Name(s):

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Partial Differential Equations

Winter Semester 2017–2018

Worksheet 7 — Tuesday, November 28, 2017

We assume all the functions to be smooth, unless stated otherwise.

Problem 1

Assume $E = (E^1, E^2, E^3)$ and $B = (B^1, B^2, B^3)$ solve the Maxwell equations:

$$\begin{cases} E_t & = \text{curl } B \\ B_t & = -\text{curl } E \\ \text{div } B & = \text{div } E = 0 \end{cases}$$

Show that $u_{tt} - \Delta u = 0$ where $u = B^i$ or E^i for $i = 1, 2, 3$.

Problem 2

Let $\phi : \mathbb{R} \to \mathbb{R}$ be C^{k+1}. Then for $k = 1, 2, ...$, prove that

(i) $(\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi(r)) = (\frac{1}{r}\frac{d}{dr})^k(r^{2k}\frac{d\phi}{dr}(r))$,

(ii) $(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi(r)) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r)$, where the constants $\beta_j^k (j = 0, ..., k-1)$ are independent of ϕ.

(iii) $\beta_0^k = 1.3.5.....(2k-1)$.

Problem 3

Let u solve

$$\begin{cases} u_{tt} - \Delta u & = 0 \quad \text{in } \mathbb{R}^3 \times (0, 1) \\ u = g, u_t & = h \quad \text{on } \mathbb{R}^3 \times t = 0, \end{cases}$$

where g, h are smooth and have compact support. Show that there exists a constant C such that

$$\sup_{x\in\mathbb{R}^3} |u(x, t)| \leq \frac{C}{t} \quad (x \in \mathbb{R}^3, t > 0).$$

Problem 4

Let $u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a solution of the wave equation $u_{tt} - u_{xx} = 0$. Show that

$$v(t, x) := \int_{-\infty}^{+\infty} \frac{\exp^{-\frac{s^2}{4t}}}{\sqrt{t}} u(s, x) ds$$

satisfies the heat equation $u_t - u_{xx} = 0$ for every $t > 0$ and $x \in \mathbb{R}$.

For December 05, 2017, at 08:15 A.M.