Name(s):

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Partial Differential Equations

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We assume all the functions to be smooth, unless stated otherwise.

Problem 1

Let u solve the initial-value problem for the wave equation:

$$\begin{cases} u_{tt} - \Delta_x u &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g, u_t &= h \quad \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Suppose g, h have compact support. Let

$$E_k(t) := \frac{1}{2} \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} |D^\alpha u_t(x,t)|^2 + |D^\alpha D_x u(x,t)|^2 dx$$

where D^α is a multi-index derivative in (x,t) of order $|\alpha|$. Prove that $E_k(t)$ is constant in t.

Problem 2

(Equipartition of energy). Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t &= h \quad \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

Suppose g, h have compact support. Let the k-th order *kinetic energy* be

$$V_k(t) := \sum_{|\alpha| \leq k} \frac{1}{2} \int_{-\infty}^{\infty} |D^\alpha u_t(x,t)|^2 dx$$

and the k-th order *potential energy* is

$$P_k(t) := \sum_{|\alpha| \leq k} \frac{1}{2} \int_{-\infty}^{\infty} |D^\alpha u_x(x,t)|^2 dx$$

where D^α is a multi-index derivative in (x,t) of order $|\alpha|$. Prove that $V_k(t) = P_k(t)$ for all large enough times t.

Problem 3

Let L be a partial differential operator of second order. Let F be a diffeomorphism on $\mathbb{R}^n$ and $y = F(x)$. Let $\tilde{L}$ be the differential operator that one gets by transforming L from the coordinates $(x_1, \ldots, x_n)$ to the coordinates $(y_1, \ldots, y_n) = F(x_1, \ldots, x_n)$. Show then that L is elliptic/hyperbolic/parabolic operator in x, if and only if $\tilde{L}$ is of the same nature in $y = F(x)$.

Problem 4

Let T be the $(n+1) \times (n+1)$ matrix defined by

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Let A be an $(n+1) \times (n+1)$ matrix such that $A \cdot T \cdot A^t = T$, where A^t is the transpose of A.

Prove that if u satisfies the wave equation for $(t, x) \in \mathbb{R}^{1+n}$ then u satisfies the wave equation in $(t', x') \in \mathbb{R}^{1+n}$ where

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = A \cdot \begin{bmatrix} t \\ x \end{bmatrix}$$