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Partial Differential Equations

Winter Semester 2017–2018

Worksheet 5 — Tuesday, November 14, 2017

Problem 1

Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(i) Show that $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Use (i) to show that $v(x,t) = x.\mathcal{D}u(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Problem 2

Assume $n = 1$ and $u(x,t) = v(\frac{x^2}{t})$.

(a) Show

$$u_t = u_{xx}$$

if and only if

$$4zv''(z) + (2+z)v'(z) = 0 \quad (z > 0). \tag{1}$$

(b) Show that the general solution of (1) is

$$v(z) = c\int_0^z \mathrm{e}^{-\frac{s}{4}} s^{-\frac{1}{2}} ds + d$$

(c) Differentiate $v(\frac{x^2}{t})$ with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for $n = 1$.

Problem 3

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times (t = 0), \end{cases}$$

where $c \in \mathbb{R}$.

Problem 4

Given $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, derive the formula

$$u(t,x) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} \mathrm{e}^{\frac{-x^2}{4(t-s)}} g(s) ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t=0\}, \\ u = g & \text{on } \{x=0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) = u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection $v(x,t) = -v(-x,t)$ for $x < 0$)