Name(s):

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Partial Differential Equations

Winter Semester 2017–2018

Worksheet 2 — Tuesday, October 24, 2017

We assume all the functions to be smooth, unless stated otherwise.

Problem 1

Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b.\mathcal{D}u + cu = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Problem 2

Prove that the Laplace's equation, $\Delta u = 0$, is invariant under rotations and translations. That is, if O is an orthogonal $n \times n$ matrix, if $y \in \mathbb{R}^n$ is a constant, and if we define

$$v(x) := u(Ox + y) \quad \text{for all } x \in \mathbb{R}^n$$

then, $\Delta v = 0$.

Problem 3

Modify the proof of the mean value formulas to show that for $n \geq 3$, we have

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f dx,$$

provided that

$$\begin{cases} -\Delta u = f & \text{in } B^o(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Problem 4

We say for $v \in C^2(\overline{U})$ is *subharmonic* if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Prove that if v is a subharmonic function, then

$$v(x) \leq \fint_{B(x,r)} v dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\overline{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and let $v := \phi(u)$. Prove that v is subharmonic.

(d) Prove that $v := |\mathcal{D}u|^2$ is subharmonic whenever u is harmonic.

For October 30, 2017