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Mathematical Systems and Control Theory – 4th Exercise Sheet.

Discussion of the solutions in the exercise on December 11, 2019.

Problem 1 (Kronecker product): Let $A = [a_{ij}] \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ and define the *Kronecker product*

$$\otimes : \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times q} \to \mathbb{R}^{np\times mq}, \quad A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np\times mq}.$$

Show the following properties for matrices A, B, C, D of conforming dimensions:

a) $(A \otimes B)(C \otimes D) = AC \otimes BD$;

b) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, if A and B are both invertible;

c) $\Lambda(A \otimes B) = \{\lambda\mu \mid \lambda \in \Lambda(A),\ \mu \in \Lambda(B)\}$.

Problem 2 (Sylvester equations): Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, and $W \in \mathbb{R}^{n\times m}$ be given matrices and consider the *Sylvester equation*

$$AX + XB = W \quad \text{for } X \in \mathbb{R}^{n\times m}. \tag{1}$$

Consider the *vectorization operator*

$$\operatorname{vec} : \mathbb{R}^{n\times m} \to \mathbb{R}^{nm}, \quad X = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

Show that for $T \in \mathbb{R}^{n\times m}$, $O \in \mathbb{R}^{m\times p}$, and $R \in \mathbb{R}^{p\times r}$ it holds that

$$\operatorname{vec}(TOR) = \left(R^{\mathsf{T}} \otimes T\right) \operatorname{vec}(O),$$

and conclude that (1) can be equivalently written as a linear system of the form

$$\Big(\left(I_m \otimes A\right) + \left(B^{\mathsf{T}} \otimes I_n\right)\Big) \operatorname{vec}(X) = \operatorname{vec}(W).$$

Problem 3 (Theorem of Stephanos):

a) Prove the *Theorem of Stephanos:* Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be given. For a bivariate polynomial $p(x,y) = \sum\limits_{i,j=0}^{k} c_{ij} x^i y^j$ we define by

$$p(A,B) := \sum_{i,j=0}^{k} c_{ij} (A^i \otimes B^j)$$

a polynomial of the two matrices. Then the spectrum of $p(A,B)$ is given by

$$\Lambda(p(A,B)) = \{p(\lambda,\mu) \mid \lambda \in \Lambda(A),\ \mu \in \Lambda(B)\}.$$

b) Use the Theorem of Stephanos to show that the Sylvester equation (1) is uniquely solvable for all $W \in \mathbb{R}^{n\times m}$, if and only if $\Lambda(A) \cap \Lambda(-B) = \emptyset$.

Problem 4 (numerical solution of Lyapunov equations): Let $A \in \mathbb{C}^{n\times n}$, $W = W^{\mathsf{H}} \in \mathbb{C}^{n\times n}$ and consider the Lyapunov equation

$$AX + XA^{\mathsf{H}} = W,$$

which is assumed to be uniquely solvable. Devise a numerical algorithm that uses at most $\mathcal{O}(n^3)$ floating point operations for computing the solution matrix $X \in \mathbb{C}^{n\times n}$.
Hint: Use the *Schur decomposition* of A, that is, there exists a unitary matrix $Q \in \mathbb{C}^{n\times n}$ (which can be numerically computed in $\mathcal{O}(n^3)$ floating point operations) such that

$$Q^{\mathsf{H}}AQ = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}.$$