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Partial Differential Equations

Winter Semester 2017–2018

Worksheet 1 — Tuesday, October 17, 2017

Problem 1

Let u be a smooth function defined on an open subset U such that

$$\begin{aligned} u : U \subset \mathbb{R}^n &\to \mathbb{R} \\ x &\to u(x) \end{aligned}$$

or

$$\begin{aligned} u : U \subset \mathbb{R}^n \times \mathbb{R} &\to \mathbb{R} \\ (x,t) &\to u(x,t) \end{aligned}$$

Classify each of the following partial differential equations as follows:

a) Is the PDE linear, semilinear, quasilinear or fully nonlinear?

b) What is the order of the PDE?

$$\begin{aligned} iu_t + \triangle u &= 0 \\ u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} &= 0 \\ |\mathcal{D}u| &= 1 \\ \operatorname{div}\Big(\frac{\mathcal{D}u}{(1+|\mathcal{D}u|^2)^{\frac{1}{2}}}\Big) &= 0 \\ \det\big(\mathcal{D}^2 u\big) &= f \\ u_{tt} - \operatorname{div}\big(\mathbf{a}(\mathcal{D}_x u)\big) &= 0, \quad \text{where } \mathbf{a}(x) \text{ is a } n \times n \text{ matrix} \end{aligned}$$

Problem 2

Prove the *Multinomial Thoerem*

$$(x_1 + ... + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$$

where $\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha}$, $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$, and $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$. The sum is taken over all multiindices $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| = k$.

Problem 3

Prove *Leibniz' formula*

$$\mathcal{D}^\alpha(uv) = \sum_{|\beta| \le \alpha} \binom{\alpha}{\beta} \mathcal{D}^\beta u \mathcal{D}^{\alpha-\beta} v$$

where $u, v : \mathbb{R}^n \to \mathbb{R}$ are smooth, $\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for all $i = 1, ..., n$.

Problem 4

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. Prove that

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \mathcal{D}^\alpha f(0) x^\alpha + O\big(|x|^{k+1}\big) \qquad \text{as} \quad x \to 0$$

for each $k = 1, 2, ...$ This is *Taylor's formula* in multiindex notation. (Hint: fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(tx)$.)