Name(s):

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Partial Differential Equations

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Worksheet 11 — Tuesday, January 09, 2018

Problem 1

Let X, Y be Banach spaces. Let $L : X \to Y$ be bounded and linear. Prove that:

(i) $(x_k) \to x \quad \text{in } X \quad \Rightarrow (Lx_k) \to Lx \quad \text{in } Y$

(ii) $(x_k) \rightharpoonup x \quad \text{in } X \quad \Rightarrow (Lx_k) \rightharpoonup Lx \quad \text{in } Y$

(iii) The operator $K : X \to Y$ is compact if and only if for any sequence (x_k) in X such that $(x_k) \rightharpoonup x$, we have $(Kx_k) \to Kx$ in Y.

Problem 2

Let H be a Hilbert space and let $K : H \to H$ be a compact operator. Then prove that:

(i) $\dim Ker(Id - K) < \infty$

(ii) $Im(Id - K)$ is closed

Problem 3

(i) Let $x \in \mathbb{R}^n$ and let δ_x be the Delta distribution centered on x. Let $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Compute the distributional derivative $(\delta_x)_{x_{i_1} x_{i_2} \ldots x_{i_k}}$.

(ii) Let H be the Heaviside function on $\mathbb{R}$, defined by

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

The map $u \mapsto (H, u)_{L^2(\mathbb{R})}$, $u \in C_c^\infty(\mathbb{R})$ defines a distribution on $\mathbb{R}$ which we again denote by H.

(a) Compute H' in the sense of distributions.

(b) Find a distribution F such that $F' = H$. Can it be realized as a function, i.e. is it of the form $u \mapsto (F, u)_{L^2(\mathbb{R})}$?

Problem 4

Let $U \subset \mathbb{R}^n$ open, bounded set with smooth boundary. Let L be the following operator with smooth coefficients and satisfy the uniform ellipticity condition:

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\ ,\ ]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu \quad \text{for all} \quad x \in U.$$