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Partial Differential Equations

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Worksheet 13 — Tuesday, January 23, 2018

Problem 1

A function $u \in H_0^2(U)$ is called a weak solution of this boundary-value problem for the *biharmonic equation*

$$\begin{cases} \Delta^2 u = f & \text{in} \quad U \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial U \end{cases} \tag{1}$$

if

$$\int_U \Delta u \cdot \Delta v \; dx = \int_U fv \; dx$$

holds for all $v \in H_0^2(U)$. Prove that for each $f \in L^2(U)$, there exists a unique weak solution of (1). (Hint: Use the Max-Milgram theorem.)

Problem 2

Assume U is connected. A function $u \in H^1(U)$ is called a weak solution of Neumann's problem

$$\begin{cases} -\Delta u = f & \text{in} \quad U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial U \end{cases} \tag{2}$$

if

$$\int_U Du \cdot Dv dx = \int_U fv dx$$

for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove that (2) has a weak solution if and only if

$$\int_U f dx = 0.$$

(Hint: Use the Poincaré inequality $\|u - \overline{u}\|_{L^2(U)} \leq C \|Du\|_{L^2(U)}$ which holds for all $u \in H^1(U)$. Here, $\overline{u} = \frac{1}{|U|} \int_U u \; dx$. Use the subspace consisting of $u \in H^1(U)$ such that $\overline{u} = 0$.)

Problem 3

Let $u \in H^1(\mathbb{R}^n)$ have a compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in} \quad \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$, and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$. (Hint: mimic the proof of interior regularity Theorem, but without the cutoff function).

Problem 4

Let $U \subset \mathbb{R}^n$ be open and bounded, L be a uniformly elliptic operator of non-divergence form (5.68) with continuous coefficients and $u \in C^2(U) \cap C(\overline{U})$ be such that $Lu = 0$ holds in the classical sense. Show that $c \geq 0$ is a nessecary condition to ensure

$$\max_U u = \max_{\partial U} u.$$

(Hint: Find a counterexample in dimension $n = 1$ when the condition $c \geq 0$ is violated.)