On the Quantum Mechanics of a Single Photon

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What is Light?

“There are phenomena which can be explained by the quantum theory but not by the wave theory. Photo-electric effect furnishes an example... There are phenomena which can be explained by the wave theory but not by the quantum theory. The bending of light around obstacles is a typical example. Finally, there are phenomena, such as the rectilinear propagation of light, which can be equally well explained by the quantum and the wave theory of light. But what is light really? Is it a wave or a shower of photons? It seems as though we must use sometimes the one theory and sometimes the other, while at times we may use either. We are faced with a new kind of difficulty. We have two contradictory pictures of reality; separately neither of them fully explains the phenomena of light, but together they do."

A. Einstein & L. Infeld, "The Evolution of Physics" (1938).

Outline

Light: Particle or Wave?

- Democritus (3rd Cent. B.C.): Light is made of particles...because *everything* is made of particles.
- Ibn-al Haytham (11th Cent.): Light is made of particles that move on straight lines.
- Descartes, Hooke, Huygens (17th Cent.): Light is made of waves.
- Newton (17th Cent.): Nope, It's made of particles.
- Young, Fresnel, ... (early 19th Cent.): Sorry Newton, it's waves: Interference, Diffraction.
- Maxwell (mid.-19th Cent.): It's waves. *Electromagnetic* waves, to be precise, and I have the equations.
- H. Hertz (late 19th Cent.): I can send and receive EM waves. They travel with the speed of light. They diffract, and reflect, just like light. They are *transverse*, not longitudinal.
- Problems with the wave theory of light: photochemistry, photoelectric effect. Energy imparted on matter by light depends on its *frequency*, not amplitude.

Particle or Wave: The Debate Rages On

- Planck (1900) Black-body radiation: The energy of radiation is absorbed or emitted in **quantized** units of $E = h\nu$.
- Einstein (1905) Light *itself* carries energy and momentum in quantized units: $E = h\nu$ and $p = \frac{h}{\lambda}$, just like a particle: Lichtquant.
- Einstein (1909): Light quanta are conceivably singularities in the electromagnetic field.
- Compton (1923): What scatters like a particle and transfers momentum like a particle, is a particle.
- Bose (1924) Light quanta as quantum particles are indistinguishable from one another, and they satisfy some weird statistics, which can be used to explain black-body radiation purely quantum mechanically. (Translation: they are bosons.)
- G. Lewis (1926): How about we call them photons instead of Lichtquant?

Particle AND Wave? (or is it a Quantized Field?)

- Einstein (ca 1920): Light quanta are *guided* by a "ghost field".
- de Broglie (1922-27): All particles, not just light, are **guided** by a wave defined on the configuration space of the particle.
- Schrödinger (1926): There are *no* particles. Everything is a matter wave, and I have the (non-relativistic) equation for it. *(To which de Broglie replied: "Don't you think it's strange to construct a wave on a space of N points that one claims don't exist?")*
- Born (1926): The modulus-squared of the wave function of a system of particles at a point in its configuration space is the probability density of detecting that configuration.
- Born, Heisenberg, Jordan (1926): Photons are the quanta of the quantized classical electromagnetic field
- Dirac (1928): Here is the relativistic wave equation for the electron.
- Oppenheimer (1931): Here is the relativistic wave equation for the photon, and here are ten reasons why the quantum mechanics of light quanta has grave problems.

Are Quantum Fields the Last Word?

- Jordan, Wigner, Heisenberg, Pauli, ... (1928-1934): Particles and waves are not fundamental. Everything is a quantum field!
- C. G. Darwin (1932): "The Compton effect, at its discovery, was regarded as a simple collision of two bodies, and yet the detailed discussion at the present time involves the idea of the annihilation of one photon and the simultaneous creation of one among an infinity of other possible ones. We would like to be able to treat the effect as a two-body problem, with the scattered photon regarded as the same individual as the incident, in just the way we treat the collisions of electrons."
- Bohm (1952): Fermions are particles guided by waves, but Bosons are fields.
- Weinberg (1995): Relativistic wave mechanics, in the sense of a relativistic quantum theory of a fixed number of particles, is an impossibility!

Our Ultimate Goal

- Inspired by Einstein's ideas and J. S. Bell's writings, we ultimately want to know whether empirical electromagnetism can be accurately accounted for in terms of a relativistic N-body quantum theory of electrons, photons, and their anti-particles, formulated as a generalization of the non-relativistic theory of de Broglie and Bohm, in which the quantum-mechanical wave function guides the actual motion of these particles.
- So, both electrons *and* photons are point-particles guided by a quantum-mechanical wave function
- This is similar in spirit to Einstein's earlier speculations about the light quanta being guided by a "ghost" field...
- ...and to the deBroglie-Bohm theory of electrons guided by their quantum-mechanical wave function.

How are we going to get there?

We envision the following steps towards accomplishing our goal:

1. Study the $N = 2$ case of one photon and one electron. This further divides into several sub-problems:
   1. Analyze the wave function and wave equation of a single electron, in order to solve the various puzzles surrounding it (negative energies, electron vs. positron, ...)
   2. Find and analyze the wave function and wave equation of a single photon.
   3. Formulate and solve the two-body quantum-mechanical problem of a photon interacting with an electron.
2. Solve the $N > 2$ problem by reducing it to the study of pair-wise interactions of the type studied in Problem 1.3
3. Prove the emergence of empirical phenomena from N-body dynamics (e.g. Compton scattering, emission/absorption of photons by atoms, Coulomb interaction of electrons, the Positronium spectrum, ...)

Bracing Ourselves for the Waves of Criticism

(Inspired by Dyson's "Three Waves" Theory of Reception)
There are three prevailing attitudes in the physics community regarding the existence of a photon wave function and a photon wave equation:

1. **This is nonsense.** There is no such thing, because:
   1. Photons don't have a position.
   2. Photons don't even exist.
   3. Weinberg and Witten proved that this is impossible
   4. "Many other reasons"
2. **This is trivial.** And, it won't change anything: The electromagnetic field is the photon wave function, and the wave equation it satisfies is equivalent to Maxwell's equations. So, photons will still be quantized electromagnetic fields: QED!
3. This is actually nontrivial and important, which is why **we did it before you did.**

- Michael Kiessling and I have formulated a relativistic quantum-mechanical wave function and wave equation for a single photon in $1 + d$-dimensional Minkowski spacetime.
  It is a direct analog of the Dirac equation for the electron, and it furnishes a conserved quantum probability current that transforms correctly under Lorentz transformations.
- Together with Matthias Lienert, we have formulated and studied a fully covariant interacting 2-body problem for an electron and a photon in 1+1-dimensional Minkowski spacetime!

THEOREM I (Kiessling & T.-Z., 2017)

Let $\mathcal{A} := \text{Cl}(1, d)_{\mathbb{C}}$ denote the complexified Clifford algebra associated with the Minkowski spacetime $\mathbb{R}^{1,d}$. (Note: For odd d, $\mathcal{A} \cong M_n(\mathbb{C})$ with $n = 2^{(d+1)/2}$.) Then,

- In d space dimensions, the wave function ψ_{PH} for a single photon (massless, chargeless, spin-1 boson with no longitudinal modes) is a section of the $\mathcal{A}$-bundle over $\mathbb{R}^{1,d}$, with the property that the two main diagonal $\frac{n}{2} \times \frac{n}{2}$ blocks of ψ_{PH} are trace-free.
- The wave equation satisfied by ψ_{PH} is a Dirac-type equation (with modifications inspired by M. Riesz (1946) and Harish-Chandra (1946)):

$$-i\hbar \mathbf{D}\psi_{\text{PH}} + m_{\not{z}} \Pi \psi_{\text{PH}} = 0, \tag{1}$$

  where $\mathbf{D} = \gamma^\mu \partial_\mu$ is the Dirac operator on $\mathbb{R}^{1,d}$, Π is the projection onto the said diagonal blocks, and $m_{\not{z}} > 0$ is a parameter to be determined.

THEOREM I (cont'd.)

- Equation (1) is fully Lorentz covariant.
- Equation (1) is invariant under gauge transformations

$$\psi_{\text{PH}} \to \psi_{\text{PH}} + (\mathbb{1} - \Pi)\Upsilon, \qquad \forall \Upsilon \text{ s.t. } \mathbf{D}\Upsilon = 0.$$

- Every component of ψ_{PH} satisfies the massless Klein-Gordon (classical wave) equation

$$\Box \psi_{\text{PH}} = \partial^\mu \partial_\mu \psi_{\text{PH}} = 0.$$

  As a result, the plane-wave solutions $\psi(t, \mathbf{x}) = \mathbf{a} e^{i(\omega t + \mathbf{k}\cdot\mathbf{x})}$ of (1) satisfy the zero-mass dispersion relation

$$\omega^2 = |\mathbf{k}|^2.$$

- The diagonal part of the wave function $\Pi\psi_{\text{PH}}$ satisfies the massless Dirac equation

$$-i\hbar \mathbf{D} \Pi \psi_{\text{PH}} = 0. \tag{2}$$

  As a result, there are no longitudinal modes.

THEOREM I (cont'd.)

- The photon wave equation (1) is derivable from a Lorentz-scalar Lagrangian. Thus the solutions of (1) enjoy the full set of Noetherian conservation laws associated with the continuous isometries of the Minkowski space.
- Equation (2) for the diagonal part of the photon wave function can be cast into the Hamiltonian form, with a Hamiltonian H that is essentially self-adjoint with respect to the L^2 innerproduct. The spectrum of H is all of $\mathbb{R}$, with generalized eigenvectors being plane waves satisfying the Einstein relations

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}.$$

- Equation (2) has a conserved timelike probability current that transforms correctly under Lorentz transformations, has a non-negative time-component, and is compatible with the Born Rule associated with the same Hilbert space structure as that of the Hamiltonian H.

The Multi-time Evolution for 2-Body Wave Functions

- A fully covariant notion of a many-body wave function requires treating each particle as a space-time *event.*
- Thus the configuration space(time) of two particles in Minkowski space $\mathbb{R}^{1,d}$ is in fact a domain $\mathcal{S}$ in $\mathcal{M} := \mathbb{R}^{1,d} \times \mathbb{R}^{1,d}$.
- Coordinates on $\mathcal{M}$: $x = (x_p, x_e) = (t_p, \mathbf{s}_p, t_e, \mathbf{s}_e)$.
- It turns out that the natural domain $\mathcal{S}$ to consider is the set of *space-like* configurations $\mathcal{S} = \{(x_p, x_e) \in \mathcal{M} \mid |t_p - t_e| < |\mathbf{s}_p - \mathbf{s}_e|\}$
- One can pose a Cauchy problem in $\mathcal{S}$ for the multi-time evolution of a 2-body wave function $\psi(x_p, x_e)$, by specifying initial values for ψ on the *initial surface* $\mathcal{I} := \{(x_p, x_e) \in \mathcal{M} \mid t_p = t_e = 0\} \subset \overline{\mathcal{S}}$.
- Since $\mathcal{S}$ has a boundary in $\mathcal{M}$, one may also need to impose boundary conditions.
- Boundary conditions in particular need to be specified on the *coincidence set* $\mathcal{C} := \{(x_p, x_e) \in \mathcal{M} \mid t_p = s_p, \mathbf{s}_p = \mathbf{s}_e\} \subset \partial\mathcal{S}$.

The Two-Body Wave Function and its Wave Equation(s)

- Recall the *one*-body wave functions for $d = 1$:
- $\psi_{\text{PH}} = \begin{pmatrix} 0 & \chi_- \\ \chi_+ & 0 \end{pmatrix} \in V_2 \subset \mathbb{C}^{2\times 2}$ and $\psi_{\text{EL}} = \begin{pmatrix} \phi_- \\ \phi_+ \end{pmatrix} \in \mathbb{C}^2$
- The 2-body WF belongs to a 4-dim. tensor product space $V_2 \otimes \mathbb{C}^2$.
- $\psi = (\psi_{--}, \psi_{-+}, \psi_{+-}, \psi_{++}) : \mathcal{S} \to \mathbb{C}^4$, so $\psi = \psi(t_p, s_p, t_e, s_e)$
- Let $\gamma_p^\mu := \gamma^\mu \otimes \mathbb{1}$, $\gamma_e^\nu := \mathbb{1} \otimes \gamma^\nu$, $\mathbf{D}_p := \gamma_p^\mu \partial_{x_p^\mu}$, $\mathbf{D}_e := \gamma_e^\nu \partial_{x_e^\nu}$.
- The (free) *multi-time equations are:*

$$\left\{ \begin{array}{rcl} -i\hbar \mathbf{D}_p \psi & = & 0 \\ -i\hbar \mathbf{D}_e \psi + m_e \psi & = & 0 \end{array} \right. \tag{3}$$

- *Note: this is ok because* $[\mathbf{D}_p, \mathbf{D}_e] = 0$.
- *We need to specify initial data:* $\psi(0, s_p, 0, s_e) = \mathring{\psi}(s_p, s_e)$.
- *We also need* boundary conditions *on* $\mathcal{C}$. *Otherwise the dynamics will only be defined in the complement of the future of* $\mathcal{C}$.

THEOREM II (Lienert, Kiessling, & T.-Z., 2018)

- Let $\mathcal{S}_1 := \{(t_p, s_p, t_e, s_e) \in \mathcal{S} \mid s_p < s_e\}$ and $X = (X^0, X^1)$ any constant timelike vector in $\mathbb{R}^{1,1}$. The following IBVP for the 2-body photon-electron multi-time wave function $\psi : \overline{\mathcal{S}_1} \to \mathbb{C}^4$,

$$\begin{cases} -i\hbar D_p \psi &= 0 & \\ -i\hbar D_e \psi + m_e \psi &= 0 & \text{in } \mathcal{S}_1 \\ \psi &= \mathring{\psi} & \text{on } \mathcal{I} \\ \psi_{+-} &= \pm i\sqrt{\frac{X^0 - X^1}{X^0 + X^1}}\psi_{-+} & \text{on } \mathcal{C} \end{cases} \tag{4}$$

  has a unique global solution that is supported in $\overline{\mathcal{S}_1}$, depends continuously on the initial data $\mathring{\psi}$, and is continuous everywhere for all times.
- (PRELIMINARY) $\exists T > 0$ such that Bohmian trajectories $Q_p(t)$ and $Q_e(t)$ for the photon and the electron exist, are unique, and $Q_p(t) \leq Q_e(t)$ for $0 \leq t < T$. (See Figure 1.)

List of Requirements for PWF and PWE

To come up with a quantum-mechanical wave function and wave equation for a single photon in position-space representation, we first list the key properties that these should have:

- Photon Wave Function needs to
  1. correspond to a spin-1 particle
  2. allow for superposition of left- and right-handed photons
  3. belong to a Hilbert space over $\mathbb{C}$
- Photon Wave Equation (for the above PWF) needs to
  1. be fully Lorentz-covariant
  2. be first order in time- and space-derivatives
  3. have zero-mass dispersion relation
  4. have only transverse modes
  5. be derivable from a Lorentz-scalar Lagrangian
  6. have a conserved Lorentz-covariant vector current, with a non-negative time-component, that is compatible with Born's Rule.

Constructing PWF

- To find the photon wave function, recall that, for $d = 3$, the wave function of an electron, which is a spin-$\frac{1}{2}$ particle, is a *rank-one bispinor*.
- That is to say, $\psi_{\mathrm{EL}} : \mathbb{R}^{1,3} \to \mathbb{C}^4$, such that, for all $\Lambda \in O(1,3)$, we have

$$\psi_{\mathrm{EL}}(\Lambda x) = L_\Lambda \psi_{\mathrm{EL}}(x)$$

  where L_Λ is the spinorial (projective) representation of Λ, given by the usual $SL(2,\mathbb{C}) \oplus SL(2,\mathbb{C})$ construction.
- By analogy, the wave function of a spin-1 particle needs to be a rank-*two* bispinor, i.e. an endomorphism of rank-one bispinors: $\psi_{\mathrm{PH}} : \mathbb{R}^{1,3} \to M_4(\mathbb{C})$, with transformation rule

$$\psi_{\mathrm{PH}}(\Lambda x) = L_\Lambda \psi_{\mathrm{PH}}(x) L_\Lambda^{-1}$$

Constructing PWF (cont'd.)

- Thus, $\psi_{\text{PH}} = \begin{pmatrix} \psi_+ & \chi_- \\ \chi_+ & \psi_- \end{pmatrix}$, with $\psi_\pm, \chi_\pm \in M_2(\mathbb{C})$ being rank-two spinors of four different types:

$$(\psi_+)^a_b, \quad (\chi_-)^a_{\dot{b}}, \quad (\chi_+)^{\dot{a}}_b, \quad (\psi_-)^{\dot{a}}_{\dot{b}}.$$

- Spinors of even rank are equivalent to ordinary vectors and tensors.
- in particular, there exist (complex-valued) vectors $\mathbf{a}_\pm$ and 2-tensors $\mathbf{f}_\pm$, such that

$$\chi_+ = \mathbf{a}^\mu_+ \sigma'_\mu, \quad \psi_+ = \frac{i}{4}\mathbf{f}^{\mu\nu}_+ \sigma_\mu \sigma'_\nu, \quad \chi_- = \mathbf{a}^\mu_- \sigma_\mu, \quad \psi_- = \frac{i}{4}\mathbf{f}^{\mu\nu}_- \sigma'_\mu \sigma_\nu,$$

- Here, $(\sigma_\mu) = (\mathbb{1}_{2\times 2}, \sigma_x, \sigma_y, \sigma_z)$ and $(\sigma'_\mu) = (\mathbb{1}_{2\times 2}, -\sigma_x, -\sigma_y, -\sigma_z)$.

Constructing PWE

- To find the right photon wave equation, we first note that *Maxwell's equations* satisfy at least *some* of the requirements we have listed.
- Moreover, it is possible to write Maxwell's equations *in spinor form*:
- Let **a** be a 1-form and **f** a 2-form on $\mathbb{R}^{1,3}$. The source-free Maxwell's equations for an EM field tensor **f** and 4-potential **a** (in Lorentz gauge,) are

$$d\mathbf{a} = \mathbf{f}, \quad \delta\mathbf{a} = 0, \quad d\mathbf{f} = 0, \quad \delta\mathbf{f} = 0. \tag{5}$$

- To put these in spinor form, define

$$\chi_+ := \mathbf{a}^\mu \sigma'_\mu, \qquad \psi_+ := \frac{i}{4}\mathbf{f}^{\mu\nu}\sigma_\mu\sigma'_\nu.$$

Constructing PWE (cont'd.)

- Then the above Maxwell's equations (5) are *almost* equivalent to

$$\sigma^\mu \partial_\mu \chi_+ = \psi_+, \qquad \sigma'^\mu \partial_\mu \psi_+ = 0. \tag{6}$$

- The only difference being their behavior under **parity**
- (5) is invariant under parity, while (6) under parity goes to

$$\sigma'^\mu \partial_\mu \chi_- = \psi_-, \qquad \sigma^\mu \partial_\mu \psi_- = 0. \tag{7}$$

- Putting (6) and (7) together *seems to* yield our PWE (1).
- But, we are not done yet!
- So far, ψ_+ and ψ_- correspond to *the same* 2-form **f**.
- This implies that $\psi_+^\dagger = \psi_-$. This PWF is *self-dual*.
- Self-duality is *not* preserved under multiplication by a complex scalar! $\implies$ We will not get a Hilbert space over $\mathbb{C}$!
- Remedy: use two different **f**'s: $\mathbf{f}_+$ for ψ_+ and $\mathbf{f}_-$ for ψ_-.
- Now we can have superposition of LH and RH photons!

Lagrangian Formulation and Conservation Laws

- Our PWE (1) is derivable from a Lorentz-scalar Lagrangian:

$$\mathcal{L} = \frac{\hbar}{16\pi i}\mathrm{tr}\left(\overline{\psi_{\mathrm{PH}}}\gamma^\mu\partial_\mu\psi_{\mathrm{PH}} - \partial_\mu\overline{\psi_{\mathrm{PH}}}\gamma^\mu\psi_{\mathrm{PH}}\right) + \frac{m_{\not{\iota}}}{8\pi}\mathrm{tr}\left(\overline{\psi_{\mathrm{PH}}}\Pi\psi_{\mathrm{PH}}\right), \quad (8)$$

- Here, $\overline{\psi} = \gamma^0\psi^\dagger\gamma^0$ is the Dirac adjoint for rank-2 bispinors, and tr is the (matrix) trace operator.
- The associated (symmetrized) canonical stress for (8) is

$$\Theta_{\mu\nu} = \frac{\hbar}{32\pi i}\mathrm{tr}\left\{\overline{\psi_{\mathrm{PH}}}\gamma_\nu\partial_\mu\psi_{\mathrm{PH}} + \overline{\psi_{\mathrm{PH}}}\gamma_\mu\partial_\nu\psi_{\mathrm{PH}} - \mathrm{adj.}\right\}$$

- It satisfies $\nabla^\mu\Theta_{\mu\nu} = 0$. Thus it can be used to generate all the Noetherian domain conservation laws of the PWE associated with the continuous symmetries of $\mathbb{R}^{1,d}$: Energy, Linear Momentum, Angular Momentum, and Centroid position.
- There are also (previously unknown) integral conservation laws associated with symmetries of the fiber, e.g. the *cross-linkage integral.*

Relationship to Maxwell Fields

Recall the 1-forms $\mathbf{a}_\pm$ and 2-forms $\mathbf{f}_\pm$ that appeared inside PWF ψ_{PH}.

- Choosing a frame, we can re-name the components of these forms: $a^0_\pm = \phi_\pm, \quad a^k_\pm = A^k_\pm, \quad -f^{0k}_\pm = e^k_\pm, \quad -*f^{0k}_\pm = b^k_\pm.$
- It will then follow that $\chi_\pm = \phi_\pm \mathbb{1} \mp \boldsymbol{\sigma} \cdot \mathbf{A}_\pm, \quad \psi_\pm = \pm i\boldsymbol{\sigma} \cdot (\mathbf{e}_\pm \pm i\mathbf{b}_\pm)$.
- As a consequence of PWE (1), the pairs $(\mathbf{e}_+, \mathbf{b}_+)$ and $(\mathbf{e}_-, \mathbf{b}_-)$ satisfy Maxwell's equations for classical E and B fields.
- Via a gauge transformation, $(\phi_\pm, \mathbf{A}_\pm)$ can be made real, and can be viewed as "scalar and vector potentials" (in Lorenz gauge) for those E and B fields.
- The conserved quantities we found can be re-expressed in terms of these "electromagnetic" quantities. In particular:
- Energy = $\frac{m_{\not{\iota}} c^2}{8\pi} \int_{\mathbb{R}^3} [\mathbf{e}_+ \cdot \mathbf{e}_- + \mathbf{b}_+ \cdot \mathbf{b}_-] \, d^3x$
- Cross Linkage = $\frac{\hbar}{8\pi m_{\not{\iota}} c} \lim_{r\to\infty} \int_{S_r} (\mathbf{a}_- \times \mathbf{a}_+) \cdot \mathbf{n} dS$.

Hamiltonian Formulation

Recall that the diagonal part of the PWF satisfies the massless Dirac eq. (2): $-i\hbar\gamma^\mu\partial_\mu\phi_{\text{PH}} = 0$, where $\phi_{\text{PH}} := \Pi\psi_{\text{PH}}$.

- Choosing a time function t for $\mathbb{R}^{1,d}$, the above has a Hamiltonian reformulation: $i\hbar\frac{\partial}{\partial t}\phi_t = H\phi_t$, where $\phi_t(\mathbf{x}) := \phi_{\text{PH}}(t,\mathbf{x})$, and

$$H := -i\hbar c\gamma^0\gamma^k\partial_k = -i\hbar c \begin{pmatrix} \boldsymbol{\sigma}\cdot\nabla & 0 \\ 0 & -\boldsymbol{\sigma}\cdot\nabla \end{pmatrix}$$

- H is easily seen to be symmetric and essentially self-adjoint with respect to the Hilbert space innerproduct

$$\langle\phi_t|\zeta_t\rangle := \int_{\mathbb{R}^3} \text{tr}\left(\phi_t^\dagger\zeta_t\right) d^3x,$$

- It follows that $\text{Spec}(H) = \mathbb{R}$, and the generalized energy and momentum eigenvalues of H satisfy the Einstein (deBroglie) energy-frequency relations: $E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}.$

Existence of a Photon Particle Current

- None of the conserved quantities mentioned so far can play the role of a *particle current* for photons.
- That is, *a future-directed causal Lorentz-vector* j^μ, *constructed only from the PWF*, such that $\partial_\mu j^\mu = 0$.
- If such a vector can be found, then upon choosing a Lorentz frame and setting $\rho := j^0 \geq 0$, $v^k := j^k/\rho$, we obtain the continuity equation $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$, which impliest that $\frac{d}{dt} \int \rho(t, \mathbf{x}) d\mathbf{x} = 0$.
- Assuming $\int_{\mathbf{R}^3} \rho(0, \mathbf{x}) d\mathbf{x} < \infty$, by normalizing the PWF we can interpret $\rho(t, \mathbf{x})$ as the *probability density of detecting the photon* at time t at position $\mathbf{x}$ in space. (When ρ is *quadratic* in the normalized WF, this is called **Born's Rule.**)
- We can further identify $\mathbf{v}$ as the (deBroglie-Bohm) velocity field of the actual photon trajectories $\mathbf{x} = \mathbf{Q}(t)$
- Thus the actual particle trajectories can be found by solving the ODE $\frac{d\mathbf{Q}}{dt} = \mathbf{v}(t, \mathbf{Q}(t))$

Constructing the Photon Particle Current

1. We show that given any (non-null) Killing field X of $\mathbb{R}^{1,d}$, there is a conserved current j^μ_X that depends only on X and $\phi_{\text{PH}} = \Pi\psi_{\text{PH}}$.
2. We show how to construct a *constant* (and therefore Killing) vectorfield X that depends *only* on ϕ_{PH}

- To do this, we first define the Riesz tensor: $\tau^{\mu\nu} = \frac{1}{4}\text{tr}\left(\overline{\phi_{\text{PH}}}\gamma^\mu\phi_{\text{PH}}\gamma^\nu\right)$
- We prove that τ is symmetric, and $\nabla^\mu\tau_{\mu\nu} = 0$ when ϕ_{PH} satisfies the massless Dirac eq. (2)
- We prove that τ satisfies the *Dominant Energy Condition*, i.e.
  1. $\tau^\mu_\nu Y^\nu$ is future directed causal whenever Y is so.
  2. $\tau_{\mu\nu}Y^\mu Z^\nu \geq 0$ whenever Y and Z are future-directed and causal.
- We now set $j^\mu_X := \tau^\mu_\nu X^\nu$ for X the given Killing field.
- This completes Step 1.

Constructing the Photon Particle Current, Step 2

- Let $\{\mathbf{e}_{(\mu)}\}$ be any fixed Lorentz-orthogonal frame of *constant* unit vectorfields for $\mathbb{R}^{1,d}$.
- By Step 1, to every Killing field Y of the Minkowski space, there corresponds a number $\pi_Y := \int_{\mathbb{R}^d} j_Y^0(t, \mathbf{x}) d\mathbf{x}$. (Note: since j_Y is a conserved current this is independent of t).
- Since each of the constant vectors $\mathbf{e}_{(\mu)}$ is itself a Killing field of Minkowski space, we can define $\pi_\mu := \pi_{\mathbf{e}_{(\mu)}}$.
- It's easy to see that under a Lorentz transformation, (π^μ) transform like the components of a Lorentz vector $\boldsymbol{\pi}$.
- Moreover, $\boldsymbol{\pi}$ is future-directed and causal.
- If $\boldsymbol{\pi}$ is not null, we set $X := \frac{1}{\pi_\mu \pi^\mu} \boldsymbol{\pi}$.
- We define the photon probability current to be

$$j^\mu_{\mathrm{PH}} = j^\mu_X = \frac{1}{4} \mathrm{tr} \left(\overline{\phi_{\mathrm{PH}}} \gamma^\mu \phi_{\mathrm{PH}} \gamma_\nu X^\nu \right)$$

with X as defined in the above.

The Two-Particle Probability Current

- Recall the currents $j^\mu_{\rm EL} := \overline{\psi_{\rm EL}}\gamma^\mu\psi_{\rm EL}$ and $j^\mu_{\rm PH} := \frac{1}{4}\mathrm{tr}\left(\overline{\psi_{\rm PH}}\gamma^\mu\psi_{\rm PH}\gamma_\nu X^\nu_{\rm PH}\right)$.
- We can define a 2-particle current by taking their tensor product:

$$j^{\mu\nu} := \frac{1}{4}\mathrm{tr}_p\left(\overline{\psi}\gamma^\mu_p\gamma^\nu_e\psi\gamma^\alpha_p X_\alpha\right)$$

- The constant timelike vector X is constructed from the *initial data* $\mathring{\psi}$ of the 2-body wave function in a similar way to $X_{\rm PH}$
- Eq. (3) implies that $j^{\mu\nu}$ is *mutually* conserved:

$$\partial_{x_p^\mu} j^{\mu\nu} = 0, \text{ for } \nu = 0,1, \text{ and } \partial_{x_e^\nu} j^{\mu\nu} = 0, \text{ for } \mu = 0,1.$$

- Since $\partial\mathcal{S} \neq \emptyset$, in order to obtain *probability conservation* (i.e., the *integral form of the current conservation law) we need to know something about the boundary values of $j^{\mu\nu}$ on $\mathcal{C}$.*

Particle Interaction Via Boundary Conditions

- One BC compatible with probability conservation is

$$\lim_{\varepsilon\to 0} \epsilon_{\mu\nu} j_X^{\mu\nu}(t, s-\varepsilon, t, s+\varepsilon) = 0 = \lim_{\varepsilon\to 0} \epsilon_{\mu\nu} j_X^{\mu\nu}(t, s+\varepsilon, t, s-\varepsilon). \quad (9)$$

- It corresponds to no current crossing the boundary, i.e. the electron and the photon not "going through" each other.
- Recall Compton Scattering: electron and photon seemed to "bounce off" of one another.
- Since the joint wave function satisfies a free evolution in each set of particle coordinates, the particles are interacting *solely* via (9).
- The requirement of Lorentz-invariance greatly restricts the possible choices for the boundary values of ψ:
- **THEOREM**: The *only* Lorentz-invariant BCs that yield (9) are

$$\lim_{\varepsilon\to 0} \psi_{+-}(t, s\mp\varepsilon, t, s\pm\varepsilon) = \pm\varsigma\, i \sqrt{\frac{X^0 - X^1}{X^0 + X^1}} \lim_{\varepsilon\to 0} \psi_{-+}(t, s\mp\varepsilon, t, s\pm\varepsilon)$$

with $\varsigma = 1$ or -1.

Solving the Initial-Boundary Value Problem

- Recall that ψ satisfies *two* equations:
  1. The massless Dirac eq., in the photon variables (t_p, s_p)
  2. The massive Dirac eq., in the electron variables (t_e, s_e)
- For $d = 1$, the photonic Dirac eq. says that each component of ψ satisfies a transport equation, so that it is either a function of $s_p + t_p$ or a function of $s_p - t_p$.
- The electronic Dirac eq. implies that each component of ψ satisfies a Klein-Gordon equation in the (t_e, s_e) variables.
- Both of the above equations are exactly solvable, at least in the non-interacting case, i.e. so long as no waves have reached the boundary $\mathcal{C}$.
- Let $\mathcal{R}$ denote the complement of the future of $\mathcal{C}$ (see figure). Existence and uniqueness of ψ at points in $\mathcal{R}$ is thus guaranteed.

Solving the IBVP (cont'd)

- For two of the components, ψ_{--} and ψ_{-+}, the backwards characteristics emanating from a point in $\mathcal{S}_1$ do not intersect the boundary $\mathcal{C}$, hence these two components are completely determined everywhere in $\mathcal{S}_1$, not just in $\mathcal{R}$.
- For ψ_{+-}, one of the backward characteristics hits the boundary $\mathcal{C}$, where the boundary condition can be used to find the values for ψ_{+-} from the known values for ψ_{-+}.
- The other backward characteristic for ψ_{+-} hits $\partial\mathcal{R}$ (which is itself a characteristic surface), where ψ_{+-} is known by continuity.
- The component ψ_{+-} thus satisfies a Goursat (characteristic Cauchy) problem, which is again exactly solvable.
- Finally, ψ_{++} can be found from ψ_{+-} by integration along a characteristic.
- Joint probability density $\rho = |\psi_{--}|^2 + |\psi_{-+}|^2 + |\psi_{+-}|^2 + |\psi_{++}|^2$. (See Movie 1.)

Existence and Uniqueness of Particle Trajectories

- The *Hypersurface Bohm-Dirac* model is a relativistic multi-particle extension of the deBroglie-Bohm guiding law. It requires:
  1. A relativistic current that is a tensor product of 1-body currents
  2. A spacelike foliation of Minkowski spacetime.
- Both of these are available to us, thanks to the current $j^{\mu\nu}$ and the constant timelike vector X built out of the initial wave function.
- Let us go into a frame in which $\hat{X} = \mathbf{e}_{(0)}$, and denote by Σ_t the leaves of the foliation induced by $\hat{X}$
- Let $Q_p(t)$ and $Q_e(t)$ denote the space coordinate of the intersections of the actual photon, resp. electron, trajectory with the Cauchy surface Σ_t. These satisfy the ODE system

$$\frac{dQ_p(t)}{dt} = \frac{j^{10}(Q_p(t), Q_e(t))}{j^{00}(Q_p(t), Q_e(t))}, \qquad \frac{dQ_e^1(t)}{dt} = \frac{j^{01}(Q_p(t), Q_e(t))}{j^{00}(Q_p(t), Q_e(t))}.$$

- It is standard that a Lipschitz right-hand-side will imply local existence and uniqueness of solutions to this ODE system.

Summary

In this talk we have shown that

- Despite previous claims to the contrary (made by some prominent physicists), a relativistic quantum-mechanical wave function and wave equation for a single photon exists, and has all the properties needed in order to treat the photon just like the electron, i.e. as a point-particle guided by its wave function defined on its configuration spacetime.
- At least in one space dimension, one can formulate and solve a fully relativistic interacting 2-body quantum-mechanical system of one photon and one electron, obtaining results that are compatible with the Compton Scattering of electrons by photons.

Outlook

We are currently working on the following extensions of the above results:

- Formulate the PWF and PWE on spacetimes that are curved and/or have non-trivial topology.
- Study the emergence of classical electromagnetic field in the limit when the number of photons goes to infinity.
- Study the emission/absorption phenomena for $d = 1$.
- Study the interaction of two electrons mediated by a photon, for $d = 1$.
- Find a way to model the interaction of photons and electrons in $d = 3$ space dimensions.

THANK YOU FOR LISTENING!