Name(s):

Lecturer: Klaus Kröncke
Trainer: Sari Ghanem

Partial Differential Equations

Winter Semester 2017–2018

Worksheet 6 — Tuesday, November 21, 2017

Problem 1

We say $v \in C^2_1(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T$$

(a) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \int\int_{E(x,t;r)} v(y,s) \frac{|x-y|}{(t-s)^2} dyds$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $\max_{\overline{U}_T} v = \max_{\Gamma_T} v$.

Problem 2

(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x + t$, $\nu = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\nu} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.

Problem 3

Tychonov's counterexample: Consider the holomorphic function $g(z) = \exp^{-\frac{1}{z^2}}$ for $z \in \mathbb{C}\backslash 0$ and denoting by $g^{(k)}$ its k-th derivative, define the function

$$\begin{cases} u(x,t) &= \sum_{k=0}^{+\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} \quad \text{if } t > 0, x \in \mathbb{R} \\ u(x,t) &= 0 \quad \text{if } t = 0, x \in \mathbb{R} \end{cases}$$

Rigorously justify that this is a solution of the heat equation with 0 Cauchy data by showing uniform convergence of the series (and of the series of its time and space derivatives involved in the equation) on any semi-strip of the type

$$(x,t) \in [-a,a] \times [\delta, +\infty)$$

with $a, \delta > 0$.

Hint: apply Cauchy's formula for the derivatives of holomorphic functions in this form

$$g^{(k)}(t) = \frac{k!}{2\pi i} \int_{\partial B(t,\frac{t}{2})} \frac{g(z)}{(z-t)^{k+1}} dz$$

to estimate $g^{(k)}(t)$. Obviously, if you find a method not using complex analysis, it is also valid!

Problem 4

Derive a representation formula for a solution of the initial/boundary-value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty), \\ u = g, u_t = h & \text{on } \mathbb{R}_+ \times \{t = 0\}, \\ u = 0 & \text{on } \{x = 0\} \times (0, \infty). \end{cases}$$

where g, h are given, with $g(0) = h(0) = 0$.

(Hint: construct a new function by *odd reflection*)