Name(s):

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Partial Differential Equations

Winter Semester 2017–2018

Worksheet 12 Tuesday, January 16, 2018

Problem 1

Consider Laplace's equation with potential function c:

$$-\Delta u + cu = 0 \tag{1}$$

and the divergence structure equation:

$$-div(a\mathcal{D}v) = 0, \tag{2}$$

where the function a is postive.

(a) Show that if u solves (1) and $w > 0$ also solves (1), then $v := \frac{u}{w}$ solves (2) for $a := w^2$.

(b) Conversely, show that if v solves (2), then $u := va^{\frac{1}{2}}$ solves (1) for some potential c.

Problem 2

Show that any function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$u(t, x) := v(x - \mathbf{b} \cdot t)$$

where $\mathbf{b}$ is a vector in $\mathbb{R}^n$ and where $v : \mathbb{R}^n \to \mathbb{R}$ is locally integrable, is a weak solution of the transport equation

$$u_t + \mathbf{b} \cdot \mathcal{D}u = 0$$

(Hint: Consider the new coordinate system $y = x - \mathbf{b} \cdot t$, $s = t$, write the transport equation in this new coordinate system and solve the exercise in this system of coordinates).

Problem 3

Show that any function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the form

$$u(t, x) := G(t - x) + F(t + x)$$

where $G, F : \mathbb{R} \to \mathbb{R}$ are locally integrable, is a weak solution of the wave equation

$$u_{tt} - \Delta u = 0$$

(Hint: consider the new coordinate system $w = t - x$, $v = t + x$, and write the wave equation in this new coordinate system - see Problem 2 of Worksheet 6)

Problem 4

Let $U = [0, L] \subset \mathbb{R}$.

(a) Find the eigenvalues and the eigenfunctions of the negative of the (1-dimensional) Laplacian, i.e. find all couples $\lambda \in \mathbb{R}$, $u \in H_0^1(U)$ such that

$$-u_{xx} = \lambda u$$

holds (Actually, any weak solution of this problem is a strong solution, i.e. $u \in C^2(U)$ and the equation holds in the usual sense).

(b) Let $f \in L^2(U)$. Give a criterion on f that ensures that the following PDE in $U = [0, L]$ has a weak solution:

$$\begin{cases} -u'' + \lambda u = f & \text{on} \quad U \\ u(0) = 0 & \text{on} \quad \partial U \end{cases}$$