote its unique steady state distribution by*

$$\pi = (\pi(n,k) : (n,k) \in E := \{0, \ldots, N+1\} \times K).$$

Then the following three properties are equivalent:

(i) (X, Y) *is ergodic on* E *with product form steady state*

$$\pi(n,k) = C^{-1} \prod_{i=0}^{n-1} \frac{\lambda^{(i)}}{\mu^{(i+1)}} \theta(k) \quad n \in \{0, \ldots, N+1\}, k \in K \tag{4.3}$$

(ii) *The equation*

$$\theta \cdot \tilde{Q}^{(0)} = 0 \tag{4.4}$$

admits a strict positive stochastic solution $\theta = (\theta(k) : k \in K)$ *which solves also*

$$\forall n \in \{0, \ldots, N+1\} : \theta \cdot \tilde{Q}^{(n)} = 0. \tag{4.5}$$

(iii) *The equation*

$$\eta \cdot \Upsilon = 0 \tag{4.6}$$

admits a strict positive stochastic solution.

The set $K_W \subseteq K$ *is a closed set for the Markov chain on state space* K *with transition matrix* R, *i.e.,*

$$\forall k \in K_W : \quad \sum_{m \in K_W} R_{km} = 1,$$

and the restriction $\eta^{(W)} := (\eta(m) : m \in K_W)$ *of* η *to* K_W *solves the equation*

$$\eta^{(W)} = \eta^{(W)} \cdot R^{(W)}, \tag{4.7}$$

where

$$R^{(W)} := (R_{km} : k, m \in K_W) \tag{4.8}$$

is the restriction of R *to* K_W*.*

Proof. The proof of **(a)** is similar to that of Theorem 2**(a)**, and in **(b)** the equivalence of **(i)** and **(ii)** is proven in almost identical way as that of Theorem 2 **(b)** (with the obvious slight changes due to having the X-component finite) and are therefore omitted.

We next show
(b) (i) ⇒ (iii):
The global balance equations of the Markov process (X, Y) are for $(n, k) \in E$

$$\begin{aligned} &\pi(n,k)\left(1_{[k\in K_W]}\cdot 1_{[n\in\{0,\dots,N\}]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}}\nu(k,m) + 1_{[k\in K_W]}1_{[n\geq 1]}\mu^{(n)}\right) \\ = \; &\pi(n-1,k)1_{[k\in K_W]}1_{[n\geq 1]}\lambda^{(n-1)} + \sum_{m\in K_W}\pi(n+1,m)R_{mk}\mu^{(n+1)}\cdot 1_{[n\in\{0,\dots,N\}]} \\ &+ \sum_{m\in K\setminus\{k\}}\pi(n,m)\nu(m,k) \end{aligned} \tag{4.9}$$

Inserting the proposed product form solution (4.3) for $\pi(n, k)$ into the global balance equations (4.9) and proceding in the same way as in the proof of Theorem 2 yields

$$\begin{aligned} 0 \;=\; &-\theta(k)\left(1_{[k\in K_W]}\cdot 1_{[n\in\{0,\dots,N\}]}\lambda^{(n)} + \sum_{m\in K\setminus\{k\}}\nu(k,m)\right) \\ &+ \sum_{m\in K_W}\theta(m)\lambda^{(n)}R_{mk}\cdot 1_{[n\in\{0,\dots,N\}]} + \sum_{m\in K\setminus\{k\}}\theta(m)\nu(m,k) \end{aligned} \tag{4.10}$$

For $n \to N + 1$ (4.10) turns to

$$\theta(k)\left(\sum_{m\in K\setminus\{k\}}\nu(k,m)\right) = \sum_{m\in K\setminus\{k\}}\theta(m)\nu(m,k)\,, \tag{4.11}$$

which verifies (4.6) with $\eta := \theta$.
For $n < N + 1$ (4.10) turns to

$$\theta(k)1_{[k\in K_W]}\lambda^{(n)} + \underbrace{\theta(k)\sum_{m\in K\setminus\{k\}}\nu(k,m)}_{(*)} = \sum_{m\in K_W}\theta(m)\lambda^{(n)}R_{mk} + \underbrace{\sum_{m\in K\setminus\{k\}}\theta(m)\nu(m,k)}_{(**)}\,,$$

where from (4.11) the expressions $(**)$ and $(*)$ cancel and we arrive at

$$\theta(k)1_{[k\in K_W]}\lambda^{(n)} \quad = \sum_{m\in K_W}\theta(m)\lambda^{(n)}R_{mk}\,. \tag{4.12}$$

Because (X,Y) is ergodic, θ is strict positive, and we conclude (set $k\in K_B$ in (4.12) which makes the left side zero)

$$R_{mk}=0, \quad \forall m\in K_W,\ k\in K_B\,,$$

which shows that K_W is a closed set for the Markov chain governed by R.

Now set $k\in K_W$ in (4.12) which makes the left side strict positive and realize that this after cancelling $\lambda^{(n)}$ is exactly (4.7).
This part of the proof is finished.

(b) (iii) $\Rightarrow$ (ii):

For proving the reversed direction we reconsider the previous part **(i)** $\Rightarrow$ **(iii)** of the proof: The strict positive stochastic solution of

$$\eta\cdot\Upsilon=0\,, \tag{4.13}$$

which is given by assumption (4.6), yields the required solution for $n\to N+1$ of

$$\theta\cdot\tilde{Q}^{(N+1)}=0\,.$$

If $K_W\subseteq K$ is a closed set for the Markov chain on state space K with transition matrix R we obtain

$$R_{mk}=0, \quad \forall m\in K_W,\ k\in K_B\,,$$

and therefore for all $n\in\{0,1,\ldots,N\}$

$$\theta\cdot\tilde{Q}^{(n)}=0$$

reduces for $k\in K_B$ to the respective expression in

$$\eta\cdot\Upsilon=0\,.$$

It remains for all $n\in\{0,1,\ldots,N\}$ and for $k\in K_W$ to show that for $k\in K_W$ the respective expression in

$$\theta\cdot\tilde{Q}^{(n)}=0$$

is valid. This follows by considering

$$\eta(k)1_{[k\in K_W]}\lambda^{(n)}+\underbrace{\eta(k)\sum_{m\in K\setminus\{k\}}\nu(k,m)}_{(*)}=\sum_{m\in K_W}\eta(m)\lambda^{(n)}R_{mk}+\underbrace{\sum_{m\in K\setminus\{k\}}\eta(m)\nu(m,k)}_{(**)}\,,$$

and remembering that the expressions $(**)$ and $(*)$ cancel. The residual terms are equal by the assumption (4.7).
This finishes the proof. $\square$

The interesting insight is that from the existence of the product form steady state π on $E = \{0, \ldots, N+1\} \times K$ implicitly restrictions on the form of the movements of the environment emerge which are not necessary in the case of infinite waiting rooms. (As indicated above, such restrictions are not necessary too in the pure queueing system framework.)

The proof of Theorem 12 has brought out the following additional, somewhat surprising, insensitivity property.

Corollary 13. *Whenever (X, Y) is ergodic with product form steady state*

$$\pi(n, k) = C^{-1} \prod_{i=0}^{n-1} \frac{\lambda^{(i)}}{\mu^{(i+1)}} \theta(k) \quad n \in \{0, \ldots, N+1\}, k \in K$$

for some (positive) parameter setting $(\lambda^{(i)} : i = 0, 1, \ldots, N), (\mu^{(i)} : i = 1, \ldots, N+1)$ with an environment (K, K_B, Υ, R), then for this same environment (X, Y) is ergodic with product form steady state with the same θ for any (positive) parameter setting for the arrival and service rates.

Proof. This becomes obvious at the step where we arrived at (4.12) and we see that the specific shape of the sequence of the $\lambda^{(i)}$ do not matter. The specific $\mu^{(i)}$ are canceled in the early steps of the proof already. □

Example 14. We describe a class of examples of environments which guarantee that the conditions of Theorem 12 are fulfilled. The construction is in three steps.

1. Take for Υ a generator of an irreducible Markov process on K with stationary distribution θ, which fulfills for all $k \in K_W$ the *partial balance condition*

$$\theta(k) \sum_{m \in K_W} \nu(k, m) = \sum_{m \in K_W} \theta(m) \nu(m, k) \tag{4.14}$$

   and $\sup(-\nu(k, k) : k \in K_W) < \infty$.

2. Denote by $\Upsilon^{(W)}$ the restriction of Υ onto K_W which has stationary distribution $\theta^{(W)} := (\theta(k)/(\sum_{m \in K_W} \theta(m)) : k \in K_W)$, see [Kel79][Exercise 1.6.2, p. 27].

3. Take for $R^{(W)}$ (see (4.8)) a uniformization chain of $\Upsilon^{(W)}$, see [Kei79][Chapter 2, Section 2.1], e.g. (with I the identity matrix on K_W)

$$R^{(W)} := I + \sup(-\nu(k, k) : k \in K_W)^{-1} \Upsilon^{(W)},$$

   which is stochastic and has equilibrium distribution $\theta^{(W)} := (\theta(k)/(\sum_{m \in K_W} \theta(m)) : k \in K_W)$ as well.
   $(R_{k,m} : k \in K_B, m \in K)$ can be arbitrarily selected, e.g. the identity matrix on K_B.

This construction ensures that the restriction $\eta^{(W)} := (\eta(m) : m \in K_W)$ of η to K_W solves the equation (4.7)

$$\eta^{(W)} = \eta^{(W)} \cdot R^{(W)}.$$

Remark 15. The construction in Example 14 may seem to produce a narrow class of examples, but this is not so: All reversible Υ fulfill the partial balance condition (4.14).

Remark 16. The construction above produces another example contributing to the discussion at the end of Section 3 on the question which particular product forms can occur, and which form of the environment and the arrival and service rate patterns may interact to result in product form equilibrium for loss systems with infinite waiting room. We only have to notice that the equations for $n < N+1$ are exactly those which occur for all $n \in \mathbb{N}_0$ in the setting of Theorem 2.

The cautious reader will already have noticed that the conditions in **(b)(iii)** of Theorem 12 provide a similar more abstract example for the discussion on Theorem 2 at the end of Section 3.

Remark 17. We should point out that in [SSD+06][Section 6] queueing-inventory models with finite waiting room are investigated with a resulting "quasi product form" steady state. The respective theorems there do not fit into the realm of Theorem 12 because the state space is **not** a product space as in Theorem 12, where we have irreducibility on $E = \{0, 1, \ldots, N, N+1\}$.

The difference is that in [SSD+06][Section 6] the element (in notation of the present paper) $(N+1, 0)$ is not a feasible state.

The results there can be considered as a truncation property of the equilibrium of the system with infinite waiting room onto the feasible state space under restriction to finite queues.

5 Applications

5.1 Inventory models

In the following we describe an M/M/1/∞-system with inventory management as it is investigated in [SSD+06].

Definition 18. *An M/M/1/∞-system with inventory management is a single server with infinite waiting room under FCFS regime and an attached inventory.*

There is a Poisson-λ-arrival stream, $\lambda \geq 0$. Customers request for an amount of service time which is exponentially distributed with mean 1. Service is provided with intensity $\mu > 0$.

The server needs for each customer exactly one item from the attached inventory. The on-hand inventory decreases by one at the moment of service completion. If the inventory is decreased to the reorder point $r \geq 0$ after the service of a customer is completed, a replenishment order is instantaneously triggered. The replenishment lead times are i.i.d. with distribution function $B = (B(t); t \geq 0)$. The size of the replenishment depends on the policy applied to the system. We consider two standard policies from inventory *management, which lead to an $M/M/1/\infty$-system with either (r,Q)-policy (size of the replenishment order is always $Q > r$) or with (r,S)-policy (replenishment fills the inventory up to maximal inventory size $S > r$).*

During the time the inventory is depleted and the server waits for a replenishment order to arrive, no customers are admitted to join the queue ("lost sales").
All service, interarrival and lead times are assumed to be independent.
Let $X(t)$ denote the number of customers present at the server at time $t \geq 0$, either waiting or in service (queue length) and let $Y(t)$ denote the on-hand inventory at time $t \geq 0$. Then $((X(t),Y(t)),t \geq 0)$, the queueing-inventory process is a continuous-time Markov process for the M/M/1/∞-system with inventory management. The state space of (X,Y) is $E = \{(n,k) : n \in \mathbb{N}_0, k \in K\}$, where $K = \mathbb{N}_0$ or $K = \{0,1,\ldots,\kappa\}$, where $\kappa < \infty$ is the maximal size of the inventory at hand.

The system described above generalizes the lost sales case of classical inventory management where customer demand is not backordered but lost in case there is no inventory on hand (see Tersine [Ter94] p. 207).

The general Theorem 2 produces as special applications the following results on product form steady states in integrated queueing inventory systems from [SSD+06].

Example 19. [SSD+06] $M/M/1/\infty$ system with (r,S)-policy, $\exp(\nu)$-distributed lead times, and lost sales. The inventory management process under (r,S)-policy fits into the definition of the environment process by setting

$$K = \{0,1,...,S\}, \qquad K_B = \{0\},$$

$$R_{0,0} = 1, \quad R_{k,k-1} = 1, \quad 1 \leq k \leq S\,, \qquad \nu(k,m) = \begin{cases} \nu, & \text{if } \; 0 \leq k \leq r, m = S \\ 0, & \text{otherwise for } \; k \neq m\,. \end{cases}$$

The queueing-inventory process (X,Y) is ergodic iff $\lambda < \mu$. The steady state distribution $\pi = (\pi(n,k) : (n,k) \in E)$ of (X,Y) has product form

$$\pi(n,k) = (1 - \frac{\lambda}{\mu})(\frac{\lambda}{\nu})^n \theta(k),$$

where $\theta = (\theta(k) : k \in K)$ with normalization constant C is

$$\theta(k) \;\; = \begin{cases} C^{-1}(\frac{\lambda}{\nu}) & k = 0, \\ C^{-1}(\frac{\lambda+\nu}{\lambda})^{k-1} & k = 1,...,r, \\ C^{-1}(\frac{\lambda+\nu}{\lambda})^{r} & k = r+1,...,S. \end{cases} \tag{5.1}$$

Example 20. [SSD+06] $M/M/1/\infty$ system with (r,Q)-policy, $\exp(\nu)$-distributed lead times, and lost sales. The inventory management process under (r,Q)-policy fits into the definition of the environment process by setting

$$K = \{0, 1, ..., r + Q\} \qquad K_B = \{0\}$$

$$R_{0,0} = 1, \quad R_{k,k-1} = 1, \quad 1 \le k \le S \qquad \nu(k, m) = \begin{cases} \nu, & \text{if } \ 0 \le k \le r, m = k + Q \\ 0, & \text{otherwise for } \ k \neq m. \end{cases}$$

The queueing-inventory process (X, Y) is ergodic iff $\lambda < \mu$. The steady state distribution $\pi = (\pi(n, k) : (n, k) \in E$ of (X, Y) has product form

$$\pi(n, k) = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\nu}\right)^n \theta(k),$$

where $\theta = (\theta(k) : k \in K)$ with normalization constant C is

$$\begin{aligned}
\theta(0) &= C^{-1}\frac{\lambda}{\nu}, \\
\theta(k) &= C^{-1}\left(\frac{\lambda + \nu}{\lambda}\right)^{k-1}, \qquad k = 1, ..., r, \\
\theta(k) &= C^{-1}\left(\frac{\lambda + \nu}{\lambda}\right)^{r}, \qquad k = r + 1, ..., Q, \\
\theta(k + Q) &= C^{-1}\left(\frac{\lambda + \nu}{\lambda}\right)^{r} - \left(\frac{\lambda + \nu}{\lambda}\right)^{k-1}, \qquad 1 = 1, ..., r.
\end{aligned}$$

Example 21. This example is taken from [KN12], the notation is adapted to that used in Section 2:

The authors study an inventory system under (r, S)-policy, which provides items for a server who processes and forwards the items in an on-demand production scheme. The processing time of each service is exponentially-μ distributed. The demand arrives in a Poisson-λ stream.

If demand arrives when the inventory is depleted it is rejected and lost to the system forever (lost sales).

The model is a supply chain where new items are added to the inventory through a second production process which is interrupted whenever the inventory at hand reaches S. The production process is resumed each time the inventory level goes down to r and continues to be on until inventory level reaches S again. The times required to add one item into the inventory (processing time + lead time) when the production is on, are exponential-ν random variables.

All inter arrival times, service times, and production times are mutually independent.

For a Markovian description we need to record the queue length of not fulfilled demand ($\in \mathbb{N}_0$), the inventory on stock ($\in \{0, 1, \ldots, S\}$), and a binary variable which indicates when the inventory level is in $\{r + 1, \ldots, S\}$, whether the second production process is on (=1) or off (= 0). (Note, that the second production process is always on, when the inventory level is in $\{0, 1, \ldots, r\}$, and is always off, when the inventory level is S.)

To fit this model into the framework of Section 2 we define a Markov process (X, Y) in continuous time with state space

$$E := \mathbb{N}_0 \times K, \quad \text{with } K := \{0, 1, \ldots, r\} \cup (\{S\} \cup (\{r+1, \ldots, S-1\} \times \{0, 1\}) \quad \text{and } K_B = \{0\}.$$

The environment therefore records the inventory size and the status of the second production process, and blocking of the production system occurs due to stock out with lost sales regime.

Starting from Example 20, Saffari, Haji, and Hassanzadeh [SHH11] proved that under (r, Q) policy the integrated queueing-inventory $M/M/1/\infty$ system with hyper-exponential lead times (= mixtures of exponential distributions) has a product-form distribution. The proof is done by solving directly the steady state equations. In [SAH10], Saffari, Asmussen, and Haji generalized this result to general lead time distributions. The proof of product form uses some intuitive arguments from related simplified systems and the marginal probabilities for the inventory position are derived using regenerative arguments.

In the following example we show that our models encompasses queueing-inventory systems with general replenishment lead times under (r, S) policy. This will allow us directly to conclude that for the ergodic system the steady state has product form and this will enable us to generalize the theorem (here Example 19) of [SSD+06] to incorporate generally distributed lead times.

In a second step we will show that the results of Saffari, Haji, and Hassanzadeh [SHH11] and of Saffari, Asmussen, and Haji [SAH10] for queueing-inventory systems under (r, Q) policy can be obtained by our method as well and can even be slightly generalized.

We will consider lead time distributions of the following phase-type which are sufficient versatile to approximate any distribution on $\mathbb{R}_+$ arbitrary close.

Definition 22 (Phase–type distributions)**.** *For $k \in \mathbb{N}$ and $\beta > 0$ let*

$$\Gamma_{\beta,k}(s) = 1 - e^{-\beta s} \sum_{i=0}^{k-1} \frac{(\beta s)^i}{i!}, \quad s \geq 0,$$

denote the cumulative distribution function of the Γ–distribution with parameters β and k. k is a positive integer and serves as a phase-parameter for the number of independent exponential phases, each with mean β^{-1}, the sum of which constitutes a random variable with distribution $\Gamma_{\beta,k}$. ($\Gamma_{\beta,k}$ is called a k–stage Erlang distribution with shape parameter β.)

We consider the following class of distributions on $\mathbb{R}_+$, which is dense with respect to the topology of weak convergence of probability measures in the set of all distributions on $(\mathbb{R}_+, \mathbb{B}_+)$ ([Sch73], section I.6). For $\beta \in (0, \infty)$, $L \in \mathbb{N}$, and probability b on $\{1, \ldots, L\}$ with $b(L) > 0$ let the cumulative distribution function

$$B(s) = \sum_{\ell=1}^{L} b(\ell) \Gamma_{\beta,\ell}(s), \quad s \geq 0, \tag{5.2}$$

denote a phase-type distribution function. With varying β, L, and b we can approximate any distribution on $(\mathbb{R}_+, \mathbb{B}_+)$ sufficiently close.

To incorporate replenishment lead time distributions of phase-type we apply the supplemented variable technique. This leads to enlarging the phase space of the system, i.e. the state space of the inventory process Y. Whenever there is an ongoing lead time, i.e., when inventory at hand is less than $r+1$ we count the number of residual successive i.i.d. $\exp(\beta)$-distributed lead time phases which must expire until the replenishment arrives at the inventory.

The state space of (X, Y) then is $E = \mathbb{N}_0 \times K$ with

$$K = \{r+1, r+2, ..., S\} \cup (\{0, 1, ...r\} \times \{L, \dots, 1\}) ,$$

and (X, Y) is irreducible on E.

Theorem 23. *$M/M/1/\infty$ system with (r, S)-policy, phase-type replenishment lead time, state dependent service rates $\mu^{(n)}$, and lost sales.*

The lead time distribution has a distribution function B from (5.2). We assume that (X, Y) is positive recurrent and denote its steady state distribution by

$$\pi = (\pi(n, k) : n \in \mathbb{N}_0 \times K).$$

The steady state π of (X, Y) is of product form. With normalization constant C

$$\pi(n, k) = C^{-1} \prod_{i=0}^{n-1} \frac{\lambda}{\mu^{(i+1)}} \cdot \theta(k) \tag{5.3}$$

where $\theta = (\theta(k) : k \in K)$ is for $r > 0$

$$\begin{aligned}
\theta(j, \ell) &= G^{-1} \left(\frac{\lambda+\beta}{\lambda}\right)^{j-1} \sum_{i=\ell}^{L} b(i) \left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell} \binom{i-\ell+r-j}{r-j}, \qquad (5.4)\\
&\quad j = 1, 2, \dots, r, \ell = 1, \dots, L\\
\theta(0, \ell) &= G^{-1} \frac{\lambda}{\beta} \left[\sum_{i=\ell}^{L} \left(\sum_{g=i}^{L} b(g)\right) \left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell} \binom{i-\ell+r-1}{r-1}\right]. \qquad (5.5)\\
&\quad \ell = 1, \dots, L,\\
\theta(r+1) &= \theta(r+2) = \dots = \theta(S) = G^{-1} \left(\frac{\lambda+\nu}{\lambda}\right)^{r}, \qquad (5.6)
\end{aligned}$$

where the normalization constant G is chosen such that

$$\sum_{k \in K} \theta(k) = 1.$$

For $r = 0$ we obtain $\theta = (\theta(k) : k \in K)$ with normalization constant G as

$$\begin{aligned}
\theta(0, \ell) &= G^{-1} \frac{\lambda}{\beta} \left[\sum_{i=\ell}^{L} b(i)\right], \quad \ell = 1, \dots, L, \qquad (5.7)\\
\theta(1) &= \theta(2) = \dots = \theta(S) = G^{-1}, \qquad (5.8)
\end{aligned}$$

Proof. The inventory management process under (r,S)-policy with distribution function B of the lead times fits into the definition of the environment process by setting

$$K = \{r+1, r+2, ..., S\} \cup (\{0, 1, ...r\} \times \{L, \dots, 1\}), \quad K_B = \{0\} \times \{L, \dots, 1\}.$$

The non negative transition rates of (X,Y) are for $(n,k) \in E$

$$\begin{aligned}
q((n,k) \to (n+1,k)) &= \lambda \qquad k \in K_W, n \geq 0\\
q((n,k) \to (n-1,m)) &= \mu^{(n)} R_{k,m} \qquad k \in K_W, m \in K, n \geq 1,\\
q((n,k) \to (n,m)) &= \nu(k,m) \in \mathbb{R}_0^+, \quad k \neq m, \quad k,m \in K,\\
q((n,k) \to (i;m)) &= 0 \qquad \text{otherwise for} \quad (n,k) \neq (i;m) \in E;
\end{aligned}$$

where

$$\begin{aligned}
R_{k,k-1} = 1 &\quad \text{if} \quad k \in \{r+2, ..., S\},\\
R_{r+1,(r,\ell)} = b(\ell) &\quad \text{if} \quad \ell \in \{L, \dots, 1\},\\
R_{(j,\ell),(j-1,\ell)} = 1 &\quad \text{if} \quad (j,\ell) \in \{1, ...r\} \times \{L, \dots, 1\},\\
R_{k,j} = 0 &\quad \text{if} \quad k,j \in K, \text{ otherwise,}
\end{aligned}$$

and

$$\begin{aligned}
\nu((j,\ell),(j,\ell-1)) = \beta &\quad \text{if} \quad j \in \{0,1,...r\}, \ell \in \{L, \dots, 2\}\\
\nu((j,1),S) = \beta &\quad \text{if} \quad j \in \{0,1,...r\},\\
\nu(k,j) = 0 &\quad \text{if} \quad k,j \in K, \text{ otherwise,}
\end{aligned}$$

Because $\lambda^{(n)} = \lambda$ for all n, Theorem 2 applies and we know that the steady state of the ergodic system is of product form

$$\pi(n,k) = C^{-1} \frac{\lambda^n}{\prod_{i=0}^{n-1} \mu^{(i+1)}} \theta(k) \quad n.k) \in E, \tag{5.9}$$

according to Corollary 6. We have to solve (3.5) which is independent of n in the present setting. By definition this is (with $R_{kk} = 0, \ \forall k \in K \backslash \{0\}, \ R_{00} = 1$)

$$\begin{aligned}
&\theta(k) \left(1_{[k \in K_W]} \lambda + \sum_{m \in K \backslash \{k\}} \nu(k,m) \right) \qquad (5.10)\\
= &\sum_{m \in K_W \backslash \{k\}} \theta(m) \left(\lambda^{(n)} R_{mk} + \nu(m,k) \right) + \sum_{m \in K_B \backslash \{k\}} \theta(m) \nu(m,k)
\end{aligned}$$

(I) For $r > 0$, (5.10) translates into

$$\begin{aligned}
\theta(S) \cdot \lambda &= \sum_{j=0}^{r} \theta(j,1) \cdot \beta, & (5.11)\\
\theta(k) \cdot \lambda &= \theta(k+1) \cdot \lambda, \qquad k = r+1, \dots, S-1 & (5.12)\\
\theta(r,\ell) \cdot (\lambda+\beta) &= \theta(r+1) \cdot \lambda b(\ell) + \theta(r,\ell+1) \cdot \beta, \qquad 1 \le \ell < L, & (5.13)\\
\theta(r,L) \cdot (\lambda+\beta) &= \theta(r+1) \cdot \lambda b(L), & (5.14)\\
\theta(j,L) \cdot (\lambda+\beta) &= \theta(j+1,L) \cdot \lambda, \qquad 1 \le j < r & (5.15)\\
\theta(j,\ell) \cdot (\lambda+\beta) &= \theta(j+1,\ell) \cdot \lambda + \theta(j,\ell+1) \cdot \beta, \quad 1 \le j < r, 1 \le \ell < L, & (5.16)\\
\theta(0,\ell) \cdot \beta &= \theta(1,\ell) \cdot \lambda + \theta(0,\ell+1) \cdot \beta, \qquad 1 \le \ell < L, & (5.17)\\
\theta(0,L) \cdot \beta &= \theta(1,L) \cdot \lambda. & (5.18)
\end{aligned}$$

From (5.12) follows

$$\theta(S) = \theta(S-1) = \cdots = \theta(r+1), \tag{5.19}$$

and from (5.14) and (5.15) follows

$$\theta(j,L) = \theta(r+1) b(L) \left(\frac{\lambda}{\lambda+\beta}\right)^{r+1-j}. \tag{5.20}$$

From (5.20) (for j=r) and (5.13) follows directly

$$\theta(r,\ell) = \theta(r+1) \frac{\lambda}{\lambda+\beta} \sum_{i=\ell}^{L} b(i) \left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell}, \quad 1 \le \ell < L. \tag{5.21}$$

Up to now we obtained the expressions for the north and west border line of the array $(\theta(j,\ell) : 1 \le j \le r, 1 \le \ell \le L)$ which can be filled step by step via (5.16). The proposed solution is

$$\theta(r-h,\ell) = \theta(r+1) \left(\frac{\lambda}{\lambda+\beta}\right)^{h+1} \sum_{i=\ell}^{L} b(i) \left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell} \binom{i-\ell+h}{h}, \tag{5.22}$$

for $h = 0, 1, \dots, r-1, \ell = 1, \dots, L$ fits with (5.21) ($h = 0$ with $\binom{i-l}{0} = 1$) and (5.20). Inserting (5.22) into (5.16) verifies (5.22) by a two-step induction with help by the elementary formula $\binom{a}{n} + \binom{a}{n-1} = \binom{a+1}{n}$.

For computing the residual boundary probabilities $\theta(0,\ell)$ we need some more effort. The proposed solution is for $\ell = 1, \dots, L$,

$$\theta(0,\ell) = \theta(r+1) \left(\frac{\lambda}{\lambda+\beta}\right)^{r} \frac{\lambda}{\beta} \left[\sum_{i=\ell}^{L} \sum_{g=i}^{L} b(g) \left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell} \binom{i-\ell+r-1}{r-1}\right]. \tag{5.23}$$

From (5.18) and (5.20) we obtain

$$\theta(0,L) = \theta(r+1) \left(\frac{\lambda}{\lambda+\beta}\right)^{r} \frac{\lambda}{\beta} b(L), \tag{5.24}$$

which fits into (5.23), and it remains to check the recursion (5.17). This amounts to compute

$$
\begin{aligned}
&\theta(1,\ell)\cdot\frac{\lambda}{\beta}+\theta(0,\ell+1)\\
=\;&\theta(r+1)\left(\frac{\lambda}{\lambda+\beta}\right)^r\sum_{i=\ell}^{L}b(i)\left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell}\binom{i-\ell+r-1}{r-1}\cdot\frac{\lambda}{\beta}+\\
&+\theta(r+1)\left(\frac{\lambda}{\lambda+\beta}\right)^r\frac{\lambda}{\beta}\left[\sum_{i=\ell+1}^{L}\sum_{g=i}^{L}b(g)\left(\frac{\beta}{\lambda+\beta}\right)^{i-(\ell+1)}\binom{i-(\ell+1)+r-1}{r-1}\right]\\
=\;&\theta(r+1)\left(\frac{\lambda}{\lambda+\beta}\right)^r\frac{\lambda}{\beta}\left[\sum_{i=\ell+1}^{L}\left\{\sum_{g=i}^{L}b(g)\left(\frac{\beta}{\lambda+\beta}\right)^{i-(\ell+1)}\binom{\overbrace{i-(\ell+1)}^{=(i-1)-\ell}+r-1}{r-1}\right.\right.\\
&\left.+b(i-1)\left(\frac{\beta}{\lambda+\beta}\right)^{(i-1)-\ell}\binom{(i-1)-\ell+r-1}{r-1}\right\}+\\
&\left.+b(L)\left(\frac{\beta}{\lambda+\beta}\right)^{L-\ell}\binom{L-\ell+r-1}{r-1}\right]\\
=\;&\theta(r+1)\left(\frac{\lambda}{\lambda+\beta}\right)^r\frac{\lambda}{\beta}\left[\sum_{i=\ell+1}^{L}\left\{\sum_{g=i-1}^{L}b(g)\left(\frac{\beta}{\lambda+\beta}\right)^{(i-1)-\ell}\binom{(i-1)-\ell+r-1}{r-1}\right\}\right.\\
&\left.+b(L)\left(\frac{\beta}{\lambda+\beta}\right)^{L-\ell}\binom{L-\ell+r-1}{r-1}\right]\\
=\;&\theta(r+1)\left(\frac{\lambda}{\lambda+\beta}\right)^r\frac{\lambda}{\beta}\left[\sum_{i=\ell}^{L-1}\left\{\sum_{g=i}^{L}b(g)\left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell}\binom{i-\ell+r-1}{r-1}\right\}\right.\\
&\left.+b(L)\left(\frac{\beta}{\lambda+\beta}\right)^{L-\ell}\binom{L-\ell+r-1}{r-1}\right]\\
=\;&\theta(0,\ell).
\end{aligned}
$$

Setting

$$\theta(r+1)=G^{-1}\left(\frac{\lambda+\nu}{\lambda}\right)^r$$

completes the proof in case of $r>0$.

(II) For $r>0$, (5.10) translates into

$$\theta(S)\cdot\lambda = \theta(0,1)\cdot\beta, \tag{5.25}$$

$$\theta(k)\cdot\lambda = \theta(k+1)\cdot\lambda, \qquad k=1,\dots,S-1 \tag{5.26}$$

$$\theta(0,\ell)\cdot\beta = \theta(1)\cdot\lambda\cdot b(L)+\theta(0,\ell+1)\cdot\beta, \qquad 1\le\ell<L, \tag{5.27}$$

$$\theta(0,L)\cdot\beta = \theta(1)\cdot\lambda\cdot b(L). \tag{5.28}$$

From (5.26) follows

$$\theta(S) = \theta(S-1) = \cdots = \theta(1), \tag{5.29}$$

and we will show that

$$\theta(0,\ell) = \theta(1)\cdot\left(\frac{\lambda}{\beta}\right)\left[\sum_{i=\ell}^{L} b(i)\right], \quad \ell = 1,\ldots,L, \tag{5.30}$$

holds. For $\ell = L$ this is immediate from (5.28), and for $\ell < L$ it follows by induction from (5.27). Setting $\theta(1) = G^{-1}$ completes the proof in case of $r = 0$. □

Remark 24. For $r > 0$ we can write (5.5) as

$$\theta(0,\ell) \quad = \quad G^{-1}\frac{\lambda}{\beta}\left[\sum_{i=\ell}^{L}\left(\sum_{g=i}^{L} b(g)\right)\left(\frac{\beta}{\lambda+\beta}\right)^{i-\ell}\binom{i-\ell+r-1}{i-\ell}\right].$$
$$\ell = 1,\ldots,L,$$

and can extend this formula to the case $r = 0$. This yields with $\binom{-1}{0} = 1$ explicitly

$$\theta(0,\ell) \quad = \quad G^{-1}\frac{\lambda}{\beta}\left[\sum_{i=\ell}^{L} b(i)\right], \quad \ell = 1,\ldots,L,$$

Corollary 25. *In steady state the marginal probabilities for the inventory at hand have the following simple representation.*

Denote by ν^{-1} the expected lead time.

Let V denote a random variable distributed according to $b = (b(\ell) : 1 \le \ell \le L)$, and let V_e denote a random variable distributed according to the "equilibrium distribution" of V, resp. b, i,e.

$$P(V_e = i) = \frac{1}{E(V)}\sum_{g=i}^{L} b(g), \quad 1 \le i \le L.$$

Let $W(u,\alpha)$ denote a random variable distributed according to a negative binomial distribution $Nb^0(u,\alpha)$ with parameters $u \in \mathbb{N}$ and $\alpha \in (0,1)$, i.e.,

$$P(W(u,\alpha) = i) = \binom{i+u-1}{u-1}\alpha^u(1-\alpha)^i, \quad i \in \mathbb{N}.$$

Let I denote a random variable distributed according to the marginal steady state probability for the inventory size. Then for $j = 1,\ldots,r$

$$P(I = j) = G^{-1}\left(\frac{\lambda+\beta}{\lambda}\right)^r \cdot P(W(r+1-j,\frac{\lambda}{\lambda+\beta}) < V), \tag{5.31}$$

and

$$P(I = 0) = G^{-1}\frac{\lambda}{\nu}\left(\frac{\lambda+\beta}{\lambda}\right)^r \cdot P(W(r,\frac{\lambda}{\lambda+\beta}) < V_e). \tag{5.32}$$

For $j = r+1, \ldots, S$ (5.6) applies directly:

$$P(I = r+1) = \cdots = P(I = S) = G^{-1} \left(\frac{\lambda + \nu}{\lambda} \right)^r .$$

Proof. For $j = 1, \ldots, r$ we have

$$\begin{aligned}
P(I = j) &= G^{-1} \left(\frac{\lambda + \beta}{\lambda} \right)^{j-1} \sum_{\ell=1}^{L} \sum_{i=\ell}^{L} b(i) \left(\frac{\beta}{\lambda + \beta} \right)^{i-\ell} \binom{i - \ell + r - j}{r - j} \\
&= G^{-1} \left(\frac{\lambda + \beta}{\lambda} \right)^{j-1} \sum_{i=1}^{L} b(i) \sum_{\ell=1}^{i} \left(\frac{\beta}{\lambda + \beta} \right)^{i-\ell} \binom{i - \ell + r - j}{r - j} \\
&= G^{-1} \left(\frac{\lambda + \beta}{\lambda} \right)^{j-1} \sum_{i=1}^{L} b(i) \sum_{g=0}^{i-1} \left(\frac{\beta}{\lambda + \beta} \right)^{g} \binom{g + r - j}{r - j} \\
&= G^{-1} \left(\frac{\lambda + \beta}{\lambda} \right)^{j-1} \left(\frac{\lambda + \beta}{\lambda} \right)^{r+1-j} \\
&\qquad \sum_{i=1}^{L} b(i) \sum_{g=0}^{i-1} \binom{g + (r+1-j) - 1}{(r+1-j) - 1} \left(\frac{\lambda}{\lambda + \beta} \right)^{r+1-j} \left(\frac{\beta}{\lambda + \beta} \right)^{g} \\
&= G^{-1} \left(\frac{\lambda + \beta}{\lambda} \right)^{r} \sum_{i=1}^{L} b(i) \cdot P(W(r+1-j, \frac{\lambda}{\lambda + \beta}) < i),
\end{aligned}$$

and for $j = 0$ we have

$$\begin{aligned}
P(I = 0) &= G^{-1} \left(\frac{\lambda}{\beta} \right) \sum_{\ell=1}^{L} \sum_{i=\ell}^{L} \sum_{g=i}^{L} b(g) \left(\frac{\beta}{\lambda + \beta} \right)^{i-\ell} \binom{i - \ell + r - 1}{r - 1} \\
&= G^{-1} \left(\frac{\lambda}{\beta} \right) \sum_{i=1}^{L} \sum_{\ell=1}^{i} \left(\frac{\beta}{\lambda + \beta} \right)^{i-\ell} \binom{i - \ell + r - 1}{r - 1} \sum_{g=i}^{L} b(g) \\
&= G^{-1} \underbrace{\left(\frac{\lambda}{\beta} \cdot E(V) \right)}_{=\lambda/\nu} \left(\frac{\lambda + \beta}{\lambda} \right)^{r} \sum_{i=1}^{L} \underbrace{\left(\frac{1}{E(V)} \sum_{g=i}^{L} b(g) \right)}_{=:P(V_e = i)} \sum_{f=0}^{i-1} \binom{f + r - 1}{r - 1} \left(\frac{\lambda}{\lambda + \beta} \right)^{r} \left(\frac{\beta}{\lambda + \beta} \right)^{f} \\
&= G^{-1} \left(\frac{\lambda}{\nu} \right) \left(\frac{\lambda + \beta}{\lambda} \right)^{r} P(W(r+1-1, \frac{\lambda}{\lambda + \beta}) < V_e).
\end{aligned}$$

□

We now revisit the results from [SHH11] and [SAH10] for queueing-inventory systems under (r, Q) policy. We allow additionally the service rate of the server to depend on the queue length of the system. We assume that the lead time distribution is of phase type.

We enlarge the phase space of the system, i.e. the state space of the inventory process Y. Whenever there is an ongoing lead time, i.e., when inventory at hand is less than

$r+1$, we count the number of residual successive i.i.d. $\exp(\beta)$-distributed lead time phases which must expire until the replenishment arrives at the inventory.

The state space of (X, Y) then is $E = \mathbb{N}_0 \times K$ with

$$K = \{r+1, r+2, ..., r+Q\} \cup (\{0, 1, ...r\} \times \{L, \ldots, 1\}),$$

and (X, Y) is irreducible on E.

Theorem 26. *$M/M/1/\infty$ system with (r, Q)-policy, phase-type replenishment lead time, state dependent service rates $\mu^{(n)}$, and lost sales.*

The lead time distribution has a distribution function B from (5.2). We assume that (X, Y) is positive recurrent and denote its steady state distribution by

$$\pi = (\pi(n, k) : n \in \mathbb{N}_0 \times K).$$

The steady state π of (X, Y) is of product form. With normalization constant C

$$\pi(n, k) = C^{-1} \prod_{i=0}^{n-1} \frac{\lambda}{\mu^{(i+1)}} \cdot \theta(k), \tag{5.33}$$

where $\theta = (\theta(k) : k \in K)$ can be obtained from formula (3) in [SAH10], and the subsquent formulas (4) - (10) there.

Proof. The proof is in its first part similar to that of Theorem 23 because the inventory management process under (r, Q)-policy with distribution function B of the lead times fits into the definition of the environment process by setting

$$K = \{r+1, r+2, ..., r+Q\} \cup (\{0, 1, ...r\} \times \{L, \ldots, 1\}), \quad K_B = \{0\} \times \{L, \ldots, 1\}.$$

Because $\lambda^{(n)} = \lambda$ for all n, Theorem 2 applies and we know that the steady state of the ergodic system is of product form

$$\pi(n, k) = C^{-1} \frac{\lambda^n}{\prod_{i=0}^{n-1} \mu^{(i+1)}} \cdot \theta(k) \quad (n, k) \in E, \tag{5.34}$$

according to Corollary 6. Thus the product form statement is proven with the required marginal queue length distribution.

In a second part we have to compute the $\theta(k)$ which is to solve (3.5). This equation is independent of n, especially independent of the $\mu^{(n)}$.

Therefore the solution in the case of state independent service rates ($\mu^{(n)} \to \mu$) from [SAH10] must be the solution in the present slightly more general setting as well. □

5.2 Unreliable servers

In [SD03] networks of queues with unreliable servers were investigated which admit product form steady states in twofold way: The joint queue length vector of the system (which in general is not a Markov process) is of classical product form as in Jackson's Theorem and the availability status of the nodes as a set valued supplementary variable process constitutes an additional product factor attached to the joint queue length vector.

We show for the case of a single server which is unreliable and breaks down due to influences from an environment that a similar product form result follows from our Theorem 2. We allow for a much more complicated breakdown and repair process as that investigated in [SD03].

Example 27. There is a single exponential server with with Poisson-λ arrival stream and state dependent service rates $\mu^{(n)}$. The server acts in a random environment which changes over time. The server breaks down with rates depending on the state of the environment and is repaired after a breakdown with repair intensity depending on the state of the environment as well. Whenever the server is broken down, new arrivals are not admitted and are lost to the system forever.

The system is described by a two-dimensional Markov process $(X, Y) = ((X(t), Y(t)) : t \in [0, \infty))$ with state space $E = \mathbb{N}_0 \times K$. K is the (countable) environment space of the process, whereas $\mathbb{N}_0$ denotes the queue length. (X, Y) is assumed to be irreducible.

The environment space of the process is partitioned into disjoint nonempty components $K := K_W + K_B$, and whenever Y enters K_B the server breaks down immediately, and will be repaired when Y enters K_W again.

The non negative transition rates of (X, Y) are for $(n, k) \in E$

$$\begin{aligned} q((n,k) \to (n+1,k)) &= \lambda \qquad k \in K_W, \\ q((n,k) \to (n-1,m)) &= \mu^{(n)} R_{km} \qquad k \in K_W, n \geq 1, \\ q((n,k) \to (n,m)) &= \nu(k,m) \in \mathbb{R}_0^+, \quad k \neq m, \\ q((n,k) \to (l,m)) &= 0 \qquad \text{otherwise for} \quad (n,k) \neq (l,m), \end{aligned} \tag{5.35}$$

and from Theorem 2 we directly obtain in case of ergodicity the product form steady state distribution.

An interesting observation is, that we can model general distributions for the successive times the system is functioning and similarly for the repair times.

By suitably selected structures for the $\nu(\cdot, \cdot)$ we can incorporate dependent up and down times.

The distinctive feature which sets the difference to the breakdown mechanism in [SD03] is that breakdowns can be directly connected with expiring service times via the "routing matrix" R, which is visible from (5.35). This widens applicability of the mechanism considerably.

5.3 Active-sleep model for nodes in wireless sensor networks

Modeling of wireless sensor networks (WSN) is a challenging task due to specific restrictions imposed on the network structure and the principles the nodes have to follow to survive without the possibility of external renewal of a node or repair. A specific task is that usually battery power cannot be renewed which strongly requires to control energy consumption. A typical way to resolve this problem is to reduce energy consumption by laying a node in sleep status whenever this is possible. In sleep status all activities of the node are either completely or almost completely interrupted.

In active mode the node undertakes several activities: Gathering data and putting the resulting data packets into its queue, receiving packets from other nodes which are placed in its queue (and relaying these packets when they arrive at the head of the node's queue), and processing the packets in the queue (usually in a FCFS regime).

The modeling approach to undertake analytical performance analysis of WSN in the literature is to first investigate a single ("referenced") node and thereafter to combine by some approximation procedure the results to investigate the behavior of interacting nodes, for a review see [WDW07]. More recent and a more detailed study of a specific node model is [Li11], other typical examples for the described procedure are [LTL05], [ZL11].

We will report here only on the first step of the procedure and follow mainly the model of a node found in [LTL05]. The functioning of the referenced node is governed by the following principles which incorporate three processes.

- Length of the packet queue of the node ($\in \mathbb{N}_0$),
- mode of the node (active $= A$, sleep $= S$)
- status of the nodes with which the referenced node is able to communicate; these nodes are called the "outer environment" and their behavior with respect to the referenced node is summarized in a binary variable (on $= 0$, off $= 1$), where "on" $= 0$ indicates that there is another active node in the neighborhood of the referenced node, while "off" $= 1$ indicates that all nodes in the neighborhood of the referenced node are in sleep mode.

It follows that the referenced node can communicate with other nodes if and only if the outer environment is on $= 0$ **and** the node itself is active $= A$.

Transforming the described behavior into the formalism of Section 2, we end with an environment

$$K = \{A, S\} \times \{0, 1\}, \quad K_W = \{(A, 0)\}.$$

and state space $E := \mathbb{N}_0 \times K$ of the joint process (X, Y).

In [LTL05] the authors assume that when the the referenced node is active ($= A$) and outer environment on ($= 0$), the stream of packets arriving at the packet queue of the node is the superposition of data gathering and receiving packets from other nodes. The superposition process is a Poisson-λ process, and processing a packet in the queue needs an exponential-μ distributed time.

For simplicity of the model we assume here that whenever the node is in sleep mode or the outer environment is off, all activities of the node are frozen. This is different from [LTL05], who allow during this periods data gathering by the node.

Because the overwhelming part of battery capacity reduction stems from the other two activities (to relay and processing packets), with respect to battery control the additional simplification can be justified for raw first approximations, especially if this is rewarded by obtaining simple to evaluate closed form expresions.

This reward is obtained by following [LTL05] and assuming that the on-off $(0-1)$ process of the outer environment is an alternating renewal process with exponential-α, resp., exponential-β holding times, and that the active-sleep $(A-S)$ process of the node is an alternating renewal process with exponential-a, resp., exponential-s holding times.

Fixing the usual overall independence assumption for all these holding times and the processing and inter arrival times, we see that this model fits precisely into the framwork of our general model from Section 2 and that the Theorem 2 provides us with an explicit steady state distribution.

Results on battery consumption obtained from the steady state distribution (similarly obtained to that in [LTL05]) clearly will then produce lower bounds for the energy consumption (which are weaker than those obtained in [LTL05] - which are obtained with expense of more computational effort).

5.4 Tandem system with finite intermediate buffer

Modeling multi-stage production lines by serial tandem queues is standard technique. In the simplest case with Poisson arrivals and with exponential production times for one unit in each stage the model fits into the realm of Jackson network models as long as the buffers between the stages have infinite capacity. Consequently, ergodic systems under this modeling approach have a product form steady state distribution.

With respect to steady state analysis the picture changes completely if the buffers between the stages have only finite capacity, no simple solutions are available. Direct numerical analysis or simulations are needed, or we have to resort to approximations. A common procedure is to use product form approximations which are developed by decomposition methods. A survey on general networks with blocking is [BDO01], special emphasis to modeling manufacturing flow lines is given in the survey [DG92].

The same class of problems and solutions are well known in teletraffic networks where finite buffers are encountered, for surveys see [Onv90] and [Per90].

A systematic study of how to use product form networks as upper or lower bounds (in a specified performance metric) is given in [Dij93]. A closed 3-station model which is related to the one given below is discussed in [Dij93][Section 4.5.1], where product form lower and upper bounds are proposed.

We consider a two-stage single server tandem queuing system where the first station has ample waiting space while the buffer between the stages has only $N \geq 0$ waiting places, $N < \infty$, i.e. there can at most $N+1$ units be stored in the system which have been processed at the first stage. It follows that for the system must be determined a blocking regime, which enforces the first station to stop production when the intermediate buffer reaches its capacity $N+1$. We apply the **blocking-before-service** regime [Per90][p.

455]: Whenever the second station is full, the server at the first station does not start serving the next customer. When a departure occurs from the second station, the first station is unblocked immediately and resumes its service. Additionally, we require that the first station, when blocked does not accept new customers, i.e., it is completely blocked.

The arrival stream is Poisson-λ, service rates are state dependent with $\mu^{(n)}$ at the first station and $\nu^{(k)}$ at the second. The standard independence assumption are assumed to hold, service at both stations is on FCFS basis.

This makes the joint queue length process (X, Y) Markov with state space $E := \mathbb{N}_0 \times \{0, 1, \ldots, N, N+1\}$. The non negative transition rates are

$$\begin{aligned} q((n,k) \to (n+1,k)) &= \lambda, & k \le N\,, \\ q((n,k) \to (n-1,k+1)) &= \mu^{(n)}, & n \ge 1, k \le N\,, \\ q((n,k) \to (n,k-1)) &= \nu^{(k)}, & 1 \le k \le N+1\,, \\ q((n,k) \to (j,m)) &= 0, & \text{otherwise for } (n,k) \neq (j,m). \end{aligned}$$

We fit this model into the formalism of Section 2 by setting

$$K = \{0, 1, ..., N+1\}, \qquad K_B = \{N+1\},$$

$$R_{k,k+1} = 1, 0 \le k \le N,\ R_{N+1,N+1} = 1\,, \qquad \nu(k,m) = \begin{cases} \nu^{(k)}, & \text{if } 1 \le k \le N+1, \\ & \quad \text{and } m = k-1 \\ 0, & \text{otherwise for } k \neq m\,. \end{cases}$$

From Theorem 2 and Corollary 6 we conclude that for the ergodic process (X, Y) the steady state distribution has product form

$$\pi(n,k) = C^{-1} \frac{\lambda^n}{\prod_{i=0}^{n-1} \mu^{(i+1)}} \theta(k) \quad (n,k) \in E, \tag{5.36}$$

with probability distribution θ on K and normalization constant

$$C = \sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{i=0}^{n-1} \mu^{(i+1)}}\,.$$

It remains to determine θ from (3.5), which is (3.9) with $\lambda = \lambda^{(n)}$.

So, the $\tilde{Q}^{(n)} := \tilde{Q}$ matrix is independent of (n) and explicitly

$$\tilde{Q} = \left(\begin{array}{c|cccccc} & 0 & 1 & 2 & & N & N+1 \\ \hline 0 & -\lambda & \lambda & & & & \\ 1 & \nu^{(1)} & -(\nu^{(1)} + \lambda) & \lambda & & & \\ 2 & & \nu^{(2)} & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ N & & & & \ddots & -(\nu^{(N)} + \lambda) & \lambda \\ N+1 & & & & & \nu^{(N)} & -\nu^{(N)} \end{array} \right)$$

This is exactly the transition rate matrix of an $M/M/1/N$ queue with Poisson-λ arrivals and service rates $\nu(n)$ and we have immediately

$$\theta(k) = G^{-1} \prod_{h=1}^{k} \frac{\lambda}{\nu_h} \qquad 0 \leq k \leq N+1$$

with normalization constant

$$G = \sum_{h=0}^{N+1} (\prod_{h=1}^{k} \frac{\lambda}{\nu_h}) \, .$$

Remark 28. The result

$$\pi(n,k) = C^{-1} \prod_{i=0}^{n-1} \frac{\lambda^n}{\mu^{(i+1)}} \cdot G^{-1} \prod_{h=1}^{k} \frac{\lambda}{\nu_h} \, , \quad (n,k) \in E,$$

is surprising, because it looks like an independence result with marginal distributions of two queues feeded by Poisson-λ streams. Due to the interruptions, neither the arrival process at the first station nor the departure stream from the first node, which is the arrival stream to the second, is Poisson-λ. There seems to be no intuitive explanation of the results.

References

[BCMP75] F. Baskett, M. Chandy, R. Muntz, and F.G. Palacios. Open, closed and mixed networks of queues with different classes of customers. *Journal of the Association for Computing Machinery*, 22:248–260, 1975.

[BDO01] S. Balsamo, V. De Nitto Persone, and R. Onvural. *Analysis of Queueing Networks with Blocking.* Kluwer Academic Publisher, Norwell, 2001.

[BK99] O. Berman and E. Kim. Stochastic models for inventory management at service facilities. *Comm. Statist.– Stochastic Models*, 15(4):695 – 718, 1999.

[Dad01] H. Daduna. Stochastic networks with product form equilibrium. In D.N. Shanbhag and C.R. Rao, editors, *Stochastic Processes: Theory and Methods*, volume 19 of *Handbook of Statistics*, chapter 11, pages 309–364. Elsevier Science, Amsterdam, 2001.

[DG92] Y. Dallery and S. B. Gershwin. Manufacturing flow line systems: a review of models and analytical results. *Queueing Systems*, 12:3–94, 1992.

[Dij93] N. M. van Dijk. *Queueing Networks and Product Forms – A Systems Approach.* Wiley, Chichester, 1993.

[Fal96] G. Falin. A heterogeneous blocking system in a random environment. *Journal of Applied Probability*, 33:211 – 216, 1996.

[GN67] W.J. Gordon and G.F. Newell. Closed queueing networks with exponential servers. *Operations Research*, 15:254–265, 1967.

[HJV05] P. Haccou, P. Jagers, and V. Vatutin. *Branching Processes: Variation, Growth, and Extinction of Populations.* Cambridge Studies in Adaptive Dynamics. Cambridge University Press, Cambridge, 2005.

[HW84] W. Helm and K.-H. Waldmann. Optimal control of arrivals to multiserver queues in a random environment. *Journal of Applied Probability*, 21:602–615, 1984.

[Jac57] J.R. Jackson. Networks of waiting lines. *Operations Research*, 5:518–521, 1957.

[Kei79] J. Keilson. *Markov chain models – Rarity and exponentiality.* Springer, New York, 1979.

[Kel76] F. Kelly. Networks of queues. *Advances in Applied Probability*, 8:416–432, 1976.

[Kel79] F. P. Kelly. *Reversibility and Stochastic Networks.* John Wiley and Sons, Chichester – New York – Brisbane – Toronto, 1979.

[Kes80] H. Kesten. Random processes in random environments. In W. Jäger, H. Rost, and P. Tautu, editors, *Biological Growth and Spread*, volume 38 of *Lecture NOtes in Biomathematics*, chapter Ib, pages 82–92. Springer, Berlin, 1980. Proceedings of the Conference on Models of Biological Growth and Spread, University of Heidelberg, 1979.

[KLM11] A. Krishnamoorthy, B. Lakshmy, and R. Manikandan. A survey on inventory models with positive service time. *OPSEARCH*, 48:153–169, 2011.

[KN12] A. Krishnamoorthy and V. C. Narayanan. Stochastic decomposition in production inventory with service time (abstract only). *SIGMETRICS Perform. Eval. Rev.*, 39(4):28–28, April 2012.

[Li11] W.W. Li. Several characteristics of active/sleep model in wireless sensor networks. In *New Technologies, Mobility and Security (NTMS), 2011 4th IFIP International Conference on*, pages 1 –5, feb. 2011.

[LTL05] J. Liu and T. Tong Lee. A framework for performance modeling of wireless sensor networks. In *Communications, 2005. ICC 2005. 2005 IEEE International Conference on*, volume 2, pages 1075 – 1081 Vol. 2, 2005.

[Neu81] M.F. Neuts. *Matrix Geometric Solutions in Stochastic Models - An Algorithmic Approach.* John Hopkins University Press, Baltimore, MD, 1981.

[Neu89] M.F. Neuts. *Structured Stochastic Matrices of M/G/1 Type and Their Applications.* Marcel Dekker, New York, 1989.

[Onv90] R. O. Onvural. Closed queueing networks with blocking. In H. Takagi, editor, *Stochastic Analysis of Computer and Communication Systems*, pages 499–528. North-Holland, Amsterdam, 1990.

[Per90] H. G. Perros. Approximation algorithms for open queueing networks with blocking. In H. Takagi, editor, *Stochastic Analysis of Computer and Communication Systems*, pages 451–498. North-Holland, Amsterdam, 1990.

[SAH10] M. Saffari, S. Asmussen, and R. Haji. The M/M/1 queue with inventory, lost sale and general lead times. Workingpaper, Thiele Centre, Institut for Matematiske Fag, Aarhus Universitet, 2010. Research Report No.11, September 2010.

[Sch73] R. Schassberger. *Warteschlangen.* Springer, Wien, 1973.

[SD03] C. Sauer and H. Daduna. Availability formulas and performance measures for separable degradable networks. *Economic Quality Control*, 18:165–194, 2003.

[SHH11] M. Saffari, R. Haji, and F. Hassanzadeh. A queueing system with inventory and mixed exponentially distributed lead times. *The International Journal of Advanced Manufacturing Technology*, 53:1231–1237, 2011.

[SSD+06] M. Schwarz, C. Sauer, H. Daduna, R. Kulik, and R. Szekli. M/M/1 queueing systems with inventory. *Queueing Systems and Their Applications*, 54:55–78, 2006.

[Ter94] R.J. Tersine. *Principles of Inventory and Materials Management.* PTR Prentice Hall, Englewood Cliffs, N.J., 4 edition, 1994.

[Vin08] K. Vineetha. *Analysis of inventory systems with positive and/negligible service time.* PhD thesis, Department of Statistics, University of Calicut,India, 2008.

[WDW07] Y. Wang, H. Dang, and H. H. Wu. A survey on analytic studies of Delay-Tolerant Mobile Sensor Networks. *IEEE Transactions on Wireless Communications*, 6(9):3287–3296, 2007.

[ZL11] Y. Zhang and W. Li. An energy-based stochastic model for wireless sensor networks. *Wireless Sensor Networks*, 3(9):322–328, 2011.