Linear perturbations of special spacetimes

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Motivation

Linear perturbations of the Kerr spacetime

- Black hole stability problem.
- Uniqueness problem.
- Self force problem.

Methods and motivation

- Covariant geometric methods based on a conformal Killing-Yano tensor or Killing spinor.
- Don't have to care much about coordinates.
- Much works for a larger class than just Kerr. Vacuum Petrov type D or Kerr-NUT. (Focus on Kerr)
- The Killing spinor can be approximated on general backgrounds.

Outline

- Spinor notation, linearized curvature notation.
- Symmetries and structure of the Kerr black hole (Petrov type D)
- Gauge invariant quantities
- Hyperbolic evolution equations for some gauge invariants
- Symmetry operators. (briefly)
- Summary and concluding remarks

Notation

Will use abstract tensor notation and 2-spinor notation.

Tensors	**Spinors**
4D real bundles	2D complex bundles
T_a	$T_{AA'}$
∇_a	$\nabla_{AA'}$
Symmetric metric g_{ab}	Antisymmetric metric ϵ_{AB}
Symmetric trace-free tensors	Symmetric spinors
Ricci scalar R	24Λ
Trace-free Ricci S_{ab}	$-2\Phi_{ABA'B'}$
Weyl tensor C_{abcd}	$\Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}$
Conformal Killing-Yano $Y_{ab} = Y_{[ab]}$ $\nabla_{(a}Y_{b)c} = \frac{1}{3}g_{c(a}\nabla^d Y_{b)d} - \frac{1}{3}g_{ab}\nabla_d Y_c{}^d.$	Killing spinor $\kappa_{AB} = \kappa_{(AB)}$ $\nabla_{A'(A}\kappa_{BC)} = 0$

Irreducible decompositions $\Rightarrow$ Symmetric spinors times ϵ $\Rightarrow$ Use symmetric spinors.

Linearized curvature

Covariant metric perturbations around a background g_{ab} metric (vacuum)

$$\tilde{g}_{ab} = g_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2)$$

- Linearized metric h_{ab}.
- Linearized Riemann in tensor form

$$\dot{R}_{abcd} = 2g_{f[d}\nabla_{c]}\nabla_{[a}h_{b]}{}^{f} + \tfrac{2}{3}R_{[ab]}{}^{f}{}_{[c}h_{d]f} - \tfrac{2}{3}R^{f}{}_{[ab][c}h_{d]f}\,,$$

- Linearized Weyl, tracefree Ricci and Ricci scalar in spinor form

$$\vartheta\Psi_{ABCD} = \tfrac{1}{2}\nabla_{(A}{}^{A'}\nabla_{B}{}^{B'}h_{CD)A'B'} - \tfrac{1}{4}\Psi_{ABCD}h^{F}{}_{F}{}^{A'}{}_{A'}\,,$$
$$\vartheta\Phi_{AB}{}^{A'B'} = \tfrac{1}{2}\nabla^{C(A'}\nabla_{(A}{}^{|C'|}h_{B)CC'}{}^{B')} + \tfrac{1}{2}\Psi_{AB}{}^{CD}h_{CD}{}^{A'B'}\,,$$
$$\vartheta\Lambda = \tfrac{1}{12}\nabla_{CB'}\nabla^{(B}{}_{A'}h^{C)}{}_{B}{}^{A'B'}\,.$$

Special spacetimes

Vacuum Petrov type D

- A Petrov type D spacetime has two repeated principal spinors o_A, ι_A, and we can write
$$\Psi_{ABCD} = \frac{1}{6}\Psi_2 o_{(A} o_B \iota_C \iota_{D)}.$$
- (Walker & Penrose 1970) Vacuum type D $\Rightarrow$ existence of
a Killing spinor κ_{AB}, $\nabla_{A'(A}\kappa_{BC)} = 0$
or a conformal Killing-Yano tensor Y_{ab}, $\nabla_{(a} Y_{b)c} = \frac{1}{3} g_{c(a}\nabla^d Y_{b)d} - \frac{1}{3} g_{ab}\nabla_d Y_c{}^d$.
- Will study such spacetimes (or more special) and build all structures from κ_{AB} or Y_{ab}.

Vacuum Petrov type D

Let $\mathcal{Y}_{ab} = \frac{1}{2}(Y_{ab} + i{*}Y_{ab})$ be anti-self dual conformal Killing-Yano tensor related to the Killing spinor κ_{AB} via

$$\mathcal{Y}_{ab} = \tfrac{3}{2} i \bar{\epsilon}_{A'B'} \kappa_{AB}, \qquad \text{and let} \qquad p = \sqrt{\mathcal{Y}_{bd}\mathcal{Y}^{bd}}.$$

From the 2-form $\mathcal{Y}_{ab}$, we can construct a Killing vector

$$\xi^c = \tfrac{2}{3} i \nabla_a \mathcal{Y}^{ca} = \nabla^{BC'} \kappa^C{}_B.$$

If ξ^c is real, we can also construct

- a Killing tensor $K_{ab} = -Y_a{}^c Y_{bc}$, $\nabla_{(a} K_{bc)} = 0$.
- a second Killing vector

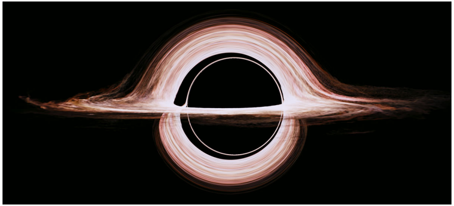
$$\zeta^a = K^{ab}\xi_b = 2\mathcal{Y}^{ab}\overline{\mathcal{Y}}_{bc}\xi^c - \tfrac{1}{4}(p^2 + \bar{p}^2)\xi^a.$$

Kerr black holes

The Kerr metric

$$g_{ab} = -\frac{(a^2\sin^2\theta - \Delta)dt_a dt_b}{\Sigma} - \frac{\Sigma dr_a dr_b}{\Delta} - \Sigma d\theta_a d\theta_b + \frac{\sin^2\theta\left(a^2\sin^2\theta\Delta - (a^2+r^2)^2\right)d\phi_a d\phi_b}{\Sigma} + \frac{2a\sin^2\theta(a^2+r^2-\Delta)dt_{(a}d\phi_{b)}}{\Sigma},$$

where $\Delta = a^2 - 2Mr + r^2$ and $\Sigma = a^2\cos^2\theta + r^2$.



- Vacuum Petrov type D spacetime describing a rotating black hole.
- M is mass and a angular momentum parameter with $|a| \leq M$.
- Will not use coordinate formulation.

Symmetries of Kerr

Kerr

- We can normalize the Killing spinor so that ξ^a is the real Killing vector $(\partial_t)^a$.
- $p = r - ia\cos\theta$ in B-L coordinates.
- $\Psi_2 p^3 = \bar{\Psi}_2 \overline{p}^3 = -M$.
- $\zeta^a = a^2(\partial_t)^a + a(\partial_\phi)^a$. (Vanish for Schwarzschild)
- Rotating case: **Killing spinor** $\Rightarrow$ All symmetries (∂_t, ∂_ϕ and K_{ab}).
- Geodesic equation integrable. (Carter constant. $Q = \dot{\gamma}^a\dot{\gamma}^b K_{ab}$)
- The **Killing spinor** can also be used for characterization of Kerr.
- Use covariant **Killing spinor** instead of coordinates for calculations.

Focus on Kerr but many results are valid for larger classes of spacetimes (Kerr-NUT).

Projectors from the Killing spinor

- $\mathcal{K}$ operators:

$$
\begin{aligned}
(\mathcal{K}^0_{k,l}\varphi)_{A_1\dots A_{k+2}} &= -6p^{-1}\kappa_{(A_1A_2}\varphi_{A_3\dots A_{k+2})} && \text{"spin raising"}\\
(\mathcal{K}^1_{k,l}\varphi)_{A_1\dots A_k} &= -3p^{-1}\kappa_{(A_1}{}^{B}\varphi_{A_2\dots A_k)B} && \text{"sign flip"}\\
(\mathcal{K}^2_{k,l}\varphi)_{A_1\dots A_{k-2}} &= \tfrac{3}{2}p^{-1}\kappa^{CD}\varphi_{A_1\dots A_{k-2}CD} && \text{"spin lowering"}
\end{aligned}
$$

The spin-0, spin-1 and spin-2 parts of the linearized curvature can be expressed by

$$
\begin{aligned}
\vartheta\Psi_2 &= \mathcal{K}^2_{2,0}\mathcal{K}^2_{4,0}\vartheta\Psi\\
&= (\dot{R}_b{}^d{}_{cd}\mathcal{Y}_a{}^c - \dot{R}_{abcd}\mathcal{Y}^{cd} - \dot{R}_{acbd}\mathcal{Y}^{cd})\mathcal{Y}^{ab}p^{-2}/6,\\
\mathcal{Z}_{ab} &= (\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0}\vartheta\Psi)_{AB}\bar{\epsilon}_{A'B'}\\
&= (-2\mathcal{Y}^{cd}\dot{R}_{[a|cd|}{}^f\mathcal{Y}_{b]f} + 3\mathcal{Y}^{cd}\dot{R}_{[a}{}^f{}_{|cd|}\mathcal{Y}_{b]f})p^{-2}/4,\\
\mathcal{W}_{abcd} &= ((\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0}\mathcal{K}^1_{4,0} - \tfrac{1}{16}\mathcal{K}^0_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^1_{2,0}\mathcal{K}^2_{4,0})\vartheta\Psi)_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'}.
\end{aligned}
$$

Expressions in a tetrad

In any principal tetrad $(l^a, n^a, m^a, \bar{m}^a)$ (aligned with principal directions of Weyl):

$$
\begin{aligned}
\vartheta\Psi_0 &= \dot{R}_{lmlm}, \\
\vartheta\Psi_1 &= \tfrac{1}{2}\dot{R}_{lmln} - \tfrac{1}{2}\dot{R}_{lmm\bar{m}}, \\
\vartheta\Psi_2 &= \tfrac{1}{6}\dot{R}_{lnln} - \tfrac{1}{3}\dot{R}_{lnm\bar{m}} + \tfrac{1}{3}\dot{R}_{lm\bar{m}n} + \tfrac{1}{6}\dot{R}_{m\bar{m}m\bar{m}}, \\
\vartheta\Psi_3 &= \tfrac{1}{2}\dot{R}_{ln\bar{m}n} - \tfrac{1}{2}\dot{R}_{m\bar{m}\bar{m}n}, \\
\vartheta\Psi_4 &= \dot{R}_{\bar{m}n\bar{m}n}, \\
\mathcal{Y}_{ab} &= ip(l_{[a}n_{b]} - m_{[a}\overline{m}_{b]}), \\
\mathcal{Z}_{ab} &= 2\vartheta\Psi_1\overline{m}_{[a}n_{b]} - 2\vartheta\Psi_3 l_{[a}m_{b]}, \\
\mathcal{W}_{abcd} &= 4\vartheta\Psi_0\bar{m}_{[a}n_{b]}\bar{m}_{[c}n_{d]} + 4\vartheta\Psi_4 l_{[a}m_{b]}l_{[c}m_{d]}.
\end{aligned}
$$

Observe that we don't formally need a frame.
Observe that $\mathcal{W}_{abcd}$ only contains $\vartheta\Psi_0, \vartheta\Psi_4$ called the Teukolsky variables.

Applications of symmetries to linearized gravity

With the help of the Killing spinor we can describe

- **All local gauge invariant quantities** from the Killing spinor and lin. curvature.
- **Field equations for gauge invariants.** Several possibilities:
  - Teukolsky master equations (TME)
  - Teukolsky Starobinsky identities (TSI)

  Both can be seen as hyperbolic.
- **Symmetry operators**.
- **Conservation laws**. (Not today.)

Benefits

- Coordinate free description.
- Characterization of Kerr in terms of Killing spinor.
- Killing spinor can be approximated on general backgrounds
  $\Rightarrow$ approximation for all structures.

Gauge invariance

- Only **gauge invariant** quantities carries physically relevant information.

Gauge invariance

- Quantities invariant under linearized diffeomorphisms $h_{ab} = \mathcal{L}_\nu g_{ab}$ for some ν^a are called (local) gauge invariant. (Linear differential operators)
- The Teukolsky scalars $\vartheta\Psi_0$ and $\vartheta\Psi_4$ (components of lin. Weyl) are gauge invariant.

Generators

- Any linear differential operator applied to a gauge invariant is also gauge invariant.
- A set of gauge invariants is **generating** if all gauge invariants can we expressed as a linear combinations of differential operators on elements of this set.
- New result: A minimal generating set of gauge invariant quantities for linearized gravity on Kerr.

Gauge invariants from Killing vectors

$$p = \sqrt{\mathcal{Y}_{bd}\mathcal{Y}^{bd}} = r - iax, \qquad U_a = -\nabla_a \log(p).$$

Proposition

Let V^a be a real Killing vector field and

$$\mathbb{I}_V = p^2 W^a \nabla_a (p^4 \vartheta \Psi_2) - \tfrac{1}{2}\mathrm{Re}(p^6 \vartheta \Psi_2 \nabla_a W^a) - 2i\mathrm{Im}(p^6 U^a W^b \mathcal{Z}_{ab}) - \tfrac{3}{2} p^6 \Psi_2 U^a W^b h_{ab}$$

where the vector field $W_a \equiv 2ip^{-3}V^b\mathcal{Y}_{ab}$ is assumed to satisfy the condition

$$\bar{p}^3 \bar{U}_{[a} \bar{W}_{b]} = -p^3 U_{[a} W_{b]}. \tag{1}$$

Then $\mathbb{I}_V$ is a local gauge invariant.

On Kerr: ξ^a and ζ^a satisfies (1) $\Rightarrow \mathbb{I}_\xi$, $\mathbb{I}_\zeta$ are gauge invariants.

All gauge invariant for perturbations of Kerr

Corollary

A set of local gauge invariant quantities for perturbations of the Kerr spacetime is given by the 18 components

Teukolsky scalars	$\vartheta\Psi_0, \vartheta\Psi_4,$	*(Second order)*	(2a)
Linearized Ricci	$\dot{R}_{ab} = \dot{R}_{acb}{}^{c},$	*(Second order)*	(2b)
Killing invariants	$\mathbb{I}_\xi, \mathbb{I}_\zeta.$	*(Third order)*	(2c)

Complete description

The set of gauge invariants above is minimal and generates all local gauge invariants for perturbations of the Kerr spacetime with $a \neq 0$.

Minkowski: All 20 components of the linearized curvature tensor.

Information carried by $\mathbb{I}_\xi, \mathbb{I}_\zeta$

$\vartheta\Psi_0 = 0$, $\vartheta\Psi_4 = 0$, $\dot{R}_{ab} = 0$, $\Rightarrow$ Linearized vacuum type D with parameters M, N, a, c (mass, NUT charge, angular momentum and c-metric)

- $\dot{M}$, $\dot{a}$ perturbations

$$\mathbb{I}_\xi = \dot{M}, \qquad \mathbb{I}_\zeta = 2a^2\dot{M} - 3Ma\dot{a},$$

- $\dot{N}$ perturbations with $x = \cos\theta$

$$\mathbb{I}_\xi = -i\dot{N} + \frac{2iM}{\bar{p}}\dot{N}, \qquad \mathbb{I}_\zeta = -ia^2\dot{N} + ax(r - 2M - \frac{Mp}{\bar{p}})\dot{N},$$

- $\dot{c}$ perturbations

$$\mathbb{I}_\xi = \frac{6M^2rx}{\bar{p}}\dot{c} + 3M\big(ia + (M-r)x\big)\dot{c}, \quad \mathbb{I}_\zeta = \frac{6M^2a^2rx^3}{\bar{p}}\dot{c} - 3iMa(p^2 - r^2x^2)\dot{c}.$$

- Important to control $\dot{M}$ and $\dot{a}$ modes for self-force problems and stability problems.

Differential relations

- The gauge invariants satisfy differential equations.
- Minkowksi: Linearized Bianchi.
- Kerr: Linearized Bianchi gives compatibility conditions between the gauge invariants.
- Important for the proof that the set generates all gauge invariants. (Not in this talk)

Hyperbolic evolution equations for lin. vacuum $\dot{R}_{ab} = 0$.

- Linearized Bianchi not always practical for estimates.
- Standard method: Apply another derivative to get wave equations for curvature components.
- This gives Teukolsky equations (TME) for $\vartheta\Psi_0, \vartheta\Psi_4$ (wave eqs) on Kerr.
- Can also get other equations: Teukolsky-Starobinsky identities (TSI) relating $\vartheta\Psi_0$ and $\vartheta\Psi_4$. (Also hyperbolic)
- Evolution equations involving $\mathbb{I}_\xi, \mathbb{I}_\zeta$ under investigation.

Field equations $\Rightarrow$ decoupled, separable *integrability conditions*

- Teukolsky Master Equations (TME):

$$\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2}+\frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\,\partial\varphi}+\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)$$

$$-2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}-2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\frac{\partial\psi}{\partial t}+(s^2\cot^2\theta-s)\psi=4\pi\Sigma T.$$

- Teukolsky-Starobinsky Identities (TSI): $\psi = e^{-i\omega t}e^{im\phi}S(\theta)R(r)$

$$\mathscr{L}_{-1}\mathscr{L}_0\mathscr{L}_1\mathscr{L}_2S_2 + 12Mi\omega S_2{}^{\dagger} = S_{-2}\,,$$

$$\mathscr{D}\mathscr{D}\mathscr{D}\mathscr{D}R_{-2} = \tfrac{1}{4}R_2\,.$$

(Properties of confluent Heun functions.)

Maxwell equations (source free)

Potential A_a		Field		Extreme components ϕ_0, ϕ_2
$\mathbf{E}A = 0$	$\longrightarrow$	$\phi = \mathbf{T}A$	$\longrightarrow$	TME: wave eq. for ϕ_0 or ϕ_2 (2nd order)
2nd order eq.		1st order		TSI: eq. relating ϕ_0 and ϕ_2 (2nd order)

TME=Teukolsky Master Equations, TSI=Teukolsky–Starobinsky Identities

Linearized gravity (vacuum)

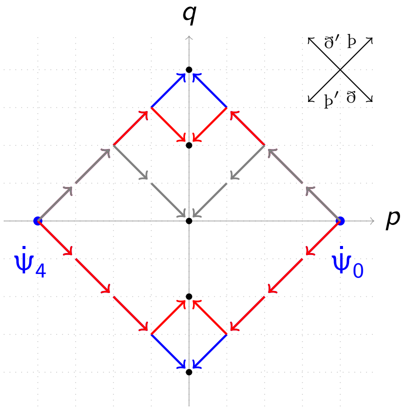
Lin. metric h_{ab}		Lin. Weyl		Extreme components $\dot{\Psi}_0$, $\dot{\Psi}_4$
$\mathbf{E}h = 0$	$\longrightarrow$	$\dot{\Psi} = \mathbf{T}h$	$\longrightarrow$	TME: wave eq. for $\dot{\Psi}_0$ or $\dot{\Psi}_4$ (2nd order)
2nd order eq.		2nd order		TSI: eq. relating $\dot{\Psi}_0$ and $\dot{\Psi}_4$ (4th order)

- Extreme components: **gauge invariant** and carry dynamical degrees of freedom.
- TME often used (gravitational waves), but TSI also carries important information.
- Not all solutions to TME correspond to solutions of linearized gravity.

- The TME is decoupled set of two hyperbolic (weighted) scalar wave equations (TME).

Completing TSI to a hyperbolic system

- One can formulate TSI as a set of two coupled differential equations.
  (Derivatives of linearized Bianchi.)
- TSI can be completed to "full TSI" with 5 equations (3 for Maxwell).
- **Remarkable:** Full TSI gives a first order symmetric hyperbolic system.
- Alternative to TME evolution.



Symmetric hyperbolic systems

First order symmetric hyperbolic systems

Any equation of the form

$$\nabla^{A}{}_{A'}\phi_{\dots A\dots} = \varphi_{\dots A'\dots} \quad \text{or} \quad \nabla_{A}{}^{A'}\phi_{\dots A'\dots} = \varphi_{\dots A\dots}$$

gives a *first order symmetric hyperbolic system*.

Maxwell

Let ϕ_{AB} be a Maxwell field, and κ_{AB} the Killing spinor ($p^2 = -\frac{9}{2}\kappa_{AB}\kappa^{AB}$.). Define

$$\varphi_{AB} = -\tfrac{1}{3}p\kappa_{(A}{}^{C}\phi_{B)C}, \qquad \psi_{AA'} = -p^2\nabla_{BA'}(p^{-2}\varphi)_{A}{}^{B}.$$

The TME and full TSI for Maxwell can be written as

$$\nabla_{(A}{}^{A'}\psi_{B)A'} = 0, \qquad \nabla^{A}{}_{(A'}\psi_{|A|B')} = 0.$$

2 TME equations, 3 TSI equations (cf Coll et. al.)

Use the commutator $\nabla^{AA'}\psi_{AA'} = -2U^{AA'}\psi_{AA'}$, with $U_{AA'} = -\nabla_{AA'}\log(p)$ to write

First order symmetric hyperbolic system for the Maxwell TME

$$\nabla^{B}{}_{A'}\varphi_{AB} = -2U^{B}{}_{A'}\varphi_{AB} + \psi_{AA'},$$
$$\nabla_{A}{}^{A'}\psi_{BA'} = \epsilon_{AB}U^{CA'}\psi_{CA'},$$

First order symmetric hyperbolic system for the Maxwell TSI

$$\nabla^{B}{}_{A'}\varphi_{AB} = -2U^{B}{}_{A'}\varphi_{AB} + \psi_{AA'},$$
$$\nabla^{A}{}_{A'}\psi_{AB'} = U^{AC'}\bar{\epsilon}_{A'B'}\psi_{AC'}.$$

New view

- Also TSI is a hyperbolic evolution system.
- Same variables for TME and TSI.
- Given an initial data surface with normal $n^{AA'}$ the difference

$$0 = -n^{D}{}_{A'}\nabla_{DB'}\psi_{A}{}^{B'} + n_{A}{}^{B'}\nabla_{DB'}\psi^{D}{}_{A'}.$$

  is a constraint for the initial data. (Spatial derivative.)
- TSI as constraint under TME evolution. (Propagates)
- TME as constraint under TSI evolution. (Propagates)

First order symmetric hyperbolic system for linearized gravity TSI

- Have a system.

$$\nabla^{A}{}_{A'} \begin{bmatrix} \varphi_{ABCD} \\ \psi_{ABCB'} \\ \chi_{ABB'C'} \\ \eta_{AB'C'D'} \\ \alpha_{AB'} \\ \beta_{AB} \end{bmatrix} = \dots$$

- Involves Teukolsky scalars and another gauge invariant ($\alpha_{AA'}$).

First order symmetric hyperbolic system for linearized gravity TME

- TME implies another hyperbolic system in *the same variables*.
- TME as constraint for TSI and vice versa under investigation.

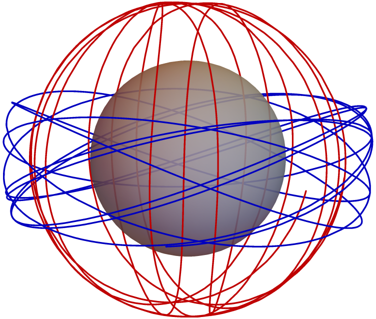
Symmetry operators

Definition

A *symmetry operator* is a linear differential operator that maps a solution of a differential equation to a solution.

Examples

- Lie derivatives along (conformal) Killing vectors for wave eq in Minkowski.



Motivation for symmetry operators

- Explains separability and integrability.
- Useful for energy estimates:
  - Sobolev norms with symmetry operators instead of partial derivatives.
  - Higher order conservation laws.
  - Morawetz estimates near orbiting null geodesics for rotating black holes. (Andersson, Blue)

Symmetry operators of the Kerr spacetime

Symmetries on the Kerr spacetime

- Classical symmetry operators: Lie derivatives along Killing vectors (∂_t and ∂_φ).
- Hidden symmetry operator for scalar wave equation:
  $\nabla_a K^{ab} \nabla_b$ (Carter) where is the Killing tensor K^{ab}, $\nabla_{(a} K_{bc)} = 0$.
- Use **Killing spinor** to derive symmetry operators for Maxwell and lin. gravity.

Symmetry operators from operator identities

Operator identities

- So far: TME and TSI in the source-free case (lin. vacuum).
- Include source terms (lin. Einstein) $\Rightarrow$ operator identities relating lin. Weyl components with lin. Einstein components. (cf Bianchi).
- Identity: $\mathbf{SE} = \mathbf{OT}$
- $\mathbf{E}$ linearized Einstein.
- $\mathbf{T} : h_{ab} \mapsto \vartheta\Psi_{ABCD}$.
- $\mathbf{O}$ TME or TSI.

Symmetry operators from TME and TSI (Maxwell and lin. gravity)

- Wald 1978: TME operator identity $\Rightarrow$ symmetry operator.
- We: Also TSI operator identity $\Rightarrow$ symmetry operator.

Adjoint operator method

Idea (Wald 1978)

Use adjoint operator method to produce solutions of Maxwell or linearized gravity from solutions of TME. (Leads to symmetry operator.)

For any linear partial differential operator $\mathbf{A}$, define

- $\mathbf{A}^\dagger$ adjoint w.r.t. $(\phi, \psi) = \int \phi \psi d\mu$, i.e. $(\mathbf{A}^\dagger \phi, \varphi) = (\phi, \mathbf{A}\varphi)$.
- $\mathbf{A}^\star$ adjoint w.r.t. $\langle \phi, \psi \rangle = \int \phi \bar{\psi} d\mu$, i.e. $\langle \mathbf{A}^\star \phi, \varphi \rangle = \langle \phi, \mathbf{A}\varphi \rangle$.

Theorem

Suppose the identity

$$\mathbf{SE} = \mathbf{OT}$$

holds for linear partial differential operators $\mathbf{S}, \mathbf{E}, \mathbf{O}$ *and* $\mathbf{T}$.
Suppose ψ *satisfies* $\mathbf{O}^\dagger \psi = 0$. *Then* $\mathbf{S}^\dagger \psi$ *satisfies* $\mathbf{E}^\dagger (\mathbf{S}^\dagger \psi) = 0$.
In particular, if $\mathbf{E}$ *is self-adjoint then* $\mathbf{E}\mathbf{S}^\dagger \psi = 0$. *(Also works with* $\star$*-adjoint.)*

Proof: $\mathbf{E}^\dagger \mathbf{S}^\dagger = \mathbf{T}^\dagger \mathbf{O}^\dagger$.

General

- Linearized Einstein operator $\mathbf{E}$ (2nd order). $\mathbf{E}^\dagger = \mathbf{E}^\star = \mathbf{E}$.
- Linearized Weyl operator $\mathbf{T} : h \mapsto \vartheta\Psi$ (2nd order).

TME and a 4th order symmetry operator

- TME equation $\mathbf{O}\vartheta\Psi = 0$ (2nd order) involves only $\vartheta\Psi_0$ and $\vartheta\Psi_4$. ($\mathbf{O}^\dagger = \mathbf{O}$)
- The identity $\mathbf{SE} = \mathbf{OT}$ holds for a 2nd order $\mathbf{S}$.
- $\mathbf{S}^\dagger\vartheta\Psi$ is a **new complex solution to the linearized gravity**.
- Symmetry operator $\mathbf{S}^\dagger\mathbf{T}$ from metric to metric is **4th order**.
- We have explicit expressions in a new powerful covariant operator formalism involving **only** covariant derivatives and the Killing spinor.

Recall

- Linearized Einstein operator $\mathbf{E}$ (2nd order). $\mathbf{E}^\dagger = \mathbf{E}^\star = \mathbf{E}$.
- Linearized Weyl operator $\mathbf{T} : h \mapsto \vartheta\Psi$ (2nd order).

TSI and a 6th order symmetry operator (Aksteiner & Bäckdahl 2016)

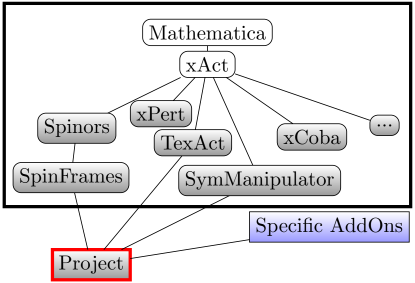
- TSI equation $\widehat{\mathbf{O}}\vartheta\Psi - \widehat{\mathbf{L}}\overline{\vartheta\Psi} = 0$ involves only $\vartheta\Psi_0$ and $\vartheta\Psi_4$.
- $\widehat{\mathbf{O}}^\star = \widehat{\mathbf{O}}$ is 4th order, and $\widehat{\mathbf{L}}^\dagger = \widehat{\mathbf{L}}$ is 1st order (Lie derivative).
- The identity $\widehat{\mathbf{S}}\mathbf{E} = \widehat{\mathbf{O}}\mathbf{T} - \widehat{\mathbf{L}}\overline{\mathbf{T}}$ holds for a 4th order $\widehat{\mathbf{S}}$.
- $\mathbf{E}(\widehat{\mathbf{S}}^\star\vartheta\Psi + \overline{\widehat{\mathbf{S}}^\star\vartheta\Psi}) = \mathbf{T}^\star(\widehat{\mathbf{O}}\vartheta\Psi - \widehat{\mathbf{L}}\overline{\vartheta\Psi}) + \overline{\mathbf{T}}^\star(\overline{\widehat{\mathbf{O}}\vartheta\Psi} - \overline{\widehat{\mathbf{L}}}\vartheta\Psi)$
- $\hat{k}_{ABA'B'} = (\widehat{\mathbf{S}}^\star\vartheta\Psi)_{ABA'B'} + (\overline{\widehat{\mathbf{S}}^\star\vartheta\Psi})_{ABA'B'}$ is a **real solution to linearized gravity**
- The symmetry operator for linearized gravity (from metric to metric) is **6th order**.

Symbolic calculations with xAct

Efficient tensor/spinor computer algebra for Mathematica

Calculations leading to the results just presented were performed with xAct.
Based on the Killing spinor $\longrightarrow$ coordinate free calculations.

- Open source, www.xAct.es
- Powerful algorithms to handle and use symmetries of tensors and spinors
- All operators of this talk are implemented
- xAct Packages: *SymManipulator*, *SpinFrames*, *TexAct*
    - Irreducible decompositions
    - Fundamental spinor operators
    - NP and GHP formalisms
    - Structured Tex output



Summary and concluding remarks

- All gauge invariant quantities for linearized gravity on Kerr.
- New view on the Teukolsky-Starobinsky Identities (TSI).
- TSI $\Rightarrow$ **hyperbolic evolution system**.
- Both TME and TSI generate symmetry operators.
- For linearized gravity we get a 4th order symmetry operator (known), and a **new** 6th order symmetry operator.
- Covariant techniques and efficient formalisms $\Rightarrow$ new insights.
- Robust operator formulation could work for "almost Kerr" backgrounds.

Thank you!