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Winter Term 2019/20

Mathematical Systems and Control Theory – 2nd Exercise Sheet.

Discussion of the solutions in the exercise on November 13, 2019.

Problem 1 (controllability Gramian): Show that the following statements are satisfied:

a) The (t_0, t_1)-controllability Gramian $P(t_0, t_1)$ of an LTV system with state equation $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ is positive definite, if and only if

$$\widehat{P}(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, t)B(t)B(t)^{\mathsf{T}}\Phi(t_1, t)^{\mathsf{T}}\mathrm{d}t$$

is positive definite.

b) An symptotically stable LTI system (with $u \equiv 0$) with the state equation $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable, if and only if the controllability Gramian

$$P = \int_0^\infty \mathrm{e}^{At}BB^{\mathsf{T}}\mathrm{e}^{A^{\mathsf{T}}t}\mathrm{d}t$$

is positive definite.

Problem 2 (properties of the matrix exponential function): show the following properties of the matrix exponential for two matrices A, $B \in \mathbb{R}^{n\times n}$:

a) $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{At} = A\mathrm{e}^{At} = \mathrm{e}^{At}A$;

b) $\mathrm{e}^{(A+B)t} = \mathrm{e}^{At}\mathrm{e}^{Bt} \quad \Leftrightarrow \quad AB = BA$.

c) If A is skew-symmetric, then e^{A} is orthogonal.

Problem 3 (Laplace transformation and frequency domain): For a function $f : [0, \infty) \to \mathbb{R}$, its *Laplace transform* is defined by

$$F(s) := \mathcal{L}\{f\}(s) := \int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$$

(assuming that the integral exists). Let now f, $g : [0, \infty) \to \mathbb{R}$ and α, $\beta \in \mathbb{C}$. Under the assumption that all of the following Laplace transforms exist, show that:

a) $\mathcal{L}\{\alpha f + \beta g\}(s) = \alpha\mathcal{L}\{f\}(s) + \beta\mathcal{L}\{g\}(s)$;

b) $\mathcal{L}\{\dot{f}\}(s) = s\mathcal{L}\{f\}(s) - f(0)$;

c) $\mathcal{L}\left\{\int_0^\bullet f(\tau)\mathrm{d}\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f\}(s)$;

d) $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{n-1}(0)$;

e) $\mathcal{L}\{\mathrm{e}^{a\cdot\bullet}\}(s) = \frac{1}{s-a}$ for $\mathrm{Re}\,(s) > a$;

f) $\mathcal{L}\{\bullet^n\}(s) = \frac{n!}{s^{n+1}}$ for $\mathrm{Re}\,(s) > 0$.

Consider now the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t),$$

with $x(0) = 0$ and assume that the Laplace transforms of $\dot{x}$, x, u, y all exist for $s \in \mathbb{C}$. Show that:

g) $\mathcal{L}\{y\}(s) = \big(C(sI - A)^{-1}B + D\big)\,\mathcal{L}\{u\}(s)$.