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Optimization of Complex Systems – 4th Exercise Sheet.

Discussion of the solutions in the exercise on November 25, 2019.

Problem 1 (weak formulation of the Poisson equation): Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Consider the Poisson equation

$$\begin{aligned} -\Delta y &= f && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma_1, && \text{(Dirichlet boundary conditions)} \\ \frac{\partial y}{\partial n} &= 0 && \text{on } \Gamma_2 && \text{(Neumann boundary conditions)} \\ \frac{\partial y}{\partial n} + \sigma y &= 0 && \text{on } \Gamma_3, && \text{(Robin boundary conditions)} \end{aligned}$$

where $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3 = \Gamma := \partial\Omega$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cap \Gamma_3 = \emptyset$, and $\Gamma_2 \cap \Gamma_3 = \emptyset$, $\sigma > 0$, and n denotes the outward normal vector. Note further that $\frac{\partial y}{\partial n} := n \cdot \nabla y$.
Show that the weak formulation of this problem is to find a $y \in V_0 := \{v \in H^1(\Omega) \ : \ v = 0 \text{ on } \Gamma_1\}$ such that

$$\int_\Omega \nabla y \cdot \nabla v \mathrm{d}x + \int_{\Gamma_3} \sigma y v \mathrm{d}s = \int_\Omega f v \mathrm{d}x \quad \forall\, v \in V_0.$$

Hint: Use the N-dimensional version of the product rule

$$\operatorname{div}(v \nabla y) = \nabla y \cdot \nabla v + v \Delta y,$$

as well as the Theorem of Gauss which gives

$$\int_\Omega \operatorname{div}(w) \mathrm{d}x = \int_\Gamma n \cdot w \mathrm{d}s.$$

Problem 2 (Poincaré-Friedrichs inequality): Let $\Omega \subset \mathbb{R}$ be a bounded domain and $y \in H_0^1(\Omega)$. Prove the Poincaré-Friedrichs inequality

$$\|y\|_{L^2(\Omega)}^2 \leq C \cdot \|y'\|_{L^2(\Omega)}^2.$$

Use the fact, that for $y \in H_0^1(\Omega)$ it holds that

$$y(x) = \int_0^x y'(z) \mathrm{d}z.$$

Problem 3 (coercivity): For finite-dimensional problems, the coercivity of the symmetric bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is equivalent to the condition $a(v, v) > 0$ for all $v \in \mathcal{V} \setminus \{0\}$. (Why?). Let now

$$\mathcal{V} = \ell_2 := \left\{ u = (u_n)_{n \in \mathbb{N}} \ : \ \sum_{n=1}^\infty u_n^2 < \infty \right\}$$

with the norm

$$\|u\|_2 := \left(\sum_{n=1}^\infty u_n^2 \right)^{1/2}.$$

a) Construct a bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ with $a(v, v) > 0$ for all $v \in \mathcal{V} \setminus \{0\}$ which is *not* coercive.

b) With the bilinear form $a(\cdot, \cdot)$ from a), construct a linear functional $F \in \mathcal{V}^*$ such that the problem

$$a(y, v) = F(v) \quad \forall\, v \in \mathcal{V}$$

does not have a unique solution $y \in \mathcal{V}$.

Problem 4 (weak convergence): Let $\mathcal{U}$ be a normed vector space and let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}$. Show the following statements:

a) If $u_n \rightharpoonup u \in \mathcal{U}$, then the weak limit u is unique.

b) Strong convergence $u_n \to u$ implies weak convergence $u_n \rightharpoonup u$ and the converse holds if $\mathcal{U}$ is finite-dimensional.

c) In a Hilbert space $\mathcal{H}$, the weak convergence $u_n \rightharpoonup u$ is equivalent to

$$(v, u_n)_{\mathcal{H}} \to (v, u)_{\mathcal{H}} \quad \forall\, v \in \mathcal{H}.$$

d) In a Hilbert space $\mathcal{H}$ we have:

$$u_n \rightharpoonup u \text{ and } \|u_n\|_{\mathcal{H}} \to \|u\|_{\mathcal{H}} \quad \Leftrightarrow \quad u_n \to u.$$