On Static Solutions of the Einstein-Vlasov System with Charged Particles

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1 Introduction – Vlasov Matter

1.1 Vlasov matter: a model for many astrophysical phenomena

Assumptions of Vlasov matter: ensemble of collisonless particles (collisionless Boltzmann equation)

- Mass distribution of globular clusters (King's model, cf. e.g. Heggie, Giersz 2007)
- (disc) galaxies: understanding e.g. velocity curves (Andréasson, Rein 2015)
- Simulation of collisions: Milkyway $\leftrightarrow$ Andromeda
- Matter accretion of black holes (e.g. Rioseco, Sarbach 2016)
- Gravitational collapse (e.g. Andréasson 2010)

1.2 Vlasov matter – with Newtonian gravity

- Matter described by a particle distribution function $f = f(t, \vec{x}, \vec{v})$ defined on *phase space*.
- Gravity modeled by gravitational (Newtonian) potential ϕ

$$\begin{cases} \Delta\phi(t,\vec{x}) = 4\pi \int_{\mathbb{R}^3} f(t,\vec{x},\vec{v})\mathrm{d}v^1\mathrm{d}v^2\mathrm{d}v^3 \\ \partial_t f(t,\vec{x},\vec{v}) + \vec{v}\cdot\nabla_{\vec{x}} f(t,\vec{x},\vec{v}) - \nabla_{\vec{x}}\phi(t,\vec{x})\cdot\nabla_{\vec{v}} f(t,\vec{x},\vec{v}) = 0 \end{cases}$$

- Characteristic system:

$$\dot{\vec{x}}(t) = \vec{v}(t), \qquad \dot{\vec{v}}(t) = -\nabla\phi(t,\vec{x}).$$

1.3 Vlasov matter – with Einstein gravity

- Relevant in the realm of concentrated matter or high particle velocities
- Vlasov-Poisson: global existence of time evolution (Lions, Perthame 1991, Pfaffelmose, Schaeffer 1991/2004)
- Einstein-Vlasov system: different behavior in certain situations
- Let $(\mathscr{M}, g)$ be a 4-dim Lorentzian manifold. Einstein-Vlasov system:

$$G_{\mu\nu}[g] = 4\pi T_{\mu\nu}[f]$$

$$T^{\mu\nu}[f] = \int_{\mathbb{R}^3} f(t,\vec{x},\vec{p}) p^\mu p^\nu \frac{\sqrt{|\det g|}}{-p_0} \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3$$

$$p^0 \partial_t f(t,\vec{x},\vec{p}) + p^i \partial_{x^i} f(t,\vec{x},\vec{p}) - \Gamma^i_{\mu\nu} p^\mu p^\nu f(t,\vec{x},\vec{p}) = 0$$

Christoffel symbol: $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma}\right)$.
Einstein-Tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$
Ricci tensor: $R_{\mu\nu}$, Ricci scalar: R.

1.4 Particle trajectories

- Particles move on geodesics
- Characteristic system

$$\begin{aligned}\dot{X}^\mu(s) &= P^\mu(s),\\ \dot{P}^\mu(s) &= -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta\end{aligned}$$

- Curves $(X^\mu(s), P^\nu(s))$ in $T\mathscr{M}$
- particles' rest mass: $m^2 := -g_{\mu\nu}p^\mu p^\nu \to$ conserved
- Curves stay in the mass shell

$$\mathscr{P}_m = \{(x^\mu, p^\mu) \in T\mathscr{M} \;:\; g_{\mu\nu}p^\mu p^\nu = -m^2,\, p\,\text{future directed}\}$$

- Massshell condition yields relation for $p^0 \to f = f(t, \vec{x}, \vec{p})$.

2 Concepts in the Analysis of Speherically Symmetric Steady States

2.1 Static solutions in spherical symmetry

Spherically symmetric + static space-time ⇒ ∃ coordinates such that

$$g = -e^{2\mu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2(\vartheta)\mathrm{d}\varphi^2.$$

A solution on $\mathbb{R}_t \times \mathbb{R}^3_x$ is characterized by

- Metric functions $\mu(r)$, $\lambda(r)$
- Matter quantities

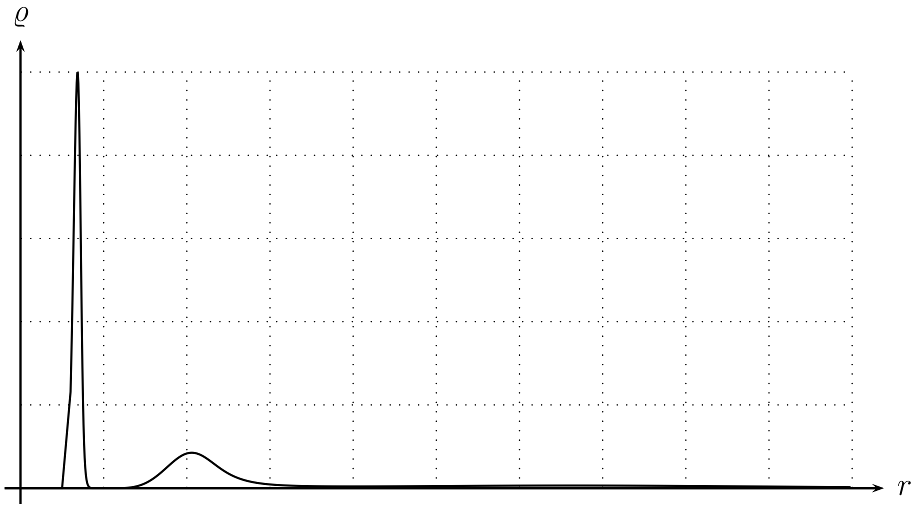
$$\varrho(r) := T_{00}(r)e^{-2\mu(r)}, \quad p(r) := T_{11}(r)e^{-2\lambda(r)}, \quad p_T(r) = g_{ij}T^{ij} - p(r).$$

Important results:

- G. Rein, A. Rendall (1992, 1993, 1999: existence, finite support),
- T. Ramming, G. Rein (2007: on finite support),
- H. Andréasson, M. Kunze, G. Rein (2014: axially symmetric solutions)

2.2 Illustration: A multishell solution

An anisotropic particle distribution forming a multishell.



The matter is arranged in shells separated by vacuum.

2.3 The Buchdahl inequality - a feature in Einstein gravity

- Hawking mass:

$$m(r) = 4\pi \int_0^r s^2 \varrho(s) \mathrm{d}s.$$

- Buchdahl inequality

$$\Gamma := \sup_{r\in(0,\infty)} \frac{2m(r)}{r} \leq \frac{8}{9}.$$

- Originally for stars where $\varrho'(r) \leq 0$, isotropic pressure (Buchdahl 1959)
- Generalized to a broad class of matter models: $\varrho(r) \geq p(r) + 2p_T(r)$ (Andréasson 2006)
- Rules out an *adiabatic black hole transition.*

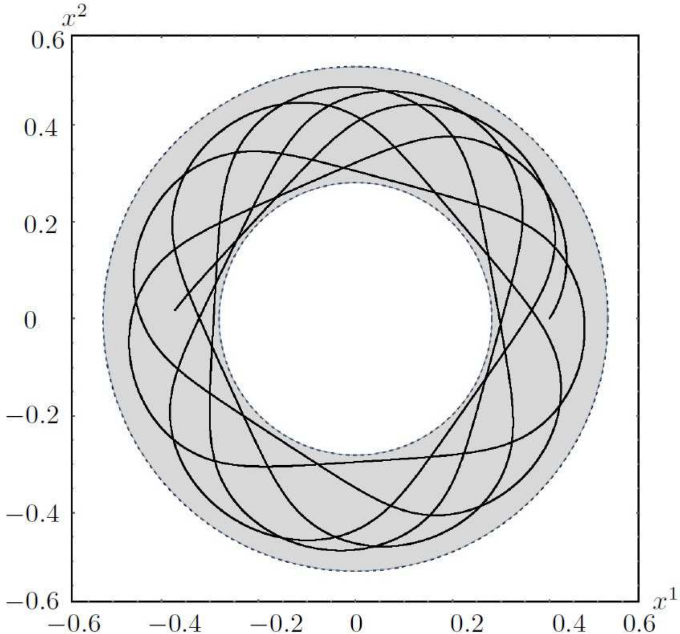
2.4 The "thin shell limit"

- Introduced by H. Andréasson in 2007 to prove that the Buchdahl inequality $\Gamma \leq \frac{8}{9}$ is sharp.
- Sequence $\{f_n, g_n, \mathbb{R}_t \times \mathbb{R}^3_x\}_{n\in\mathbb{N}}$ of static, spherically symmetric solutions; the matter quantities are supported on $[R_{1,n}, R_{2,n}]$.
- This sequence then converges to the *thin shell limit* if

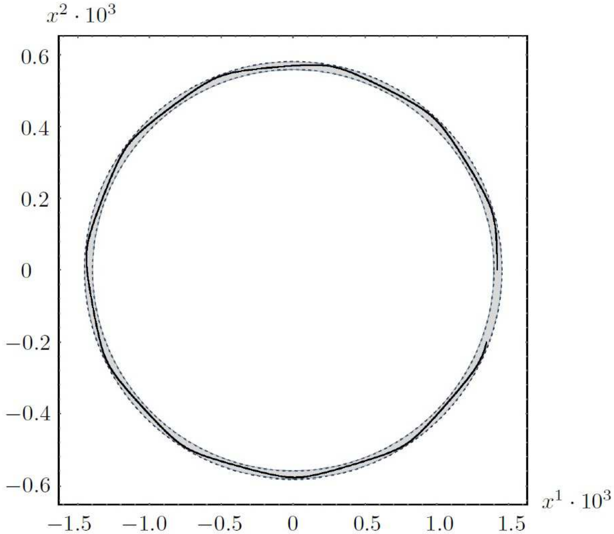
$$\frac{R_{1,n}}{R_{2,n}} \to 1, \quad \text{as} \quad n \to \infty.$$

- Properties of the thin shell limit
  - The inequality $\Gamma \leq \frac{8}{9}$ becomes sharp
  - the energy condition $\varrho \geq p + 2p_T$ becomes sharp
  - the transversal pressure p_T dominates the radial pressure p
- Important for: Buchdahl-type inequalities, massless particles, stability questions (?)

2.5 Geodesics in a shell



2.6 Geodesics in a thin shell

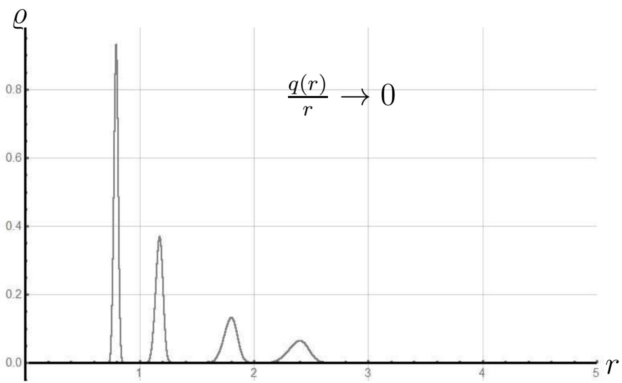


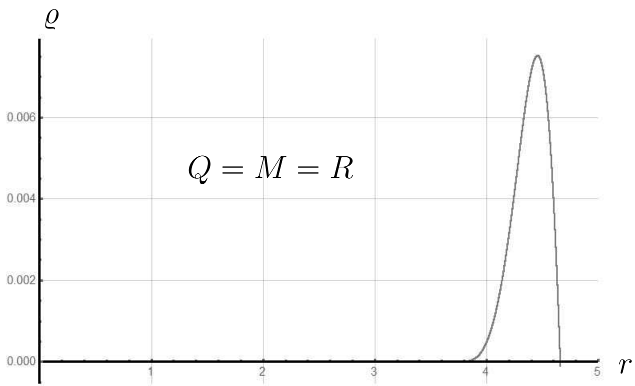
The behavior is very different from the "Einstein cluster", which has $\frac{2M}{R} \leq \frac{2}{3}$.

3 Charged Particles

3.1 Charge - a poor man's angular momentum

- With angular momentum: Stationary solutions with ergoregions observed (Andréasson, Ames, Logg 2016)
- In the charged case: Einstein-Vlasov-Maxwell system $\rightarrow$ solution characterized by $(\mu, \lambda, q, \varrho, p, p_T)$ – a Buchdahl-like inequality holds (Andréasson 2007)
- Coordinate singularity if: $\frac{q(r)}{r} \rightarrow 1$ & Buchdahl inequality saturated
- Saturation observed in two scenarios: (Andréasson, Eklund, Rein 2009)





3.2 The Einstein-Vlasov-Maxwell system

A solution of the EVM-system for particles with mass $m_0 \geq 0$ and charge $q_0 \geq 0$ is a 4-dim. Lorentzian manifold $(\mathscr{M}, g)$, a particle distribution function $f \in C(\mathscr{P}_{m_0}; \mathbb{R}_+)$, and an electro-magnetic field tensor $F \in \Lambda^2(T\mathscr{M})$ such that the EVM-system,

$$
\begin{aligned}
R_{\mu\nu}[g] - \frac{1}{2}R[g]g_{\mu\nu} &= 8\pi \left(T_{\mu\nu}[f] + \tau_{\mu\nu}[F]\right), \\
T_{\mu\nu}[f] &= g_{\mu\alpha}g_{\nu\beta} \int_{\mathscr{P}_x} f(x,p)p^\alpha p^\beta \mu_{\mathscr{P}_x}, \\
\tau_{\mu\nu}[F] &= \frac{1}{4\pi}\left(-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + F_{\nu\alpha}F_\mu{}^\alpha\right), \\
\left(v^\mu \partial_{x^\mu} + \left(q_0 F^i{}_\nu p^\nu - \Gamma^i_{\alpha\beta}p^\alpha p^\beta\right)\partial_{v^i}\right) f &= 0, \\
\mathrm{d}F &= 0, \\
\nabla_\alpha F^{\alpha\beta} &= -4\pi q_0 \int_{\mathscr{P}_x} f(x,p)p^\beta \mu_{\mathscr{P}_x}
\end{aligned}
$$

is satisfied.

3.3 The reduced static system in spherical symmetry

- Simplifications due to *staticity* and *spherical symmetry*

$$
\begin{aligned}
g &= -e^{2\mu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2,\\
f(t,x,p) &= f(r,w,L),\\
F(t,x) &= F(r).
\end{aligned}
$$

- "Elimination" of the Vlasov equation with the *method of characteristics*
  - Recall: f solves the Vlasov equation iff. $\frac{\mathrm{d}}{\mathrm{d}s}f(X^\mu(s),P^\nu(s)) = 0$
  - Identify conserved quantities E, L
  - Construct a solution of the Vlasov equation $f = \Phi(E,L)$
  - Matter quantities become explicite functions of $r, \mu, \lambda, \ldots$, like e.g.

$$\varrho(r) = g_\Phi(r, \mu(r), \ldots).$$

3.4 The system in spherical symmetry

The system reduces to a system of three coupled ordinary integro-differential equations.

An asympt. flat solution of the EVM-system is a triple $(\mu, \lambda, q) \in \left(C^1([0,\infty))\right)^3$ such that

$$\begin{aligned} e^{-2\lambda(r)} &= 1 - \frac{8\pi}{r}\int_0^r s^2 g_\Phi(s,\mu(s),I_q(s))\mathrm{d}s - \frac{1}{r}\int_0^r \frac{q^2(s)}{s^2}\mathrm{d}s, \\ \mu'(r) &= e^{2\lambda(r)}\left(4\pi r h_\Phi(r,\mu(r),I_q(r)) + \frac{4\pi}{r^2}\int_0^r s^2 g_\Phi(s,\mu(s),I_q(s))\mathrm{d}s - \frac{q^2(r)}{2r^3} + \frac{1}{2r^2}\int_0^r \frac{q^2(s)}{s^2}\mathrm{d}s\right), \\ q'(r) &= 4\pi r^2 q_0 e^{\lambda(r)} k_\Phi(r,\mu(r),I_q(r)), \end{aligned}$$

and

$$\mu(0) = \mu_c \quad \text{and} \quad \lambda(0) = q(0) = \lim_{r\to\infty}\mu(r) = \lim_{r\to\infty}\lambda(r) = 0.$$

3.5 Existence of solutions

Theorem. *Let* $(\mu_0, \lambda_0)_{\mu_c}$ *be a uncharged background solution corresponding to the central value* $\mu_c < 0$ *with matter quantities supported on* $[0, R_{\text{vac}}]$. *Let* $\Delta > 0$, *such that* $(\mu_0, \lambda_0)_{\mu_c}$ *has a vacuum region on* $[R_{\text{vac}}, R_{\text{vac}} + \Delta]$.
Then, if q_0 *is chosen sufficiently small there exists a spherically symmetric, asymptotically flat, static solution* $(\mu, \lambda, q)_{\mu_c}$ *of the Einstein-Vlasov-Maxwell system whose matter quantities are supported on* $[0, R_{\text{vac}} + \Delta]$.

3.6 Perturbation method

1. Local existence: Picard iteration. $\Psi^{\mu_c} := (\mu, \lambda, q)_{\mu_c}$ exists on the intervall $[0, \delta]$.
2. Continuation criterion: Assume the solution exists on $[0, R_c)$. If
$$e^{2\lambda(r)} = \frac{1}{1 - \frac{2m_\lambda(r)}{r}} \leq C < \infty$$
for all $r \in [0, R_c)$, then $\exists \delta > 0$ s.t. solution exists on $[0, R_c + \delta)$.
3. Consider a background solution $\Psi_0^{\mu_c}$ with vaccum region $[R_{vac}, \infty)$.
4. Bootstrap assumption: Let $R > 0$ be s.t. for some $d > 0$
$$|\Psi^{\mu_c}(r) - \Psi_0^{\mu_c}(r)| \leq d \quad \text{for all } r \in [0, T].$$
5. Improve with Grönwall inequality, provided q_0 is small. Derive
$$\|\Psi^{\mu_c} - \Psi_0^{\mu_c}\|_\infty \leq d \quad \text{on} \quad [0, R_{\text{vac}} + \Delta],$$
for some $\Delta > 0$. Deduce that Ψ^{μ_c} has a vacuum region on $[R_{\text{vac}} + \Delta, \infty)$ due to closeness to $\Psi_0^{\mu_c}$.

3.7 Thin shells

Theorem. *Let* $q_0 > 0$ *be arbitrary and let* $\hat{\mu}_c < 0$ *be chosen such that* $-\hat{\mu}_c$ *is sufficiently large. Then for all* $\mu_c \leq \hat{\mu}_c$ *there exists a solution* $(\mu, \lambda, q)_{\mu_c}$ *of the Einstein-Vlasov-Maxwell system with the same particle charge* q_0*. The matter quantities of these solutions are supported on* $[R_1(\mu_c), R_2(\mu_c)]$ *and we have*

$$\lim_{\mu_c \to -\infty} \frac{R_2}{R_1} = 1.$$

Furthermore we have

$$\lim_{\mu_c \to -\infty} \frac{q(r)}{r} = 0.$$

3.8 Constraints on the particles' support

- The particle energy is given by

$$E = e^{\mu(r)}\sqrt{m_0^2 + w^2 + \frac{L}{r^2}} - I_q(r).$$

- Assuming an ansatz Φ such that $\Phi(E) = 0$, for $E \geq E_0$ implies constraints on the support.
- Analyze the *characteristic function*

$$\gamma(r) = \ln(E_0 + I_q(r)) - \mu(r) - \frac{1}{2}\ln\left(m_0^2 + \frac{L_0^2}{r^2}\right).$$

- For small r there is a close connection to the matter quantities:

$$\varrho \approx C\frac{\gamma^\kappa}{r^4}, \quad p \approx C\frac{\gamma^{\kappa+1}}{r^4}, \quad \varrho_q \approx C q_0 e^{\lambda(r)}\frac{\gamma^\kappa}{r^3}.$$

- Choose central data μ_c such that R_1 small $\Rightarrow \exists$ radius r^* such that $2m(r^*)/r^* > 0.8$.
- γ has a zero after r^*, a vacuum exterior can be glued to the matter region.

3.9 Outlook and conclusion

Achieved results

- Static solutions, with particles of small charge parameter, close to chargeless solutions
- For arbitrary particle charges: existence of a sequence of charged solutions approaching the *thin shell limit*, saturating the Buchdahl inequality

Open questions

- New classes of saturating solutions of the EVM-system with $Q = M = R$
- (In)Stability of thin shell solutions