Name(s):

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Partial Differential Equations

Winter Semester 2017–2018

Worksheet 4 — Tuesday, November 07, 2017

Problem 1

Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show that Du is not bounded near $x = 0$.

Hint: estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$.

Problem 2

Assume $g \in C(\partial B(0,r))$ and define u by

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) \quad (x \in B^0(0,r)).$$

Prove that

(i) $u \in C^\infty(B^0(0,r))$,

(ii) $\Delta u = 0$ in $B^0(0,r)$,

(iii) $\lim_{x \to x^0, x \in B(0,r)} u(x) = g(x^0)$ for each point $x^0 \in \partial B(0,r)$.

Problem 3

Fix $R > 0$ and consider a *nonnegative* function $u \geq 0$ which is harmonic in the ball $B(0,R)$. Prove the following **Harnack's inequality**: for every x such that $|x| < R$ one has

$$R^{N-2} \frac{R - |x|}{(R + |x|)^{N-1}} u(0) \leq u(x) \leq R^{N-2} \frac{R + |x|}{(R - |x|)^{N-1}} u(0).$$

Deduce that

$$\sup_{B(0,\frac{R}{2})} u \leq 3^N \inf_{B(0,\frac{R}{2})} u.$$

Problem 4

Assume that $u : B(0,1) \subset \mathbb{R}^2 \to \mathbb{R}$ is harmonic. Let $v(x) = u(\frac{x}{|x|^2})$ defined in $\mathbb{R}^2 \backslash B(0,1)$. Show that v is harmonic.

For November 14, 2017, at 08:15 A.M.