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Stability and Attraction in ODEs – An Overview

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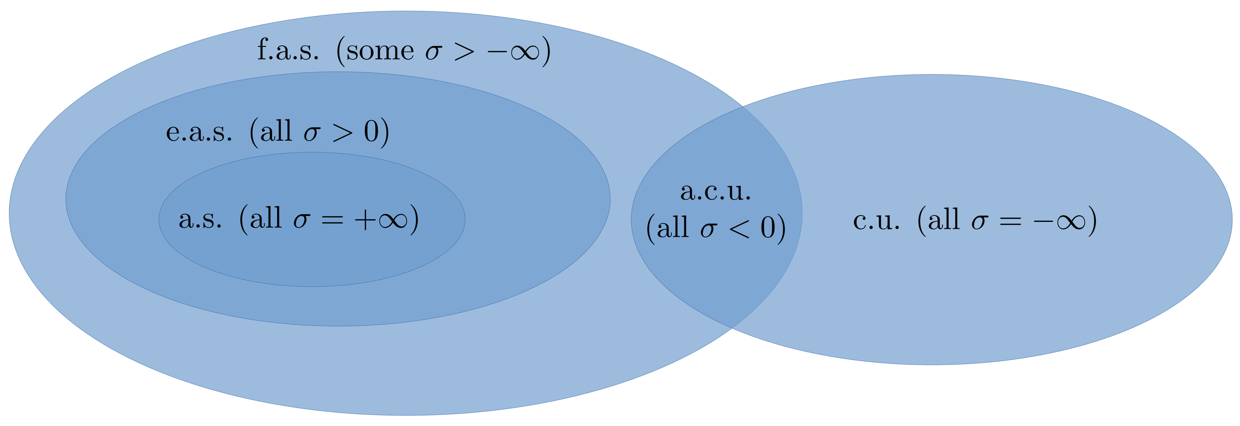
Let $\dot{x} = f(x)$ and $X \subset \mathbb{R}^n$ be a compact, invariant set for the flow $\phi_t(x)$, $\mathcal{B}_\varepsilon(X) = \{x \mid \omega(x) \subset X \text{ and } \phi_t(x) \in B_\varepsilon(X)\ \forall t \geq 0\}$ the ε-local basin of attraction, and $\ell(\cdot)$ Lebesgue measure. In the following, X is primarily thought of as a *heteroclinic cycle* or *network* (a union of equilibria and non-empty intersections of their stable and unstable manifolds), even though most definitions apply to more general X.

Notions of Stability/Attraction — from "pretty much everything" to "almost nothing"

- X is called *asymptotically stable (a.s.)* if for all $\varepsilon > 0$ there is $\delta > 0$ such that $B_\delta(X) \subset \mathcal{B}_\varepsilon(X)$.
- X is called *essentially asymptotically stable (e.a.s.)* if it is a.s. relative to a set $N \subset \mathbb{R}^n$ such that

$$\frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} \xrightarrow{\varepsilon \to 0} 1. \qquad (1)$$

- X is called *fragmentarily asymptotically stable (f.a.s.)* if there is $\varepsilon > 0$ such that $\ell(\mathcal{B}_\varepsilon(X)) > 0$.
- X is called *almost completely unstable (a.c.u.)* if it is c.u. relative to a set $N \subset \mathbb{R}^n$ satisfying (1).
- X is called *completely unstable (c.u.)* if there is $\varepsilon > 0$ such that $\ell(\mathcal{B}_\varepsilon(X)) = 0$.



Stability Indices — quantifying basins of attraction

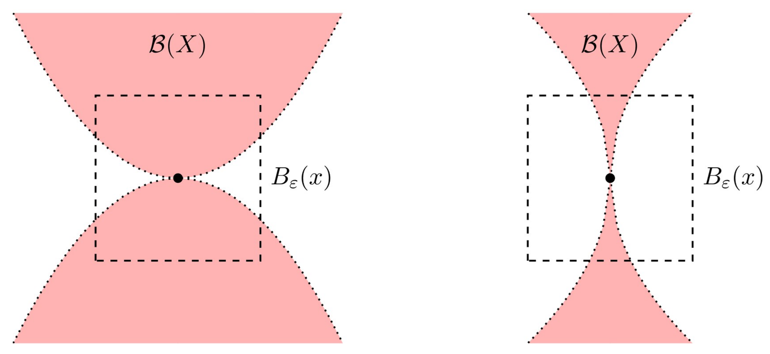
For $x \in X$ the *(local) stability index* $\sigma(x) := \sigma_+(x) - \sigma_-(x) \in [-\infty, +\infty]$ quantifies the size of the (local) basin of attraction, where $\sigma_+(x) := \lim_{\varepsilon,\delta \to 0} \frac{\ln(1-\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}$ and $\sigma_-(x) := \lim_{\varepsilon,\delta \to 0} \frac{\ln(\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}$ with $\Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \,\cap\, \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}$.

→ can be computed relative to a cross section and is constant along trajectories

→ relation to stability properties for heteroclinic cycles/networks: Lohse 2015.

→ computation results for simple cycles in $\mathbb{R}^4$: Podvigina & Ashwin 2011

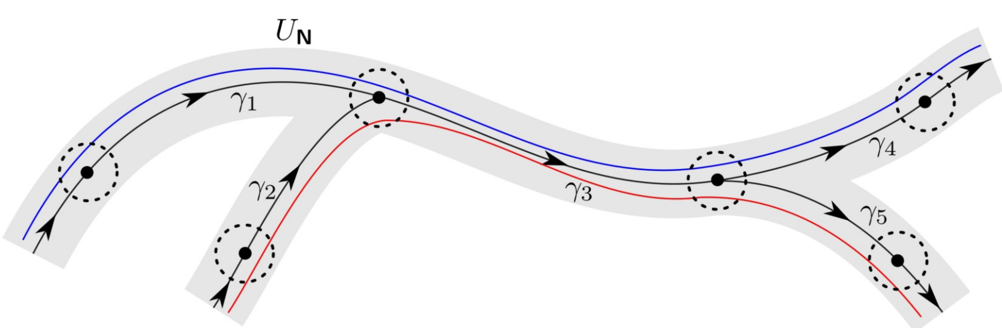
→ computation results for quasi-simple cycles in $\mathbb{R}^n$: Garrido-da-Silva & Castro 2019



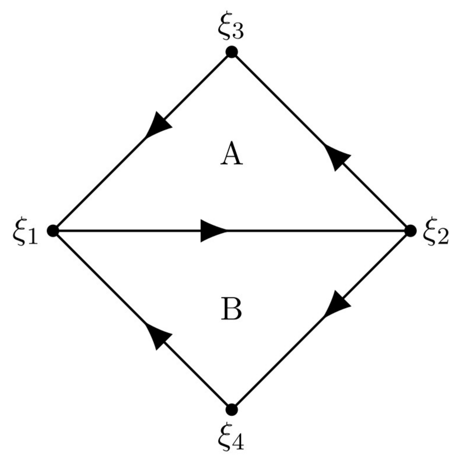
Typical basins of attraction (red) for positive (left) and negative (right) stability index.

Omnicycles and Sequences – ω-limit sets are not enough

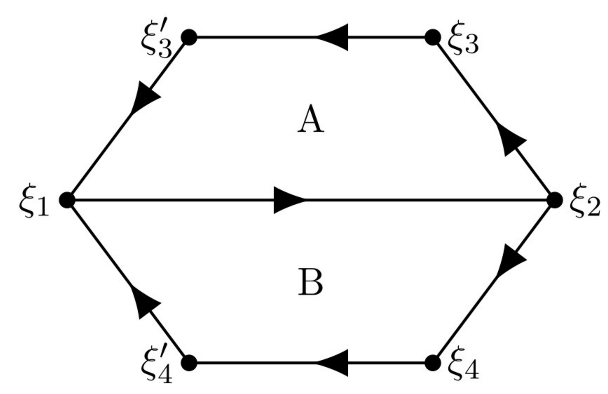
Trajectories attracted to a heteroclinic network X converge to it by following a unique sequence of equilibria and connections in X. While doing so, they do not necessarily limit to a single cycle or subnetwork. This prompts the definition of an *omnicycle* (Podvigina 2023) as a periodic sequence on X, in which equilibria and connections may occur more than once. An omnicycle / a periodic sequence $\mathbf{q}$ on X is *trail-stable* (or *f.a.s.* by slight abuse of terminology) if there is a positive measure set of initial conditions such that the corresponding trajectories converge to X and follow $\mathbf{q}$ while doing so. This is stronger than demanding that the underlying subset of X is f.a.s.



Trajectories near a heteroclinic network following different sequences.



The Figure-8-omnicycle $\overline{AB}$ in Kirk & Silber 1994 cannot be f.a.s.



The Figure-8-omnicycle $\overline{AB}$ in this network from Podvigina 2023 is f.a.s. for suitable parameter values.

More examples of f.a.s. periodic sequences (that are not cycles!) can be found in the Rock-Paper-Scissors-Lizard-Spock game (Postlethwaite & Rucklidge 2022).

Aperiodic Sequences — irregular convergence to heteroclinic structures

The notion of (local) basin of attraction can be generalized for aperiodic sequences on a network X (Bick & Lohse 2025). For a sequence $\mathbf{q}$ and $\delta > 0$ let $S_\delta(\mathbf{q})$ be the *δ-stable set of* $\mathbf{q}$, consisting of all points in $B_\delta(X)$ that follow $\mathbf{q}$. Requiring convergence to X yields the subset $\mathcal{A}_\delta(\mathbf{q}) := \{x \in S_\delta(\mathbf{q}) \mid \lim_{t \to \infty} \mathrm{d}(X, \phi_t(x)) = 0\}$, and finally the *asymptotic basin of* $\mathbf{q}$

$$\mathcal{B}_\delta(\mathbf{q}) := \bigcup_{k \in \mathbb{N}} \mathcal{A}_\delta(\Sigma^k \mathbf{q}),$$

where Σ is the shift along the sequence $\mathbf{q}$. We call $\mathbf{q}$ *f.a.s.* if there is $\delta > 0$ such that $\ell(\mathcal{B}_\delta(\mathbf{q})) > 0$. Alternatively, we can write $\mathcal{B}_\delta(\mathbf{q})$ as a disjoint union

$$\mathcal{B}_\delta(\mathbf{q}) = \bigcup_{k \in \mathbb{N}} \mathcal{D}_\delta^{(k)}(\mathbf{q}) \quad \text{with} \quad \mathcal{D}_\delta^{(k)}(\mathbf{q}) := \mathcal{A}_\delta(\Sigma^k \mathbf{q}) \setminus \mathcal{A}_\delta(\Sigma^{k-1} \mathbf{q}).$$

- An f.a.s. sequence $\mathbf{q}$ is called *f.a.s. of finite type (f-f.a.s.)* if for any $\delta > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ we have $\ell(\mathcal{D}_\delta^{(k)}(\mathbf{q})) = 0$.
- An f.a.s. sequence $\mathbf{q}$ is called *f.a.s. of infinite type (i-f.a.s.)* if for any $\delta > 0$ and all $k \in \mathbb{N}$ we have $\ell(\mathcal{D}_\delta^{(k)}(\mathbf{q})) > 0$.

All f.a.s. (pre)periodic sequences are f-f.a.s. There are at most countably many f.a.s. sequences. **Q: Are there examples of i-f.a.s. sequences?**

Visibility — what do we observe in physical systems or numerics?

Even the refined stability/attraction properties do not always reveal which substructures we should expect to "see" in reality: for example, the Kirk-Silber network can be f.a.s. as a set, even though no trajectory has the full network as its ω-limit. This prompts notions of *visibility* (Castro, Postlethwaite & Rucklidge 2025).

- X is called *visible* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for all $\bar{x} \in X$ and for all $x \in B_\delta(X) \setminus X$ there is $T > 0$ such that $\mathrm{d}(\phi_t(x), X) < \varepsilon$ for all $t > T$ and there is an increasing sequence $t_n \to \infty$ such that $\mathrm{d}(\phi_{t_n}(x), \bar{x}) < \varepsilon$ for all $n \in \mathbb{N}$.

Analogous notions exist for *fragmentary visibility*, *essential visibility* etc. Their relation to stability/attraction properties is subtle.

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